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Weak Variation of Gaussian Processes

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Let $X(t)$ $(t \in \mathbf{R})$ be a real-valued centered Gaussian process with stationary increments. We assume that there exist positive constants δ_0, c_1, and c_2 such that for any $t \in \mathbf{R}$ and $h \in \mathbf{R}$ with $|h| \leqslant \delta_0$

$$E[(X(t+h) - X(t))^2] \leqslant c_1 \sigma^2(|h|)$$

and for any $0 \leqslant r < \min\{|t|, \delta_0\}$

$$\mathrm{Var}(X(t) \mid X(s): r \leqslant |s-t| \leqslant \delta_0) \geqslant c_2 \sigma^2(r)$$

where $\sigma: [0, \delta_0) \to [0, \infty)$ is regularly varying at zero of order α $(0 < \alpha < 1)$. Let τ be an inverse function of σ near zero such that $\phi(s) = \tau(s) \log\log(1/s)$ is increasing near zero. We obtain exact estimates for the weak ϕ-variation of $X(t)$ on $[0, a]$.

KEY WORDS: Weak variation; Gaussian processes; local times; symmetric Lévy processes.

1. INTRODUCTION

The elegant result of P. Lévy on the quadratic variation of Brownian motion has been extended in different ways (see Taylor,[21] Kôno,[10] Kawada and Kôno,[9] Marcus and Rosen,[12, 13] and references therein). Let

$$\pi = \{0 = t_0 < t_1 < \cdots < t_{k(\pi)} = a\}$$

be a partition of $[0, a]$, and let $m(\pi) = \sup_{1 \leqslant i \leqslant k(\pi)}(t_i - t_{i-1})$ denote the length of the largest interval in π ($m(\pi)$ is called the mesh of π). Let $Q_a(\delta)$

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be the family of the partitions π of $[0, a]$ with $m(\pi) < \delta$. For any function $f: \mathbf{R}_+ \to \mathbf{R}^d$ and any function $\phi: [0, \delta] \to \mathbf{R}_+$ with $\phi(0) = 0$, the weak ϕ-variation of f on $[0, a]$ is defined by

$$\underline{V}_{\phi, a}(f) = \lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(f; \pi) \tag{1.1}$$

where

$$V_\phi(f; \pi) = \sum_{t_i \in \pi} \phi(R(f; [t_{i-1}, t_i])) \tag{1.2}$$

and

$$R(f; [a, b]) = \sup_{s, t \in [a, b]} |f(t) - f(s)|$$

The sum in (1.2) and all that follows is taken over all the terms in which both t_{i-1} and t_i are contained in π. The strong ϕ-variation of f on $[0, a]$ is defined by

$$\bar{V}_{\phi, a}(f) = \lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{t_i \in \pi} \phi(|f(t_i) - f(t_{i-1})|) \tag{1.3}$$

Let $B = \{B(t), t \in \mathbf{R}_+\}$ be a Brownian motion in $\mathbf{R}^d$. Taylor[21] proved the following exact estimates for the weak and strong variation of B: with probability 1

$$\lim_{\delta \to 0} \inf_{\pi \in Q_A(\delta)} V_{\phi_1}(B; \pi) = \lambda_d \tag{1.4}$$

$$\lim_{\delta \to 0} \sup_{\pi \in Q_1(\delta)} \sum_{t_i \in \pi} \phi_2(|B(t_i) - B(t_{i-1})|) = 2 \tag{1.5}$$

where $\phi_1(s) = s^2 \log\log 1/s$, $\phi_2(s) = s^2/\log\log 1/s$ and λ_d is a positive finite constant. Strong ϕ-variation result similar to (1.5) has been obtained by Kawada and Kôno[9] for certain Gaussian processes (see also Marcus and Rosen[12]). The main purpose of this paper is to generalize (1.4) to strongly locally nondeterministic Gaussian processes.

Let $X(t)$ $(t \in \mathbf{R})$ be a real-valued, centered Gaussian process with $X(0) = 0$. We assume that $X(t)$ $(t \in \mathbf{R})$ has stationary increments and continuous covariance function $R(t, s) = EX(t)\, X(s)$ given by

$$R(t, s) = \int_{\mathbf{R}} (e^{it\lambda} - 1)(e^{-is\lambda} - 1)\, \Delta(d\lambda) \tag{1.6}$$

where $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbf{R}\backslash\{0\}$ satisfying

$$\int_{\mathbf{R}} \frac{\lambda^2}{1+\lambda^2} \Delta(d\lambda) < \infty$$

Then there exists a centered complex-valued Gaussian random measure $W(d\lambda)$ such that

$$X(t) = \int_{\mathbf{R}} (e^{it\lambda} - 1)\, W(d\lambda) \tag{1.7}$$

and for any Borel sets $A, B \subseteq \mathbf{R}$

$$E(W(A)\,\overline{W(B)}) = \Delta(A \cap B) \qquad \text{and} \qquad W(-A) = \overline{W(A)}$$

