Properties of Local Nondeterminism of Gaussian and Stable Random Fields and Their Applications

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August 30, 2005

Abstract

In this survey, we first review various forms of local nondeterminism and sectorial local nondeterminism of Gaussian and stable random fields. Then we give sufficient conditions for Gaussian random fields with stationary increments to be strongly locally nondeterministic (SLND). Finally, we show some applications of SLND in studying sample path properties of (N, d)-Gaussian random fields. The class of random fields to which the results are applicable includes fractional Brownian motion, the Brownian sheet, fractional Brownian sheets and so on.

Running head: Local Nondeterminism of Gaussian Random Fields

2000 AMS Classification Numbers: Primary 60G15, 60G17, 60G60. Secondary 60G18, 28A80, 28A78.

Key words: Local nondeterminism, Gaussian or stable random field, fractional Brownian motion, the stable sheet, small ball probability, local times, level set, Hausdorff dimension and Hausdorff measure.

1 Introduction

The most important example of self-similar (non-Markovian) Gaussian processes is fractional Brownian motion (fBm) which was first introduced, as a moving average Gaussian process, by Mandelbrot and Van Ness (1968)

$$B_H(t) = \kappa_H \int_{-\infty}^{t} \left[((t-s)_+)^{H-1/2} - ((-s)_+)^{H-1/2}\right] dB(s),$$

where $t_+ = \max\{t, 0\}$, B is the ordinary Brownian motion and $\kappa_H > 0$ is the normalizing constant so that $\mathbb{E}(B^H(1)^2) = 1$, where $H \in (0, 1)$ is called the self-similarity index, or Hurst index. Except the case $H = 1/2$, fBm does not have

*Research partially supported by NSF grant DMS-0404729.

independent increments, it is not a Markov process, nor a semimartingale; see Lin (1995) or Rogers (1997) for a proof of this last fact. Due to its self-similarity and long-range dependence (as $H > 1/2$), it has been applied to model various phenomena in telecommunications, turbulence, image processing and finance. As a result, the theory on fractional Brownian motion has been developed significantly. We refer to Doukhan et al. (2003) for further information.

Moreover, in recent years, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models in several different scientific areas; see e.g. Addie et al. (1999), Anh et al. (1999), Benson et al (2004), Bonami and Estrade (2003), Cheridito (2004), Mannersalo and Norros (2002), Mueller and Tribe (2002), just to mention a few. Such applications have raised many interesting theoretical questions about Gaussian random fields in general.

One of the major difficulties in studying the probabilistic, analytic or statistical properties of Gaussian random fields is the complexity of their dependence structures. As a result, many of the existing tools from theories on Brownian motion, Markov processes or martingales fail for Gaussian random fields; and one often has to use general principles for Gaussian processes or to develop new tools. In this paper, we show that in many circumstances, the properties of *local nondeterminism* can help us to overcome this difficulty so that many elegant and deep results of Brownian motion (and Markov processes) can be extended to Gaussian (or stable) random fields.

The rest of this paper is organized as follows. In Section 2, we recall the definitions of various forms of local nondeterminism. In Section 3, we give sufficient conditions for ordinary or strong local nondeterminism to hold for Gaussian random fields with stationary increments. In Section 4, we show applications of the properties of local nondeterminism in studying small ball probabilities, Hausdorff dimension and exact Hausdorff measure functions of the sample paths, and local times of Gaussian random fields.

We end this section with some general notation. Throughout this paper (except in Section 2.4), $X = \{X(t), t \in \mathbb{R}^N\}$ will denote an (N, d)-Gaussian random field, where for every $t \in \mathbb{R}^N$,

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \tag{1.1}$$

and we will assume $\mathbb{E}(X_j(t)) \equiv 0$ for every $1 \le j \le d$. When $N = 1$, X is called a Gaussian process in $\mathbb{R}^d$.

A parameter $t \in \mathbb{R}^N$ is written as $t = (t_1, \ldots, t_N)$ and if $t_1 = t_2 = \cdots = t_N = c \in \mathbb{R}$, then we write t as $\langle c \rangle$. There is a natural partial order, "$\preccurlyeq$", on $\mathbb{R}^N$. Namely, $s \preccurlyeq t$ if and only if $s_\ell \le t_\ell$ for all $\ell = 1, \ldots, N$. When $s \preccurlyeq t$, we define the *closed interval* or *rectangle*,

$$[s, t] = \prod_{\ell=1}^{N} [s_\ell, t_\ell].$$

We will let $\mathcal{A}$ denote the class of all N-dimensional closed intervals $T \subset \mathbb{R}^N$.

