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Hausdorff measure of the graph of fractional Brownian motion

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Abstract

Let $X(t)$ $(t \in \mathbf{R}^N)$ be a fractional Brownian motion in $\mathbf{R}^d$ of index α. Let $\mathrm{Gr}\, X([0,1]^N)$ be the graph of X and let

$$\phi_1(s) = s^{N/\alpha} \log\, \log \frac{1}{s}$$

and

$$\phi_2(s) = s^{N+(1-\alpha)s} \left(\log\, \log \frac{1}{s}\right)^{\alpha d/N} .$$

It is proved that if $N < \alpha d$, then almost surely

$$K_1 \leqslant \phi_1\text{-}m\big(\mathrm{Gr}\, X([0,1]^N)\big) \leqslant K_2$$

and if $N > \alpha d$, then almost surely

$$K_1 \leqslant \phi_2\text{-}m\big(\mathrm{Gr}\, X([0,1]^N)\big) \leqslant K_2 \ ,$$

where ϕ-m is the ϕ-Hausdorff measure and $K_1,\ K_2$ are positive finite constants. The exact Hausdorff measure of the image and graph of certain Gaussian random fields with independent fractional Brownian motion components are also obtained.

1. *Introduction*

Given $0 < \alpha < 1$, let $Y = \{Y(t),\ t \in \mathbf{R}^N\}$ be the real-valued, centred gaussian random field with covariance function

$$E(Y(t)Y(s)) = \tfrac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}),$$

where $|\cdot|$ denotes the Euclidean norm. Consider the gaussian random field $X = \{X(t),\ t \in \mathbf{R}^N\}$ in $\mathbf{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)) \ ,$$

where $X_1, \ldots, X_d$ are independent copies of Y. Then X is called the (N, d, α) gaussian process or the d-dimensional fractional Brownian motion (FBM) of index α (see [**K**], chapter 18). If $N = 1,\ \alpha = \frac{1}{2}$, it is the ordinary d-dimensional Brownian motion; if $\alpha = \frac{1}{2},\ d = 1$, it is the multiparameter Lévy Brownian motion. It is easy to see that X is a self-similar process of exponent α, i.e. for any $a > 0$,

$$X(a\,\cdot) \overset{\mathrm{d}}{=} a^{\alpha} X(\cdot) \ ,$$

where $X \overset{\mathrm{d}}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions.

The Hausdorff dimension of the image and graph of X and more general gaussian random fields were obtained by Adler ([**A**]). See Kahane ([**K**]) for more geometric properties of the image, graph and level sets of fractional Brownian motion. The exact Hausdorff measure of the image set $X([0,1]^N)$ of transient fractional Brownian motion (that is $N < \alpha d$) was obtained by Goldman ([**G**]) for $\alpha = \frac{1}{2}$, and by Talagrand ([**T**]) for all $0 < \alpha < 1$. Their results were generalized by the author ([**X2**]) to strongly locally nondeterministic gaussian random fields.

In this paper, we consider the exact Hausdorff measure of the graph of fractional Brownian motion X. Let

$$\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) \colon t \in [0,1]^N\} .$$

Alder ([**A**]) proved that

$$\dim \mathrm{Gr}X([0,1]^N) = \begin{cases} N/\alpha & \text{if } N \leqslant \alpha d \\ N + (1-\alpha)d & \text{if } N > \alpha d, \end{cases}$$

where dim denotes Hausdorff dimension. Let

$$\phi_1(s) = s^{N/\alpha} \log \log \frac{1}{s} ,$$

$$\phi_2(s) = s^{N+(1-\alpha)d} \left(\log \log \frac{1}{s}\right)^{\alpha d/N} .$$

We shall show that there exist positive and finite constants K_1 and K_2 such that almost surely

$$K_1 \leqslant \phi_1\text{-}m(\mathrm{Gr}X([0,1]^N)) \leqslant K_2 \quad \text{if } N < \alpha d , \tag{1·1}$$

and

$$K_1 \leqslant \phi_2\text{-}m(\mathrm{Gr}X([0,1]^N)) \leqslant K_2 \quad \text{if } N > \alpha d , \tag{1·2}$$

where ϕ-m is the ϕ-Hausdorff measure.

In the critical case of $N = \alpha d$, except for the planar Brownian motion (i.e. $N = 1,\ \alpha = \frac{1}{2},\ d = 2$) for which Pruitt and Taylor ([**PT**]) showed that

$$\phi(s) = s^2 \log \frac{1}{s} \log \log \log \frac{1}{s}$$

is the correct Hausdorff measure function for the graph, the Hausdorff measure of the graph $\mathrm{Gr}X([0,1]^N)$ is not known.