It follows from (1.6) that

$$E[(X(t+h) - X(t))^2] = 2\int_{\mathbf{R}} (1 - \cos h\lambda)\, \Delta(d\lambda)$$

We assume that there exist constants $\delta_0 > 0$, $0 < c_1$, $c_2 < \infty$ and a nondecreasing continuous function σ: $[0, \delta_0) \to [0, \infty)$ which is regularly varying at zero of order α $(0 < \alpha < 1)$ such that for any $t \in \mathbf{R}$ and $h \in \mathbf{R}$ with $|h| \leqslant \delta_0$

$$E[(X(t+h) - X(t))^2] \leqslant c_1 \sigma^2(|h|) \tag{1.8}$$

and for all $t \in \mathbf{R}$ and any $0 \leqslant r < \min\{|t|, \delta_0\}$

$$\operatorname{Var}(X(t) \mid X(s)\colon r \leqslant |s-t| \leqslant \delta_0) \geqslant c_2 \sigma^2(r) \tag{1.9}$$

If (1.8) and (1.9) hold, we shall say that $X(t)$ $(t \in \mathbf{R})$ is strongly locally σ-nondeterministic. A typical example of strongly locally nondeterministic Gaussian process is the so-called fractional Brownian motion of index α $(0 < \alpha < 1)$, the centered, real-valued Gaussian process $X(t)$ $(t \in \mathbf{R})$ with covariance

$$EX(t)\, X(s) = \tfrac{1}{2}(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha})$$

We refer to Berman[2-4] and Cuzick and Du Peez[5] for more information on (strongly) locally nondeterminism.

Based on an isomorphism theorem of Dynkin,[6,7], Marcus and Rosen,[12-14] studied the sample path properties of the local times of strongly symmetric Markov processes through their associated Gaussian processes. In particular, they proved results similar to (1.5) for the (strong)

p-variation of the local times of symmetric Lévy processes in the spatial variable (see Refs. 13 and 14). It is natural to consider the weak ϕ-variation of the local times of symmetric Lévy processes. In this case, the argument of Marcus and Rosen seems not enough and we have not been able to prove an analogous result for the local times of symmetric Lévy processes.

Throughout the paper, we let $\tau(s)$ to be an inverse of $\sigma(s)$ near zero. Then $\tau(s)$ is regularly varying at zero of order $1/\alpha$. We may and will choose $\tau(s)$ to be continuous and such that $\phi(s) = \tau(s) \log\log(1/s)$ is strictly increasing near zero (see e.g., Senata,[17], p. 22).

2. 0-1 LAW

Let $X(t)$ $(\in \mathbf{R})$ be a real-valued Gaussian process with stationary increments and $X(0) = 0$. We assume that $X(t)$ satisfies (1.8), where $\sigma(s)$ is regularly varying at zero of order α $(0 < \alpha < 1)$, and it has a representation (1.7). It is well known that $X(t)$ has continuous sample paths almost surely [see e.g., Jain and Marcus,[8] (Sect. IV, Thm. 1.3)]. The following lemma is a corollary of Proposition 2 of Wichura.[22]

Lemma 1. Let

$$Z_n(t) = \int_{n-1 \leqslant |t| < n} (e^{i\lambda t} - 1)\, W(d\lambda)$$

Then with probability 1

$$Y(t) = \sum_{n=1}^{\infty} Z_n(t)$$

converges uniformly on $[0, 1]$.

Now we prove a 0-1 law about the weak ϕ-variation of $X(t)$ on $[0, 1]$. 0-1 laws for other types of variation were obtained by Kôno[10] and by Kawada and Kôno.[9]

Theorem 1. Let $X(t)$ $(t \in \mathbf{R})$ be a real-valued Gaussian process with stationary increments and $X(0) = 0$ satisfying (1.8). Then there exists a constant $0 \leqslant \lambda \leqslant \infty$ such that

$$P\{\lim_{\delta \to 0} \inf_{\pi \in Q_1(\delta)} V_\phi(X; \pi) = \lambda\} = 1 \tag{2.1}$$

Proof. Let $(\Omega, \mathscr{B}, P)$ be a basic probability space such that for every $\omega \in \Omega$, the series $\sum_{n=1}^{\infty} Z_n(t, \omega)$ in Lemma 1 converges uniformly in t on

$[0, 1]$. Let $\mathscr{B}_N$ be the σ-algebra generated by $\{Z_n(t), n = N, N+1, \cdots\}$. An event which is measurable with respect to $\mathscr{B}_\infty = \bigcap_{N=1}^{\infty} \mathscr{B}_N$ is called a tail event. Since $\{Z_n(t), n \geqslant 1\}$ is a sequence of independent random variables, it follows from Kolmogorov's 0-1 law that for any $A \in \mathscr{B}_\infty$, $P(A) = 0$ or 1.

Let

$$A = \{\lim_{\delta \to 0} \inf_{\pi \in Q_1(\delta)} V_\phi(X; \pi) = \text{constant}\}$$

We will show that A is a tail event and then (2.1) follows.