We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively, no matter the value of the integer m.

Unspecified positive and finite constants will be denoted by c which may have different values from line to line. Specific constants in Section i will be denoted by $c_{i,1}, c_{i,2}, \ldots$. For two non-negative functions f and g on $\mathbb{R}^N$, we denote $f \asymp g$ if there exists a finite constant $c \geq 1$ such that $c^{-1} f(x) \leq g(x) \leq c\, f(x)$ for all x in some neighborhood of 0.

Acknowledgement. The author thanks Professors Serge Cohen and Jacques Istas for their invitation.

2 Definitions of local nondeterminism

In this section, we recall the definitions of different forms of local nondeterminism for Gaussian and stable random fields.

2.1 Local nondeterminism for Gaussian random fields

The concept of local nondeterminism (LND, in short) of a Gaussian process was first introduced by Berman (1973) to unify and extend his methods for studying the existence and joint continuity of local times of real-valued Gaussian processes. Berman's definition was later extended by Pitt (1978) and Cuzick (1982a) to (N, d)-Gaussian random fields and by Cuzick (1978) to local ϕ-nondeterminism for an arbitrary positive function ϕ.

Berman's definition of LND for Gaussian processes Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a real-valued, separable Gaussian process with mean 0 and let $T \subset \mathbb{R}_+$ be an open interval. Assume that $\mathbb{E}[X(t)^2] > 0$ for all $t \in T$ and there exists $\delta > 0$ such that

$$\sigma^2(s,t) = \mathbb{E}\big[(X(s) - X(t))^2\big] > 0 \quad \text{for} \;\; s,\; t \in T \;\text{ with }\; 0 < |s-t| < \delta.$$

Recall from Berman (1973) that X is called *locally nondeterministic* on T if for every integer $n \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_n - t_1 \leq \varepsilon} V_n > 0, \tag{2.1}$$

where V_n is the relative prediction error:

$$V_n = \frac{\mathrm{Var}\big(X(t_n) - X(t_{n-1}) | X(t_1), \ldots, X(t_{n-1})\big)}{\mathrm{Var}\big(X(t_n) - X(t_{n-1})\big)} \tag{2.2}$$

and the infimum in (2.1) is taken over all ordered points $t_1 < t_2 < \cdots < t_n$ in T with $t_n - t_1 \leq \varepsilon$. Roughly speaking, (2.1) means that a small increment of the process X is not almost relatively predictable based on a finite number of observations from the immediate past.

It follows from Berman (1973, Lemma 2.3) that (2.1) is equivalent to the following property which says that X has *locally approximately independent increments*: for any positive integer $n \geq 2$, there exist positive constants c_n and δ (both may depend on n) such that

$$\mathrm{Var}\Big(\sum_{j=1}^{n} u_j \left(X(t_j) - X(t_{j-1})\right)\Big) \geq c_n \sum_{j=1}^{n} u_j^2 \, \sigma^2(t_{j-1}, t_j) \tag{2.3}$$

for all ordered points $0 = t_0 < t_1 < t_2 < \cdots < t_n$ in T with $t_n - t_1 < \delta$ and all $u_j \in \mathbb{R}$ $(1 \leq j \leq n)$. We refer to Nolan (1989, Theorem 2.6) for a proof of the above equivalence in much more general setting.

<u>*Local nondeterminism for Gaussian random fields*</u> In order to study the joint continuity of the local times of an (N, d)-Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$, Pitt (1978) extended Berman's definition (2.1) of LND to the random field setting; see also Geman and Horowitz (1980). Assume that $T \in \mathcal{A}$ is an interval and for all $s \neq t \in T$, the covariance matrix of $X(s) - X(t)$ is positive definite and is denoted by $\Sigma^2(s,t)$. Then there is a non-singular matrix $\Sigma(s,t)$ such that $\Sigma(s,t)\Sigma'(s,t) = \Sigma^2(s,t)$.

According to Pitt (1978), a Gaussian random field X as above is called *locally nondeterministic* on T if for every integer $n \geq 2$, there exist positive constants c_n and δ_n such that for all $\overline{u} = (u^1, \ldots, u^n) \in \mathbb{R}^{nd} \backslash \{0\}$,

$$\mathrm{Var}\Big(\sum_{j=1}^{n} \langle u^j,\ \Sigma_j^{-1}(X(t^j) - X(t^{j-1}))\rangle\Big) \geq c_n \sum_{j=1}^{n} |u^j|^2 \tag{2.4}$$

whenever the points $t^1, t^2, \ldots, t^n$ are distinct and all lie in a sub-interval of T with side-length at most δ_n, and satisfy

$$|t^j - t^{j-1}| \leq |t^j - t^i| \quad \text{for all} \;\; 1 \leq i < j \leq n. \tag{2.5}$$

Note that (2.5) introduces a partial order among $t^1, \ldots, t^n \in \mathbb{R}^N$; and there are at least n different ways to order them using (2.5).