The Hausdorff measure of the graph of Brownian motion and Lévy stable processes were evaluated by Jain and Pruitt ([**JP**]) in the transient case, by Pruitt and Taylor ([**PT**]) in the recurrent cases. In the case of Brownian motion, their results are recovered by (1·1) and (1·2). Since their proofs depend on the specific properties of Brownian motion such as the strong Markov property, we can not prove (1·1) and (1·2) by modifying their arguments.

Another important example of gaussian random fields is the Brownian sheet or N-parameter Wiener process $W(t)$ ($t \in \mathbf{R}^N_+$), see Orey and Pruitt ([**OP**]). The Hausdorff measure of the image and graph of the Brownian sheet in $\mathbf{R}^d$ were obtained by

Ehm ([**E**]). Since the dependence structure of fractional Brownian motion is more complicated than that of the Brownian sheet, we can not apply his method either.

Let $X_i = \{X_i(t),\ t \in \mathbf{R}^N\}$ $(i = 1, 2, \ldots, d)$ be d independent fractional Brownian motion in $\mathbf{R}$ of index α_i with

$$0 = \alpha_0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_d < 1\ .$$

Let $Z(t) = (X_1(t), \ldots, X_d(t))$. Then $Z = \{Z(t),\ t \in \mathbf{R}^N\}$ is a gaussian random field in $\mathbf{R}^d$ with stationary increments. If α_i $(i = 1, \ldots, d)$ are not all equal, then Z is not self-similar. Sample path properties of Z have been studied by Cuzick ([**C**]), Pitt ([**P**]) and the author ([**X1**]). It follows from theorem 2·1 in [**X1**] that almost surely

$$\begin{aligned}\dim\ Z([0,1]^N) &= \min\left\{d;\ \frac{N + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\quad i = 1, \ldots, d\right\}\\ &= \begin{cases} \dfrac{N + \sum_{i=1}^{k}(\alpha_k - \alpha_i)}{\alpha_k} & \text{if } \sum_{i=0}^{k-1}\alpha_i < N \leqslant \sum_{i=0}^{k}\alpha_i \\ d & \text{if } N > \sum_{i=1}^{d}\alpha_i \end{cases}\end{aligned}$$

$$\dim\ \mathrm{Gr}\, Z([0,1]^N) = \begin{cases} \dim\ Z([0,1]^N) & \text{if } N \leqslant \sum_{i=1}^{d}\alpha_i \\ N + \sum_{i=1}^{d}(1-\alpha_i) & \text{if } N > \sum_{i=1}^{d}\alpha_i\ . \end{cases}$$

In Section 4, we consider the exact Hausdorff measure for the image $Z([0,1]^N)$ and the graph $\mathrm{Gr}\, Z([0,1]^N)$.

We shall use K to denote unspecified positive finite constant, it may be different in each appearance.

2. *Preliminaries*

Let Φ be the class of functions ϕ: $(0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leqslant K, \quad \text{for } 0 < s < \tfrac{1}{2}\delta.$$

For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbf{R}^N$ is defined by

$$\phi\text{-}m(E) = \lim_{\epsilon \to 0} \inf\left\{\sum_i \phi(2r_i)\colon\ E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\quad r_i < \epsilon\right\},$$

where $B(x, r)$ denotes the open ball of radius r centred at $x \in \mathbf{R}^N$. The Hausdorff dimension of E is defined by

$$\begin{aligned}\dim\ E &= \inf\{\alpha > 0\colon\ s^\alpha\text{-}m(E) = 0\}\\ &= \sup\{\alpha > 0\colon\ s^\alpha\text{-}m(E) = \infty\}\ .\end{aligned}$$

We refer to [**F**] for more properties of Hausdorff measure and Hausdorff dimension.

Lemma 2·1 is derived from a density theorem for Hausdorff measure ([**RT**]), which gives a way to get a lower bound for ϕ-$m(E)$. For any Borel measure μ on $\mathbf{R}^N$ and

$\phi \in \Phi$, the upper ϕ-density of μ at $x \in \mathbf{R}^N$ is defined by

$$\overline{D}_\mu^\phi(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{\phi(2r)} .$$

LEMMA 2·1. *For a given $\phi \in \Phi$ there exists a positive constant K such that for any Borel measure μ on $\mathbf{R}^N$ and every Borel set $E \subseteq \mathbf{R}^N$, we have*

$$\phi\text{-}m(E) \geqslant K\mu(E) \inf_{x \in E} \{\overline{D}_\mu^\phi(x)\}^{-1} .$$

The following lemma will be useful in our study on the sojourn time of fractional Brownian motion and other strongly locally nondeterministic gaussian random fields.