Let N be an arbitrary positive integer and set

$$X_N(t) = \sum_{n=N}^{\infty} Z_n(t), \qquad Y_N(t) = \sum_{n=1}^{N-1} Z_n(t)$$

Let $\delta_n \downarrow 0$ be a decreasing sequence. For any $0 < \eta < \min\{1/\alpha - 1, 1/3\}$ and any positive integer n, let

$$\Omega_\eta^n = \{\omega\colon R(Y_N; [s, t]) \leqslant \eta^2 \sigma(|t-s|^{1+\eta}) \text{ for all } k \geqslant n \text{ and all } [s, t] \subseteq [0, 1] \text{ with } |t-s| \leqslant \delta_k\}.$$

Clearly

$$\Omega_\eta^n \subseteq \Omega_\eta^{n+1} \subseteq \cdots$$

Since for each $\omega \in \Omega$, $Y_N(t, \omega)$ is continuously differentiable on $[0, 1]$ and $\alpha(1+\eta) < 1$, we have

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_\eta^n$$

We note that $\phi(s)$ is regularly varying at zero of order $1/\alpha$, then for any $\varepsilon > 0$ there exists $s_0 > 0$ such that for $0 < s < s_0$

$$\left| \frac{\phi(xs)}{\phi(s)} - x^{1/\alpha} \right| < \varepsilon \qquad \text{for all} \quad x \in [\eta, 3/2] \tag{2.2}$$

[see e.g., Seneta,[17] Thm. 1.1]. For any $\omega \in \Omega$, there exists n_1 such that $\omega \in \Omega_\eta^n$ for all $n \geqslant n_1$ and hence for any $\pi \in Q_1(\delta_n)$ and any t_{i-1}, $t_i \in \pi$, we have

$$R(Y_N; [t_{i-1}, t_i]) \leqslant \eta^2 \sigma(|t_{i-1} - t_i|^{1+\eta}) \tag{2.3}$$

$$\sigma(|t_{i-1} - t_i|^{1+\eta}) < s_0, \qquad R(X_N; [t_{i-1}, t_i]) < s_0 \tag{2.4}$$

and

$$|t_{i-1}-t_i|^\eta \log\log \frac{1}{\sigma(|t_{i-1}-t_i|^{1+\eta})} \leqslant 1 \tag{2.5}$$

Let

$$K_\pi = \{i: R(X_N; [t_{i-1}, t_i]) \leqslant \eta\sigma(|t_{i-1}-t_i|^{1+\eta})\}$$

Then by (2.2)–(2.5) we have

$$\begin{aligned}
&\sum_{i\in K_\pi} \phi(R(X; [t_{i-1}, t_i])) \\
&\quad \leqslant \sum_{i\in K_\pi} \phi(R(X_N; [t_{i-1}, t_i]) + R(Y_N; [t_{i-1}, t_i])) \\
&\quad \leqslant \sum_{i\in K_\pi} \phi((\eta+\eta^2)\,\sigma(|t_{i-1}-t_i|^{1+\eta})) \\
&\quad \leqslant \sum_{i\in K_\pi} ((\eta+\eta^2)^{1/\alpha}+\varepsilon)|t_i-t_{i-1}|^{1+\eta} \log\log \frac{1}{\sigma(|t_i-t_{i-1}|^{1+\eta})} \\
&\quad \leqslant (\eta+\eta^2)^{1/\alpha}+\varepsilon
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\sum_{i\notin K_\pi} \phi(R(X; [t_{i-1}, t_i])) \\
&\qquad \leqslant \sum_{i\notin K_\pi} \phi\left(R(X_N; [t_{i-1}, t_i])\left(1+\frac{R(Y_N; [t_{i-1}, t_i])}{R(X_N; [t_{i-1}, t_i])}\right)\right) \\
&\qquad \leqslant \sum_{i\notin K_\pi} \phi((1+\eta)\, R(X_N; [t_{i-1}, t_i])) \\
&\qquad \leqslant ((1+\eta)^{1/\alpha}+\varepsilon) \sum_{i\notin K_\pi} \phi(R(X_N; [t_{i-1}, t_i]))
\end{aligned} \tag{2.7}$$

Similarly we have

$$\sum_{i\notin K_\pi} \phi(R(X; [t_{i-1}, t_i])) \geqslant ((1-\eta)^{1/\alpha}-\varepsilon) \sum_{i\notin K_\pi} \phi(R(X_N; [t_{i-1}, t_i])) \tag{2.8}$$

Combining (2.6)–(2.8) and noticing that $\eta > 0$, $\varepsilon > 0$ are arbitrary, we have

$$\lim_{\delta\to 0} \inf_{\pi\in Q_1(\delta)} V_\phi(X; \pi) = \lim_{n\to\infty} \inf_{\pi\in Q_1(\delta_n)} V_\phi(X_N; \pi)$$

for any positive integer N. Hence A is a tail event. This completes the proof of (2.1). □

3. WEAK ϕ-VARIATION FOR GAUSSIAN PROCESSES

Let $X(t)$ $(t \in \mathbf{R})$ be a real-valued Gaussian process with stationary increments and $X(0)=0$ satisfying (1.8) and (1.9), where $\sigma(s)$ is regularly varying at zero of order α $(0<\alpha<1)$. In this section, we study the weak ϕ-variation of $X(t)$ on $[0, a]$.