Cuzick (1982a) gives another definition of local nondeterminism: an (N, d)-Gaussian random field X is locally nondeterministic on T if for all integers $n > 1$, there exist $c_n > 0$ and $\delta_n > 0$ (depending only on n) such that for any $t^1, \ldots, t^n \in T$ with $|t^j - t^n| \leq \delta_n$, the conditional vector $X(t^n)$ given $X(t^j)$, $j = 1, \ldots, n-1$ satisfies

$$\mathrm{detCov}\big(X(t^n)\big|X(t^j), 1 \leq j \leq n-1\big) \geq c_n \,\mathrm{detCov}\big(X(t^n) - X(t^*)\big), \tag{2.6}$$

where $t^* = t^i$ if $|t^i - t^n| = \inf_{j<n} |t^j - t^n|$ and $\mathrm{detCov}(Z)$ denote the determinant of the covariance matrix of the random vector Z.

Note that when $d = 1$ or, $d > 1$ and X has independent components, Theorem 2.6 of Nolan (1989) implies that (2.4) and (2.6) are equivalent. In general, however, it does not seem clear how these two definitions are related.

Remark 2.1 Both definitions of Pitt and Cuzick are applicable to all (N, d)-Gaussian random fields. Even though so far most authors have been working only with (N, d)-Gaussian random fields with independent components, it has become clear that one also needs to study (N, d)-Gaussian fields with dependent components. An interesting example of such Gaussian random fields is the operator fractional Brownian motion defined in Mason and Xiao (2002). It would be interesting to know whether it is LND in the sense of Pitt and/or Cuzick. An affirmative answer will be useful to establish many interesting sample path properties of operator fractional Brownian motion. □

The inequalities (2.3) and (2.4) have played significant roles in the works of Berman (1969–1973) and Pitt (1978) on local time theory of a large class of Gaussian random fields. Their results, in turn, imply irregularity and fractal properties of the sample paths of Gaussian random fields. See Geman and Horowitz (1980), Adler (1981), Geman et al. (1984) and the references therein for further information. Moreover, local nondeterminism has been applied by Rosen (1984) and Berman (1991) to study the existence and regularity of self-intersection local times, by Kahane (1985) to study the image and level sets of fractional Brownian motion, and by Monrad and Pitt (1987) to prove uniform Hausdorff dimension results for the image and inverse image of Gaussian random fields. Because of its various applications, it has been an interesting question to determine when a given Gaussian process is locally nondeterministic. Some sufficient conditions for real-valued Gaussian processes to be locally nondeterministic can be found in Berman (1973, 1988, 1991), Cuzick (1978), Pitt (1978).

<u>*k-th order local nondeterminism*</u> Berman's definition of LND was extended by Cuzick (1978) who defined local ϕ-nondeterminism for real-valued Gaussian processes by replacing the variance function $\sigma^2(t_n, t_{n-1})$ in (2.2) by $\phi(t_n - t_{n-1})$, where ϕ is an arbitrary positive function. Furthermore, he has defined the so-called kth order local ϕ-nondeterminism, not for the process X itself, but for the k-th divided differences of X; see Cuzick (1978, p.73) for details. He has given sufficient conditions for a stationary Gaussian processes to have this kth order LND property and then applied it to estimate the moments of the number $N(0, T)$ of zero crossings of a smooth stationary Gaussian process X in time interval $[0, T]$. In particular, he has provided verifiable sufficient conditions for the finiteness of the k-th factorial moment $M_k(0, T)$ of $N(0, T)$; see also Cuzick (1975) and Miroshin (1977). Even though the rest of this paper will not discuss the k-th order local ϕ-nondeterminism any further, we mention that, in order to study the rate of growth of $M_k(0, T)$ as a function of k, Cuzick (1978, p. 81) has noticed that the k-th order local ϕ-nondeterminism is not enough and has suggested to use a notion of k-th order strong ϕ-local nondeterminism. See Cuzick (1977) for some partial results along this direction on a stationary Gaussian process X such that X' exists in the quadratic mean sense. It would be interesting to study this problem under the more general setting of Section 2.2.