LEMMA 2·2. *Let $0 < \beta < N$ and $r > 0$. Then for any distinct $t_1, \ldots, t_n \in B(0,r)$, we have*

$$\int_{B(0,\,r)} \frac{dt}{\min\{|t-t_i|^\beta, \quad i = 1, \ldots, n\}} \leqslant K n^{\beta/N} r^{N-\beta} , \tag{2·1}$$

where K is a positive finite constant depending on N and β only.

Proof. Let

$$\Gamma_i = \{t \in B(0,r) \colon |t-t_i| = \min\{|t-t_j|, \quad j = 1, \ldots, n\}\} .$$

Then

$$B(0,r) = \bigcup_{i=1}^n \Gamma_i \quad \text{and} \quad L_N(B(0,r)) = \sum_{i=1}^n L_N(\Gamma_i) , \tag{2·2}$$

where L_N is the Lebesgue measure in $\mathbf{R}^N$. For any $t \in \Gamma_i$, we write $t = t_i + \rho\theta$, where $\theta \in S_{N-1}$ the unit sphere in $\mathbf{R}^N$ and $0 \leqslant \rho \leqslant \rho_i(\theta)$. Then

$$\begin{aligned} L_N(\Gamma_i) &= C_N \int_{S_{N-1}} \nu(d\theta) \int_0^{\rho_i(\theta)} \rho^{N-1} d\rho \\ &= \frac{C_N}{N} \int_{S_{N-1}} \rho_i(\theta)^N \nu(d\theta) , \end{aligned} \tag{2·3}$$

where ν is the normalized surface area in S_{N-1} and C_N is a positive finite constant depending on N only. Hence by (2·3), Jensen's inequality and (2·2), we have

$$\begin{aligned} &\int_{B(0,\,r)} \frac{dt}{\min\{|t-t_i|^\beta, \quad i = 1, \ldots, n\}} \\ &\qquad = \sum_{i=1}^n \int_{\Gamma_i} \frac{dt}{|t-t_i|^\beta} \\ &\qquad = \sum_{i=1}^n C_N \int_{S_{N-1}} \frac{1}{N-\beta}\, \rho_i(\theta)^{N-\beta} \nu(\theta) \\ &\qquad \leqslant K \sum_{i=1}^n \left(\int_{S_{N-1}} \frac{1}{N}\, \rho_i(\theta)^N \nu(\theta) \right)^{(N-\beta)/N} \\ &\qquad \leqslant K \sum_{i=1}^n (L_N(\Gamma_i))^{(N-\beta)/N} \end{aligned}$$

$$\leqslant Kn\left(\frac{1}{n}\sum_{i=1}^{n} L_N(\Gamma_i)\right)^{(N-\beta)/N}$$
$$= Kn^{\beta/N} r^{N-\beta} .$$

This proves (2·1).

We also need some specific facts about fractional Brownian motion. In the following, Lemma 2·3 and Lemma 2·4 were proved by Pitt ([**P**]) and Talagrand ([**T**]) respectively.

LEMMA 2·3. *There exists a positive finite constant K, depending on N and α only, such that for any $t \in \mathbf{R}^N$ and any $0 \leqslant r \leqslant |t|$,*

$$\operatorname{Var}(Y(t)|Y(s), |s-t| \geqslant r) = Kr^{2\alpha} .$$

LEMMA 2·4. *There exists a constant $\delta > 0$ such that for any $0 < r_0 \leqslant \delta$, we have*

$$P\left\{\exists r \in [r_0^2,\ r_0] \quad \textit{such that} \sup_{|t|\leqslant r} |X(t)| \leqslant Kr^{\alpha}\left(\log\log\frac{1}{r}\right)^{-\alpha/N}\right\}$$
$$\geqslant 1-\exp\left(-\left(\log\frac{1}{r_0}\right)^{\frac{1}{2}}\right) . \qquad (2·4)$$

3. *Hausdorff measure of the graph of FBM*

Let $X(t)$ $(t \in \mathbf{R}^N)$ be the fractional Brownian motion of index α $(0 < \alpha < 1)$ in $\mathbf{R}^d$ and let

$$\operatorname{Gr} X([0,1]^N) = \{(t, X(t)) \colon t \in [0,1]^N\}$$

be the graph of X. In this section, we consider the exact Hausdorff measure of $\operatorname{Gr} X([0,1]^N)$.

For any $0 < r < 1$, $t \in I = [0,1]^N$ and $y \in \mathbf{R}^d$, let

$$T_{t,y}(r,r) = \int_{B(t,r)} 1_{\{|X(s)-y|<r\}}(s)ds . \qquad (3·1)$$

If $(t,y) = (0,0)$, we write $T_{t,y}(r,r)$ as $T(r)$.