We will need several lemmas. Lemma 2 is proved in Xiao,[23] which generalizes a result of Talagrand.[20] Lemma 3 is from Talagrand.[19] Lemma 4 gives an estimate for the small ball probability of the Gaussian processes satisfying (1.8) and (1.9), which is a modification of Theorem 2.1 in Monrad and Rootzén[15] (see also Shao[18]).

We will use K to denote an unspecified positive constant, which may be different in each appearance.

Lemma 2. There exists a constant $\delta_1>0$ such that for any $0<r_0\leqslant\delta_1$, we have

$$P\left\{\exists r\in[r_0^2, r_0] \text{ such that } \sup_{|t|\leqslant r}|X(t)|\leqslant K\sigma\left(r\left(\log\log\frac{1}{r}\right)^{-1/N}\right)\right\}$$
$$\geqslant 1-\exp\left(-\left(\log\frac{1}{r_0}\right)^{1/2}\right) \tag{3.1}$$

Let $Z(t)$ $(t\in S)$ be a Gaussian process. We provide S with the following metric

$$d(s,t)=\|Z(z)-Z(t)\|_2$$

where $\|Z\|_2=(E(Z^2))^{1/2}$. We denote by $N_d(S,\varepsilon)$ the smallest number of open d-balls of radius ε needed to cover S.

Lemma 3. Consider a function Ψ such that $N_d(S,\varepsilon)\leqslant\Psi(\varepsilon)$ for all $\varepsilon>0$. Assume that for some constant $C>0$ and all $\varepsilon>0$ we have

$$\Psi(\varepsilon)/C\leqslant\Psi\left(\frac{\varepsilon}{2}\right)\leqslant C\Psi(\varepsilon)$$

Then

$$P\{\sup_{s,\, t \in S} |Z(z) - Z(t)| \leqslant u\} \geqslant \exp(-K\Psi(u))$$

where K depends on C only.

Lemma 4. For any $0 < r < \delta_0$ and any $\varepsilon < \sigma(r)$, we have

$$\exp\left(-\frac{Kr}{\tau(\varepsilon)}\right) \leqslant P(\sup_{t \in [0,\, r]} |X(t)| \leqslant \varepsilon) \leqslant \exp\left(-\frac{r}{K\tau(\varepsilon)}\right) \tag{3.2}$$

where $K > 0$ is an absolute constant.

Proof. The left-hand side follows immediately from Lemma 3. The proof of the right-hand side by using (1.9) is the same as that of Theorem 2.1 in Monrad and Rootzén.[15]

Proposition 3. For any $t \in \mathbf{R}$, with probability 1

$$\liminf_{h \to 0} \frac{R(X; [t, t+h])}{\xi(h)} \geqslant \gamma \tag{3.3}$$

where $\xi(s)$ is the inverse function of $\phi(s) = \tau(s) \log\log(1/s)$ near zero and $\gamma > 0$ is a finite constant.

Proof. Clearly,

$$R(X; [t, t+h]) \geqslant M(X; [t, t+h]) \equiv \max_{s \in [t,\, t+h]} |X(s) - X(t)|.$$

Then (3.3) follows from (3.2) and the Borel-Cantelli lemma in a standard way.

Remark. By using an argument similar to the proof of Theorem 3.3 in Monrad and Rootzén,[15] we can strengthen (3.3) to

$$\liminf_{h \to 0} \frac{R(X\colon\ [t, t+h])}{\xi(h)} = K \qquad a.s. \tag{3.4}$$

for some constant $K > 0$. But we will not need (3.4) in the present paper.

Proposition 2. For any $t \in \mathbf{R}$, with probability 1

$$\lim_{\delta \to 0} \inf_{\substack{u, v \geqslant 0 \\ 0 < u + v \leqslant \delta}} \frac{R(X; [t-v, t+u])}{\xi(u+v)} \geqslant \gamma, \tag{3.5}$$

where $\xi(s)$ and $\gamma > 0$ are as in Proposition 1.

Proof. Since the proof of (3.5) is similar to that of Lemma 3.4 in Taylor,[21] there seems no need in reproducing it.

We are now in a position to prove the main result of this section.