LEMMA 3·1. *If $N > \alpha d$, then there exists a positive finite constant K, depending on N, α, and d only, such that for any integer $n \geqslant 1$ we have*

$$E(T(r)^n) \leqslant K^n (n!)^{\alpha d/N} r^{(N+(1-\alpha)d)n} . \qquad (3·2)$$

Proof. Let $0 < r < 1$. By a change of variable, we have

$$E(T(r)) = \int_{B(0,r)} P\{|X(t)| < r\}dt$$
$$\leqslant \int_{B(0,r)} \min\{1,\ K(r/|t|^{\alpha})^d\}dt$$
$$= K\int_0^r \min\{1,\ K(r/\rho^{\alpha})^d\}\rho^{N-1}d\rho$$
$$\leqslant Kr^{N+(1-\alpha)d} , \qquad (3·3)$$

since $N > \alpha d$. For $n \geqslant 2$,

$$E(T(r)^n) = \int_{B(0,r)^n} P\{|X(t_1)| < r, \ldots, |X(t_n)| < r\} dt_1 \cdots dt_n \,. \tag{3·4}$$

Consider $t_1, \ldots, t_n \in B(0,r)$ satisfying

$$t_j \neq 0 \quad \text{for} \quad j = 1, \ldots, n, \quad t_j \neq t_k \quad \text{for} \quad j \neq k \,.$$

Let $\eta = \min\{|t_n|,\ |t_n - t_j|, \quad j = 1,\ \ldots, n-1\}$. Then by Lemma 2·3 we have

$$\operatorname{Var}(X_i(t_n)|X_i(t_1), \ldots, X_i(t_{n-1})) \geqslant K\eta^{2\alpha} \quad \text{for} \quad i = 1, \ldots, d \,. \tag{3·5}$$

Since conditional distributions in gaussian processes are still gaussian, it follows from (3·5) that

$$\begin{aligned} &P\big\{|X(t_n)| < r|X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}\big\} \\ &\leqslant K \int_{|u|<r} \frac{1}{\eta^{\alpha d}} \exp\left(-\frac{|u|^2}{K\eta^{2\alpha}}\right) du \\ &\leqslant K r^d / \eta^{\alpha d} \,. \end{aligned} \tag{3·6}$$

By (3·6) and (2·1), we have

$$\begin{aligned} &\int_{B(0,r)} P\big\{|X(t_n)| < r|X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}\big\} dt_n \\ &\leqslant K \int_{B(0,r)} \frac{r^d}{\min\{|t_n|,\ |t_n - t_i|, \quad i = 1,\ \ldots, n-1\}^{\alpha d}}\, dt_n \\ &\leqslant K n^{\alpha d/N} r^{N+(1-\alpha)d} \,. \end{aligned} \tag{3·7}$$

Finally, by (3·4), (3·7) and and an easy argument of conditioning, we obtain

$$E(T(r))^n \leqslant K n^{\alpha d/N} r^{N+(1-\alpha)d} E(T(r))^{n-1} \,.$$

Hence (3·2) is proved by induction.

It follows from (3·2) and Jensen's inequality that there exists $0 < b < 1/K$ such that

$$E \exp\left(b(r^{-(N+(1-\alpha)d)} T(r))^{N/(\alpha d)}\right) < \infty \,, \tag{3·8}$$

see, e.g. Ehm [**E**].

PROPOSITION 3·1. *If $N > \alpha d$, then with probability 1,*

$$\limsup_{r\to 0} \frac{T(r)}{r^{N+(1-\alpha)d}(\log\log 1/r)^{\alpha d/N}} \leqslant \frac{1}{b} \,.$$

Proof. This can be proved by using (3·8) and the Borel-Cantelli lemma in a standard way (see e.g. [**X2**]).

Since $X(t)$ $(t \in \mathbf{R}^N)$ has stationary increments, the same arguments as above yield the following result.

PROPOSITION 3·2. *If $N > \alpha d$, then for any $t_0 \in I$ we have almost surely*

$$\limsup_{r\to 0} \frac{T_{t_0, X(t_0)}(r)}{r^{N+(1-\alpha)d}(\log\log 1/r)^{\alpha d/N}} \leqslant \frac{1}{b} \,. \tag{3·9}$$

Now we prove the main result in this section.

THEOREM 3·1. *Let $X(t)$ $(t \in \mathbf{R}^N)$ be the fractional Brownian motion in $\mathbf{R}^d$ of index $\alpha(0 < \alpha < 1)$ and $N > \alpha d$. Then there exist positive finite constants K_1 and K_2 such that with probability 1*

$$K_1 \leqslant \phi_2\text{-}m(\operatorname{Gr}X([0,1]^N)) \leqslant K_2\ , \tag{3·10}$$

where $\phi_2(s) = s^{N+(1-\alpha)d}(\log\ \log 1/s)^{\alpha d/N}$.

Proof. We define a random Borel measure μ on $\operatorname{Gr}X([0,1]^N) \subseteq \mathbf{R}^{N+d}$ by

$$\mu(B) = L_N\{t \in I\colon\ (t, X(t)) \in B\} \quad \text{for any} \quad B \subseteq \mathbf{R}^{N+d}\ .$$

Then $\mu(\mathbf{R}^{N+d}) = \mu(\operatorname{Gr}X([0,1]^N)) = 1$. It follows from Proposition 3·2 that for any fixed $t_0 \in I$, almost surely

$$\limsup_{r\to 0} \frac{\mu(B((t_0, X(t_0)), r))}{r^{N+(1-\alpha)d}(\log\ \log 1/r)^{\alpha d/N}} \leqslant \frac{1}{b}\ . \tag{3·11}$$

Let $E(\omega) = \{(t_0, X(t_0))\colon\ t_0 \in I \text{ and } (3·11) \text{ holds}\}$. Then $E(\omega) \subseteq \operatorname{Gr}X([0,1]^N)$ and a Fubini argument shows that $\mu(E(\omega)) = 1$ almost surely. By Lemma 2·1, we have almost surely

$$\phi_2\text{-}m(\operatorname{Gr}X([0,1]^N)) \geqslant Kb\ . \tag{3·12}$$

To prove the upper bound, we shall make use of Lemma 2·4. For any integer $k \geqslant 1$, let

$$R_k = \big\{t \in I\colon \exists r \in [2^{-2k}, 2^{-k}]$$

such that

$$\sup_{|s-t|\leqslant r} |X(s) - X(t)| \leqslant K2^{-\alpha k}(\log\ \log 2^k)^{-\alpha/N}\big\}\ .$$

Then it follows from Lemma 2·4 that

$$P\{t \in R_k\} \geqslant 1 - \exp\left(-\sqrt{(\tfrac{1}{2}k)}\right)\ . \tag{3·13}$$

By Fubini Theorem, this implies that $P(\Omega_0) = 1$, where

$$\Omega_0 = \{\omega\colon\ L_N(R_k) \geqslant 1 - \exp\left(-\sqrt{(\tfrac{1}{4}k)}\right) \quad \text{infinitely often}\}\ .$$

On the other hand, it is well known that (see e.g. section IV, theorem 1·3 in Jain and Marcus [**JM**]) there exists an event Ω_1 such that $P(\Omega_1) = 1$ and for all $\omega \in \Omega_1$, there exists $n_1 = n_1(\omega)$ large enough such that for all $n \geqslant n_1$ and any dyadic cube C of order n in I, we have

$$\sup_{s,\,t\in C} |X(t) - X(s)| \leqslant K2^{-n\alpha}\sqrt{n}\ . \tag{3·14}$$

Now fix an $\omega \in \Omega_0 \bigcap \Omega_1$, we show that $\phi_2\text{-}m(\operatorname{Gr}X([0,1]^N)) < \infty$. Consider $k \geqslant 1$ such that

$$L_N(R_k) \geqslant 1 - \exp\left(-\sqrt{(\tfrac{1}{4}k)}\right)\ .$$

For any $x \in R_k$ we can find n with $k \leqslant n \leqslant 2k + k_0$ (where k_0 depends on N only) such that

$$\sup_{s,\,t\in C_n(x)} |X(t) - X(s)| \leqslant K2^{-n\alpha}(\log\ \log 2^n)^{-\alpha/N}\ , \tag{3·15}$$

where $C_n(x)$ is the unique dyadic cube of order n containing x. Thus we have

$$R_k \subseteq V = \bigcup_{n=2k}^{2k+k_0} V_n$$

where V_n $(n = k, \ldots, 2k + k_0)$ are disjoint and each V_n is a union of dyadic cubes C_{nl} of order n for which (3·15) holds. Clearly $\mathrm{Gr}\, X(C_{nl}) \subseteq C_{nl} \times D_{nl}$, where D_{nl} is a ball of radius

$$\rho_n = K 2^{-n\alpha} (\log\, \log\, 2^n)^{-\alpha/N} .$$

Hence $\mathrm{Gr}\, X(C_{nl})$ can be covered by

$$M_n = K \left(\frac{2^{-n\alpha} (\log\, \log\, 2^n)^{-\alpha/N}}{2^{-n}} \right)^d = K 2^{(1-\alpha)dn} (\log\, \log\, 2^n)^{-\alpha d/N} \tag{3·16}$$

cubes $\{E_m^{(nl)}\}$ of side 2^{-n} in $\mathbf{R}^{N+d}$. Now

$$\mathrm{Gr}\, X(R_k) \subseteq \bigcup_{n=2k}^{2k+k_0} \bigcup_{l} \bigcup_{m=1}^{M_n} E_m^{(nl)}$$

and

$$\sum_n \sum_l M_n \phi_2(\surd(N + d)\, 2^{-n}) \leqslant \sum_n \sum_l K L_N(C_{nl}) = K L_N(V) < \infty . \tag{3·17}$$