Theorem 2. Let $X(t)$ $(t \in \mathbf{R})$ be a Gaussian process with stationary increments and $X(0) = 0$ satisfying (1.8) and (1.9). Let $\phi(s) = \tau(s) \log \log(1/s)$. Then there exists a positive finite constant λ such that for any $a > 0$ with probability 1

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(X; \pi) = \lambda a \tag{3.6}$$

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(X^2; \pi) = \lambda 2^{1/\alpha} \int_0^a |X(t)|^{1/\alpha}\, dt \tag{3.7}$$

Proof. For Eq. (3.6), we start by proving that with probability 1, for δ small enough

$$\inf_{\pi \in Q_1(\delta)} V_\phi(X; \pi) \leqslant K < \infty \tag{3.8}$$

For $k \geqslant 1$, consider the set

$$R_k = \left\{ t \in [0, 1] : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that} \right.$$
$$\left. \sup_{|s-t| \leqslant r} |X(s) - X(t)| \leqslant K\sigma\left(r\left(\log\log\frac{1}{r}\right)^{-1}\right)\right\}$$

By Lemma 2, we have that for $k \geqslant \log 1/\delta_1$

$$P\{t \in R_k\} \geqslant 1 - \exp(-\sqrt{k/2})$$

Denote by $|A|$ the Lebesgue measure of A. It follows from Fubini's theorem that $P(\Omega_0) = 1$, where

$$\Omega_0 = \{\omega : |R_k| \geqslant 1 - \exp(-\sqrt{k/4}) \text{ infinitely often}\}$$

On the other hand, it is well known [see e.g., Jain and Marcus,[8], Sect. IV, Thm. 1.3] that there exists an event Ω_1 such that $P(\Omega_1)=1$ and for all $\omega \in \Omega_1$, there exists $n_4 = n_4(\omega)$ large enough such that for all $n \geqslant n_4$ and any dyadic cube C of order n in $[0, 1]$, we have

$$\sup_{s,\, t \in C} |X(t) - X(s)| \leqslant K\sigma(2^{-n}) \sqrt{n} \tag{3.9}$$

Now fix an $\omega \in \Omega_0 \cap \Omega_1$, we show that $V_\phi(X; \pi) \leqslant K < \infty$.

For any $x \in \mathbf{R}$, we denote by $C_l(x)$ the unique dyadic interval of order l containing x. Consider $k \geqslant 1$ such that

$$|R_k| \geqslant 1 - \exp(-\sqrt{k/4})$$

For any $x \in R_k$ we can find l with $k \leqslant l \leqslant 2k$ such that

$$R(X; C_l(x)) \leqslant K\sigma(2^{-l}(\log\log 2^l)^{-1}) \tag{3.10}$$

Let V_l be the union of such dyadic intervals C_l, then $\{V_l\}$ are disjoint and $R_k \subseteq V = \bigcup_{l=k}^{2k} V_l$. Let π be the partition of $[0, 1]$ formed by the end points of the dyadic intervals in V and the end points of the dyadic intervals of order $2k$ that do not meet R_k. Then

$$V_\phi(X; \pi) = \sum{}' \phi(R(X; C_l)) + \sum{}'' \phi(R(X; C'_l)) \tag{3.11}$$

where $\sum'$ sums over all the intervals in V and $\sum''$ sums over all the dyadic intervals of order $2k$ that do not intersect R_k. By (3.10) we have

$$\sum{}' \phi(R(X; C_l)) \leqslant \sum_l \sum_{C \in V_l} K 2^{-l} = K < \infty \tag{3.12}$$

On the other hand, the number of the dyadic intervals of order $2k$ that do not meet R_k is at most

$$2^{2k} |[0, 1] \backslash V| \leqslant K 2^{2k} \exp(-\sqrt{k}/4)$$

For each C'_l of these intervals, by (3.9) we have

$$R(X; C'_l) \leqslant K\sigma(2^{-2k}) \sqrt{k}$$

It follows that

$$\begin{aligned} \sum{}'' \phi(R(X; C'_l)) &\leqslant K 2^{2k} \exp(-\sqrt{k}/4)\, \phi(K\sigma(2^{-2k})\sqrt{k}) \\ &\leqslant K \exp\left(-\frac{\sqrt{k}}{4}\right) k^{(1/2\alpha)+\varepsilon} \leqslant 1 \end{aligned} \tag{3.13}$$

for k large enough. Since k can be arbitrarily large, (3.8) follows from (3.11)–(3.13).

To prove the opposite inequality for $a=1$, we set for any $\varepsilon>0$, $\delta>0$,

$$E_{\varepsilon,\delta}=\left\{(t,\omega)\in[0,1]\times\Omega:\ \inf_{\substack{u,v\geqslant 0\\ 0<u+v\leqslant\delta}}\frac{R(X;[t-v,t+u])}{\xi(u+v)}\geqslant\gamma-\varepsilon\right\}$$

Then $E_{\varepsilon,\delta}$ is measurable in (t,ω). Define $I_\delta(t,\omega)=1$ for $(t,\omega)\in E_{\varepsilon,\delta}$ and $I_\delta(t,\omega)=0$ otherwise. By Proposition 2, for any fixed $t\in[0,1]$ almost surely

$$\lim_{\delta\to 0} I_\delta(t,\omega)=1$$

It follows from Fatou's lemma that

$$\begin{aligned}E\left(\liminf_{\delta\to 0}\int_0^1 I_\delta(t,\omega)\,dt\right)&\geqslant E\int_0^1\liminf_{\delta\to 0} I_\delta(t,\omega)\,dt\\&=\int_0^1 E(\liminf_{\delta\to 0} I_\delta(t,\omega))\,dt=1\end{aligned}$$