On the other hand, $I \backslash V$ is contained in a union of dyadic cubes of order $q = 2k + k_0$, none of which meets R_k. There can be at most

$$2^{Nq} L_N([0,1]^N \backslash V) \leqslant K 2^{Nq} \exp\,(-\surd(\tfrac{1}{4}k))$$

of such cubes. For each C of these cubes, it follows from (3·14) that $\mathrm{Gr}\, X(C)$ is contained in the union of at most $K 2^{(1-\alpha)dq} q^{d/2}$ cubes of side 2^{-q} and

$$\begin{aligned} &\sum \phi_2(\surd(N + d) 2^{-q})) \\ &\leqslant K 2^{Nq} \exp\,(-\surd(\tfrac{1}{4}k)) 2^{(1-\alpha)dq} q^{d/2} \times 2^{-(N+(1-\alpha)d)q} (\log\, \log\, 2^q)^{\alpha d/N} \\ &\leqslant K \exp\,(-\surd(\tfrac{1}{4}k)) q^{d/2} (\log\, q)^{\alpha d/N} \\ &\leqslant 1 \end{aligned} \tag{3·18}$$

for k large enough. Since k can be arbitrarily large, the upper bound in (3·10) follows from (3·17) and (3·18). This completes the proof of Theorem 3·1.

Now we consider the transient case $N < \alpha d$.

THEOREM 3·2. *Let $X(t)$ $(t \in \mathbf{R}^N)$ be the fractional Brownian motion in $\mathbf{R}^d$ of index $\alpha(0 < \alpha < 1)$ and $N < \alpha d$. Then there exist positive finite constants K_1 and K_2 such that with probability 1*

$$K_1 \leqslant \phi_1\text{-}m(\mathrm{Gr}\, X([0,1]^N)) \leqslant K_2 , \tag{3·19}$$

where $\phi_1(s) = s^{N/\alpha} \log\, \log\, 1/s$.

Proof. The lower bound in (3·19) follows from

$$\phi_1\text{-}m(X([0,1]^N)) \leqslant \phi_1\text{-}m(\mathrm{Gr}\, X([0,1]^N)) \tag{3·20}$$

and a theorem of Talagrand ([**T**]). The proof of the upper bound is similar to that of Theorem 3·1 except that we cover $\mathrm{Gr}\, X(C_{nl})$ by cubes of side $2^{-n\alpha}(\log\log 2^n)^{-\alpha/N}$ if the dyadic cube C_{nl} of order n intersects R_k, by cubes of side $2^{-n\alpha}\sqrt{n}$ otherwise.

Remark. With a little more effort, the above results can be extended to more general strongly locally nondeterministic gaussian random fields. Since the main ingredients have been obtained in [**X2**] and in the above, we will not give all the details.

4. *Random fields with FBM components*

Let $X_i = \{X_i(t),\ t \in \mathbf{R}^N\}$ $(i = 1, 2, \ldots, d)$ be d independent fractional Brownian motions in $\mathbf{R}$. We assume that X_i has index α_i and

$$0 = \alpha_0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_d < 1 \ .$$

Let $Z(t) = (X_1(t), \ldots, X_d(t))$. Then $Z = \{Z(t),\quad t \in \mathbf{R}^N\}$ is a gaussian random field in $\mathbf{R}^d$ with stationary increments. If α_i $(i = 1, \ldots, d)$ are not all equal, then Z is not self-similar. It follows from theorem 2·1 in [**X1**] that almost surely

$$\begin{aligned}\dim\ Z([0,1]^N) &= \min\left\{d;\ \frac{N + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\quad i = 1, \ldots, d\right\}\\ &= \begin{cases} \dfrac{N + \sum_{i=1}^{k}(\alpha_k - \alpha_i)}{\alpha_k} & \text{if } \sum_{i=0}^{k-1}\alpha_i < N \leqslant \sum_{i=0}^{k}\alpha_i \\ d & \text{if } N > \sum_{i=1}^{d}\alpha_i \end{cases}\end{aligned}$$

$$\dim\ \mathrm{Gr}\, Z([0,1]^N) = \begin{cases} \dim\ Z([0,1]^N) & \text{if } \ N \leqslant \sum_{i=1}^{d}\alpha_i \\ N + \sum_{i=1}^{d}(1 - \alpha_i) & \text{if } \ N > \sum_{i=1}^{d}\alpha_i \ . \end{cases}$$

In this section we evaluate the exact Hausdorff measure of the image set $Z([0,1]^N)$ and the graph $\mathrm{Gr}\, Z([0,1]^N)$.

THEOREM 4·1. *If for some* $1 \leqslant k \leqslant d$

$$\sum_{i=0}^{k-1}\alpha_i < N < \sum_{i=1}^{k}\alpha_i \quad \textit{or} \quad N = \sum_{i=0}^{k-1}\alpha_i \quad \textit{and} \quad \alpha_{k-1} = \alpha_k \ . \tag{4·1}$$

Then there exist positive finite constants K_1 *and* K_2 *such that with probability 1*

$$K_1 \leqslant \psi_k\text{-}m(Z([0,1]^N)) \leqslant K_2 \ , \tag{4·2}$$

where

$$\psi_k(s) = s^{(N + \sum_{i=1}^{k}(\alpha_k - \alpha_i))/\alpha_k} \log\log 1/s \ . \tag{4·3}$$

To prove the lower bound, we need to make use of Lemma 2·1. For any $r > 0$ and any $y \in \mathbf{R}^d$, let

$$T_y(r) = \int_{\mathbf{R}^N} 1_{\{|Z(t) - y| < r\}}(t)dt$$

be the sojourn time of $Z(t)$ $(t \in \mathbf{R}^N)$ in the open ball $B(y, r)$ in $\mathbf{R}^d$.

PROPOSITION 4·1. *If (4·1) holds for some* $1 \leqslant k \leqslant d$*, then there exists a positive finite constant* γ *such that almost surely*

$$\limsup_{r \to 0} \frac{T_0(r)}{\psi_k(r)} \leqslant \gamma \, . \tag{4·4}$$

Proof. It is sufficient to show that for any integer $n \geqslant 1$ and $0 < r < 1$

$$E(T_0(r)^n) \leqslant K^n n! r^{\beta n} \, , \tag{4·5}$$

where $\beta = (N + \sum_{i=1}^{k} (\alpha_k - \alpha_i))/\alpha_k$.

For $n = 1$, by simple calculation we have

$$\begin{aligned} E(T_0(r)) &= \int_{\mathbf{R}^N} P\{|X(t)| < r\} \, dt \\ &\leqslant \int_{\mathbf{R}^N} \prod_{i=1}^{d} P\{|X_i(t)| < r\} \, dt \\ &= \int_{\mathbf{R}^N} \prod_{i=1}^{d} P\left\{|X_i(1)| < \frac{r}{|t|^{\alpha_i}}\right\} \, dt \\ &\leqslant K \int_0^\infty \rho^{N-1} \prod_{i=1}^{d} P\left\{|X_i(1)| < \frac{r}{\rho^{\alpha_i}}\right\} \, d\rho \\ &= K \Bigg(\sum_{i=1}^{d} \int_{r^{1/\alpha_{i-1}}}^{r^{1/\alpha_i}} \frac{r^{i-1}}{\rho^{\sum_{j=1}^{i-1} \alpha_j}} \, \rho^{N-1} d\rho + \int_{1/\alpha_d}^{\infty} \frac{r^d}{\rho^{\sum_{j=1}^{d} \alpha_j}} \, \rho^{N-1} d\rho \Bigg) \\ &\leqslant \begin{cases} K r^\beta & \text{if (4·1) holds} \\ K r^\beta \log \frac{1}{r} & \text{if } N = \sum_{i=0}^{k-1} \alpha_i \quad \text{and} \quad \alpha_{k-1} < \alpha_k \, , \end{cases} \end{aligned} \tag{4·6}$$

where the last inequality follows from the easily verified fact that if (4·1) holds, then

$$\beta = \min \left\{ d; \ \frac{N + \sum_{j=1}^{i} (\alpha_i - \alpha_j)}{\alpha_i} \, , \quad i = 1, \ldots, d \right\} .$$

For $n \geqslant 2$, by using Lemma 2·3 and modifying the argument of Goldman [**G**] we have

$$E(T_0(r)^n) \leqslant \begin{cases} K n r^\beta E(T_0(r)^{n-1}) & \text{if (4·1) holds} \\ K n r^\beta \log \frac{1}{r} E(T_0(r)^{n-1}) & \text{if } N = \sum_{i=0}^{k-1} \alpha_i \quad \text{and} \quad \alpha_{k-1} < \alpha_k \, , \end{cases} \tag{4·7}$$

see e.g. [**X2**]. Hence (4·5) follows from (4·6), (4·7) and induction. This finishes the proof of (4·4).