Hence with probability 1

$$\lim_{\delta\to 0}\int_0^1 I_\delta(t,\omega)\,dt=1$$

This implies that for any $\varepsilon'>0$, there exists almost surely $\delta_5=\delta_5(\omega)>0$ such that for any $0<\delta<\delta_5$, $|E_{\varepsilon,\delta}(\omega)|\geqslant 1-\varepsilon'$, where

$$E_{\varepsilon,\delta}(\omega)=\{t\in[0,1]:(t,\omega)\in E_{\varepsilon,\delta}\}$$

For any partition $\pi\in Q_1(\delta)$, $\pi=\{0=t_0<t_1<\cdots<t_{k(\pi)}=1\}$, let

$$\Gamma(\omega)=\{i:[t_{i-1},t_i]\cap E_{\varepsilon,\delta}(\omega)\neq\varnothing\}$$

Then we have

$$\begin{aligned}V_\phi(X;\pi)&=\sum_{t_i\in\pi}\phi(R(X;[t_{i-1},t_i]))\\&\geqslant\sum_{i\in\Gamma(\omega)}K(t_i-t_{i-1})\\&\geqslant K(1-\varepsilon')\end{aligned}$$

This proves that

$$\lim_{\delta \to 0} \inf_{\pi \in Q_1(\delta)} V_\phi(X; \pi) \geqslant K > 0 \tag{3.14}$$

It follows from (3.8), (3.14) and Theorem 1 that (3.6) holds for $a = 1$.

For any $a > 0$, let $G(t) = X(at)$. By applying this result to $G(t)$ ($t \in [0, 1]$) we get (3.6).

For Eq. (3.7), we divide $[0, a]$ into $m \geqslant 1$ equal subintervals $I_{j,m}(a) = [(j-1/m)\, a, (j/m)\, a]$, $j = 1,..., m$ and denote $Q(I_{j,m}(a); \delta)$ the family of partitions π_j of $I_{j,m}(a)$ with $m(\pi_j) < \delta$. For any partition $\pi = \{0 = t_0 < t_1 < \cdots < t_{k(\pi)} = a\}$ of $[0, a]$, define

$$t_{k(j)}(\pi) = \sup_k \left\{ t_k : t_k \leqslant \frac{j}{m} a \right\} \qquad j = 1,..., m$$

Consider the partitions of $I_{j,m}(a)$ given by

$$\pi(I_{j,m}(a)) = \{ t_{k(j-1)} < t_{k(j-1)+1} < \cdots < t_{k(j)} \} \qquad j = 1,..., m \tag{3.15}$$

and

$$\tilde{\pi}(I_{j,m}(a)) = \left\{ \frac{j-1}{m} a < t_{k(j-1)+1} < \cdots < t_{k(j)} \leqslant \frac{j}{m} a \right\} \qquad j = 1,..., m \tag{3.16}$$

Thus for any $\pi \in Q_a(\delta)$, we can write

$$\begin{aligned} V_\phi(X^2; \pi) &= \sum_{j=1}^{m} V_\phi(X^2; \pi(I_{j,m}(a))) \\ &\geqslant \sum_{j=1}^{m} V_\phi(X^2; \tilde{\pi}(I_{j,m}(a))) - \sum_{j=1}^{m-1} \phi\left(R\left(X^2; \left[\frac{j}{m} a, t_{k(j)+1} \right] \right) \right) \\ &\quad - \sum_{j=1}^{m} \phi\left(R\left(X^2; \left[t_{k(j)}, \frac{j}{m} a \right] \right) \right) \\ &\geqslant \sum_{j=1}^{m} V_\phi(X^2; \tilde{\pi}(I_{j,m}(a))) - 2m\phi(\omega(\delta) \sup_{t \in [0, a]} |2X(t)|) \end{aligned} \tag{3.17}$$

where

$$\omega(\delta) = \sup_{\substack{s, t \in [0, a] \\ |s-t| \leqslant \delta}} |X(s) - X(t)|$$

It follows from (3.17) that

$$\inf_{\pi \in Q_a(\delta)} V_\phi(X^2;\pi) \geqslant \sum_{j=1}^{m} \inf_{\pi_j \in Q(I_{j,m}(a);\delta)} V_\phi(X^2;\pi_j) - 2m\phi(\omega(\delta) \sup_{t\in[0,a]} |2X(t)|) \tag{3.18}$$

Since ϕ is regularly varying at zero, for any $\varepsilon > 0$, $v > u > 0$, for s sufficiently small, we have

$$\phi(sb) \geqslant (1-\varepsilon)\,\phi(x)\,|b|^{1/\alpha} \qquad \text{for all} \quad b \in [u, v] \tag{3.19}$$