Remark. By using (4·6) and (4·7), we can prove that if $N = \sum_{i=0}^{k-1} \alpha_i$ and $\alpha_{k-1} < \alpha_k$, then almost surely

$$\limsup_{r \to 0} \frac{T_0(r)}{\phi_3(r)} \leqslant \gamma \, , \tag{4·8}$$

where $\phi_3(r) = r^{k-1} \log 1/r \log \log 1/r$. But (4.8) is not the best possible (see [**PT**]). In this case, the exact Hausdorff measure functions for the image and graph of Z remain unknown.

By the stationarity of the increments of X and Proposition 4·1 we have

Corollary 4·1. *If (4.1) holds for some* $1 \leqslant k \leqslant d$, *then there exists a positive finite constant* γ *such that for every* $t_0 \in I$ *almost surely*

$$\limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\psi_k(r)} \leqslant \gamma\ . \tag{4·9}$$

In order to prove the upper bound in (4·2), we need the following lemma, which can be proved in the same way as that of proposition 4·1 in [**T**].

Lemma 4·1. *There exists a constant* $\delta > 0$ *such that for any* $0 < r_0 < \delta$, *we have*

$$P\left\{\exists r \in [r_0^2,\ r_0]\ \ \textit{such that}\ \ \sup_{|t| \leqslant r} |X_i(t)| \leqslant K r^{\alpha_i} (\log\log 1/r)^{-\alpha_i/N}, \quad i = 1, \dots, d\right\} \geqslant 1 - \exp\left(-(\log 1/r_0)^{\frac{1}{2}}\right)\ . \tag{4·10}$$

Proof of Theorem 4·1. The lower bound in (4·2) follows from Lemma 2·1 and (4·9). To prove the upper bound in (4·2), we let

$$\begin{aligned} \overline{R}_n = \{ t \in I :\ & \exists r \in [2^{-2n}, 2^{-n}], \\ & \sup_{|s-t| \leqslant r} |X_i(s) - X_i(t)| \leqslant K 2^{-n\alpha_i} (\log\log 2^n)^{-\alpha_i/N},\ 1 \leqslant i \leqslant d \}\ . \end{aligned}$$

Then by Lemma 4·1, we have

$$P\{t \in \overline{R}_n\} \geqslant 1 - \exp\left(-\sqrt{\tfrac{1}{2} n}\right)\ .$$

The rest of the proof is quite similar to that of Theorem 3·1. We only remark that for each dyadic cube C of order l that intersects $\overline{R}_n$, $X(C)$ can be covered by a super-rectangle of sides $K 2^{-n\alpha_i} (\log\log 2^n)^{-\alpha_i/N}$ $(i = 1, \dots, d)$, and hence can be covered by

$$K 2^{l \sum_{j=1}^{k} (\alpha_k - \alpha_j)} (\log\log 2^l)^{\sum_{j=1}^{k} (\alpha_k - \alpha_j)/N}$$

cubes of side $K 2^{-n\alpha_k} (\log\log 2^n)^{-\alpha_k/N}$. For each dyadic cube C of order l that does not meet $\overline{R}_n$, $X(C)$ can be covered by $K 2^{l \sum_{j=1}^{k} (\alpha_k - \alpha_j)}$ cubes of side $K 2^{-l\alpha_k} \sqrt{l}$.

The following theorem gives the exact Hausdorff measure for the graph of Z.

Theorem 4·2. *If (4·1) holds, then there exist positive finite constants* K_1 *and* K_2 *such that with probability 1*

$$K_1 \leqslant \psi_k\text{-}m(\mathrm{Gr}\, Z([0,1]^N)) \leqslant K_2\ , \tag{4·11}$$

where $\psi_k(s)$ *is given is (4·3).*

If $N > \sum_{i=1}^{d} \alpha_i$ *and*

$$\psi(s) = s^{N + \sum_{i=1}^{d} (1 - \alpha_i)} (\log\log 1/s)^{\sum_{i=1}^{d} \alpha_i / N}\ .$$

Then there exist positive finite constants K_1 *and* K_2 *such that with probability 1*

$$K_1 \leqslant \psi\text{-}m(\mathrm{Gr}\, Z([0,1]^N)) \leqslant K_2\ . \tag{4·12}$$

Proof. The upper bounds in (4·11) and (4·12) can be proved by using Lemma 4·1. The lower bound in (4·11) follows from (3·20) and (4·2). The proof of the lower bound in (4·12) is similar to that of Theorem 3·1.

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