Since with probability 1, $X(t)$ is uniformly continuous on $[0, a]$, there exists almost surely $\delta_6 = \delta_6(\omega) > 0$ such that for any $0 < \delta < \delta_6$, we have, by (3.18) and (3.19)

$$\begin{aligned}
&\inf_{\pi \in Q_a(\delta)} V_\phi(X^2;\pi)\\
&\quad \geqslant \sum_{j=1}^{m} \inf_{\pi_j \in Q(I_{j,m}(a);\delta)} V_\phi(X^2;\pi_j)\cdot I(\inf_{t\in I_{j,m}(a)} |X(t)| \geqslant u)\\
&\qquad - 2m\phi(\omega(\delta) \sup_{t\in[0,a]} |2X(t)|)\\
&\quad \geqslant (1-\varepsilon) \sum_{j=1}^{m} \inf_{\pi_j \in Q(I_{j,m}(a);\delta)} V_\phi(X;\pi_j) \inf_{t\in I_{j,m}(a)} |2X(t)|^{1/\alpha}\\
&\qquad \times I(\inf_{t\in I_{j,m}(a)} |X(t)| \geqslant u) - 2m\phi(\omega(\delta) \sup_{t\in[0,a]} |2X(t)|)
\end{aligned} \tag{3.20}$$

where $I(A)$ is the indicator function of the set A. It follows from Jain and Marcus,[8] [Sect. IV, Thm. 1.3] that

$$\limsup_{\delta\to 0} \frac{\omega(\delta)}{\sigma(\delta)\sqrt{\log 1/\delta}} \leqslant K \qquad a.s.$$

for some constant K. Let $\delta \to 0$ in (3.20), by (3.6) we have

$$\begin{aligned}
&\lim_{\delta\to 0} \inf_{\pi \in Q_a(\delta)} V(X^2;\pi)\\
&\quad \geqslant (1-\varepsilon)\,\lambda 2^{1/\alpha} \sum_{j=1}^{m} \frac{a}{m} \inf_{t\in I_{j,m}(a)} |X(t)|^{1/\alpha}\cdot I(\inf_{t\in I_{j,m}(a)} |X(t)| \geqslant u)
\end{aligned} \tag{3.21}$$

Let $m \to \infty$, we see that the left-hand side of (3.21) is

$$\geqslant (1-\varepsilon)\,\lambda 2^{1/\alpha} \int_0^a |X(t)|^{1/\alpha}\, I(|X(t)| \geqslant u)\, dt$$

$$\geqslant (1-\varepsilon)\,\lambda 2^{1/\alpha} \left(\int_0^a |X(t)|^{1/\alpha}\, dt - a u^{1/\alpha} \right)$$

Since $\varepsilon > 0$, $u > 0$ are arbitrary, we get

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(X^2; \pi) \geqslant \lambda 2^{1/\alpha} \int_0^a |X(t)|^{1/\alpha}\, dt \tag{3.22}$$

To prove the opposite inequality, we note that for any $\varepsilon > 0$, $v > u > 0$, for s sufficiently small, we have

$$\phi(sb) \leqslant (1+\varepsilon)\, \phi(s)\, |b|^{1/\alpha} \qquad \text{for all} \quad b \in [u, v]$$

However for b small enough, $\phi(sb) \leqslant \phi(s)\, |b|^{1/\alpha}$. Therefore for any $0 < v < \infty$ and any $\varepsilon > 0$, if s is sufficiently small we have

$$\phi(sb) \leqslant (1+\varepsilon)\, \phi(s)\, |b|^{1/\alpha} \qquad \text{for all} \quad b \leqslant v \tag{3.23}$$

Similar to (3.18), by (3.23) we have

$$\inf_{\pi \in Q_a(\delta)} V_\phi(X^2; \pi) \cdot I\left(\sup_{t \in [0, a]} |X(t)| \leqslant \frac{v}{2} \right)$$

$$\leqslant \sum_{j=1}^{m} \inf_{\pi_j \in Q(I_{j,m}(a);\, \delta)} V_\phi(X^2; \pi_j) \cdot I\left(\sup_{t \in [0, a]} |X(t)| \leqslant \frac{v}{2} \right)$$

$$\leqslant (1+\varepsilon) \sum_{j=1}^{m} \inf_{\pi_j \in Q(I_{j,m}(a);\, \delta)} V_\phi(X; \pi_j) \big(\sup_{s \in I_{j,m}(a)} |2X(s)| \big)^{1/\alpha}$$

Let $\delta \to 0$, it follows from (3.6) that

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V(X^2; \pi) \cdot I\left(\sup_{t \in [0, a]} |X(t)| \leqslant \frac{v}{2} \right)$$

$$\leqslant (1+\varepsilon)\, \lambda 2^{1/\alpha} \sum_{j=1}^{m} \frac{a}{m} \sup_{t \in I_{j,m}(a)} |X(t)|^{1/\alpha}$$

Letting $m \to \infty$ and noticing that $\varepsilon > 0$ and $v > 0$ are arbitrary, we obtain (3.22) but with a less than or equal to sign. This completes the proof of (3.7). □

The argument in Taylor[21] can be applied to prove Corollary 1.

Corollary 1. Let

$$U_\phi(X; [0, a]) = \inf_\pi V_\phi(X; \pi)$$

where the infimum is taken over all the partitions of $[0, a]$. Then with probability 1, $0 < U_\phi(X; [0, 1]) \leqslant \lambda a$.

Remark. If $X(t)$ $(t \in \mathbf{R})$ is the fractional Brownian motion of index α $(0 < \alpha < 1)$ in $\mathbf{R}$, then (1.8) and (1.9) are satisfied with $\sigma(t) = t^\alpha$. It follows from Theorem 2 and Corollary 1 immediately that with probability 1

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(X; \pi) = \lambda a \tag{3.24}$$

and

$$0 < U_\phi(X; [0, 1]) \leqslant \lambda a \tag{3.25}$$

where $\phi(s) = s^{1/\alpha} \log\log 1/s$. In the case of Brownian motion, i.e., $\alpha = 1/2$, (3.24) and (3.25) recover the result of Taylor[21] mentioned in (1.4).

4. CONCLUSIONS

Let $Y = \{Y(t), t \in \mathbf{R}_+\}$ be a symmetric real-valued Lévy process with characteristic function

$$E \exp(i\lambda Y(t)) = \exp(-t\psi(\lambda))$$

and Lévy exponent

$$\psi(\lambda) = 2 \int_0^\infty (1 - \cos u\lambda)\, dv(\lambda)$$

where v is a Lévy measure, i.e., $\int_0^\infty \min\{1, u^2\}\, dv(u) < \infty$. If $\psi(\lambda) = \lambda^2/2$, then Y is Brownian motion. We assume that $\psi(\lambda)$ is regularly varying at infinity of order $1 < \beta \leqslant 2$. It follows from a result of Barlow[1] that Y has an almost surely jointly continuous local time, which is denoted by $L=\{L_t^x, (t, x) \in \mathbf{R}_+ \times \mathbf{R}\}$ normalized such that

$$E^0\left(\int_0^\infty e^{-t}\, dL_t^x\right) = u^1(x)$$

where

$$u^1(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos x\lambda}{1+\psi(\lambda)}\, d\lambda$$

is the 1-potential density of Y.

The mean zero Gaussian process $G=\{G(x)\, x \in \mathbf{R}\}$ with covariance $E(G(x)\, G(y)) = u^1(x-y)$ is said to be associated with Y. Then we have

$$E(G(x)-G(y))^2 = \sigma^2(|x-y|)$$

where

$$\sigma^2(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos x\lambda}{1+\psi(\lambda)}\, d\lambda$$

By a result of Pitman[16] we know that $\sigma^2(x)$ is regularly varying at zero of order $\beta - 1$. Marcus and Rosen[12] proved a corollary of the Dynkin Isomorphism Theorem,[6,7] which enables them[13,14] to obtain almost sure results for the (strong) variations of the local times of symmetric Lévy processes from analogous results about the associated Gaussian processes. It would be interesting to study the weak variation of the local times L of symmetric Lévy processes.

Let $G(x)$ $(x \in \mathbf{R})$ be the associated Gaussian process and let (Ω_G, P_G) be the probability space of G. Then $G(x)$ $(x \in \mathbf{R})$ is strongly locally σ-nondeterministic if: (i) $\sigma^2(x)$ is concave near zero and $\sigma^2(x) \to 0$ as $x \to 0$ (see Marcus[11]; or (ii) $f(\lambda) \geqslant K\,|\lambda|^{-\beta}$ for large $|\lambda|$, where

$$f(\lambda) = \frac{2}{\pi}\frac{1}{1+\psi(\lambda)}\frac{\lambda^2}{1+\lambda^2}$$

and

$$F(x) = \int_0^x f(\lambda)\, d\lambda$$

(see Berman[2]). Hence Theorem 2 holds under the condition (i) or (ii). It follows from Lemma 4.3 in Ref. 12, and (3.7) that for almost all $\omega \in \Omega_G$,

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi\left(L_t^{\cdot} + \frac{G_\omega^2(\cdot)}{2}; \pi\right) = \lambda 2^{1/(\beta-1)} \int_0^a \left|L_t^x + \frac{G_\omega^2(x)}{2}\right|^{1/(\beta-1)} dx \tag{4.1}$$

for almost all $t \in \mathbf{R}_+$ almost surely. However, it is not known whether the following weak variation result for the local time L holds or not: almost surely

$$\lim_{\delta \to 0} \inf_{\pi \in Q_a(\delta)} V_\phi(L_t^{\cdot}; \pi) = \lambda 2^{1/(\beta-1)} \int_0^a |L_t^x|^{1/(\beta-1)} dx$$

for almost all $t \in \mathbf{R}_+$. This can not be derived from (3.7) and (4.1) by using the argument of Marcus and Rosen.[13]

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