2.2 Strong ϕ-local nondeterminism for Gaussian random fields

There are some drawbacks in the definitions of local nondeterminism in Section 2.1: one is that the liminf in (2.1) and the constant c_n in (2.3) depend on the number of "time" points; the other is that there are many different ways to order n points in $\mathbb{R}^N$ using (2.5). Because of these, the properties of local nondeterminism defined by Berman (1973), Pitt (1978) and Cuzick (1978, 1982a) are not enough for establishing fine regularity properties such as the law of the iterated logarithm and the modulus of continuity for the local times or self-intersection local times of Gaussian random fields. For studying these and many other problems on Gaussian random fields, the concept of *strong local nondeterminism* (SLND) has proven to be more appropriate. See Cuzick (1982b), Monrad and Pitt (1987), Csörgő et al. (1995), Monrad and Rootzén (1995), Talagrand (1995, 1998), Xiao (1996, 1997a, b, c), Kasahara et al. (1999), Xiao and Zhang (2002), just to mention a few. In Section 4, we will address some of these aspects.

The following definition of the strong local ϕ-nondeterminism (SLϕND) was essentially given by Cuzick and DuPreez (1982) for Gaussian processes (i.e., $N = 1$). For Gaussian random fields, Definition 2.2 is more general than the definition of *strong local α-nondeterministism* of Monrad and Pitt (1987).

Definition 2.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued Gaussian random field with $0 < \mathbb{E}[X(t)^2] < \infty$ for all $t \in T$, where $T \in \mathcal{A}$ is an interval. Let ϕ be a given function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Then X is said to be* strongly locally ϕ-nondeterministic *(SLϕND) on T if there exist positive constants $c_{2,1}$ and r_0 such that for all $t \in T$ and all $0 < r \le \min\{|t|,\ r_0\}$,*

$$\mathrm{Var}\big(X(t)|X(s) : s \in T,\ r \le |s-t| \le r_0\big) \ge c_{2,1}\, \phi(r). \tag{2.7}$$

Remark 2.3 By modifying the proof of Proposition 7.2 of Pitt (1978), we can verify that if (2.7) holds and T is bounded away from 0, then for all $n \ge 2$ there exists a constant $c_{2,2} = c_{2,2}(n) > 0$ such that

$$\mathrm{Var}\Big(\sum_{j=1}^n u_j(X(t_j) - X(t_{j-1}))\Big) \ge c_{2,2} \sum_{j=1}^n u_j^2\, \phi(|t_j - t_{j-1}|) \tag{2.8}$$

for all $u_j \in \mathbb{R}$ and $t_j \in T$ $(j = 1, \ldots, n)$ satisfying (2.5). That is, X is locally ϕ-nondeterministic on T in the sense of Section 2.1. On the other hand, Cuzick (1977) has given an example of stationary Gaussian process $X = \{X(t), t \in \mathbb{R}\}$ in $\mathbb{R}$ that satisfies (2.8) for each fixed integer n and a function $\phi \asymp \sigma^2$, while the conditional variance in the left-hand side of (2.7) equals 0. Hence SLND (2.7) is strictly stronger than Berman's LND (2.1) or (2.3). $\square$

Remark 2.4 When $N = 1$, one could also define X to be strongly locally ϕ-nondeterministic when the constant c_n in (2.3) (with σ^2 replaced by ϕ) is independent of n. Clearly, this condition implies (2.7). It is not known whether the converse is true; see Remark 2.3 for a weaker result. Even though this alternative way of defining SϕLND is not needed for Gaussian processes, a modification of this is useful for stable processes; see Section 2.4. $\square$

Remark 2.5 We mention that in the studies of Gaussian processes $X = \{X(t), t \in \mathbb{R}\}$, due to the simple order structure of $\mathbb{R}$, it is sometimes enough to assume that X is *one-sided* strongly locally ϕ-nondeterministic, namely, for some constant $c_{2,3} > 0$

$$\mathrm{Var}\big(X(t)|X(s) : s \in T,\ r \le t - s \le r_0\big) \ge c_{2,3}\, \phi(r); \tag{2.9}$$

see Cuzick (1978), Berman (1972, 1978), Monrad and Rootzén (1995). When $X = \{X(t), t \in \mathbb{R}\}$ is a Gaussian process with stationary increments, some sufficient conditions in terms of the variance function $\sigma^2(h) = \mathbb{E}\big[\big(X(t+h) - X(t)\big)^2\big]$ for the *one-sided* strong local nondeterminism have been obtained earlier. Marcus (1968a) and Berman (1978) have proved that if $\sigma(h) \to 0$ as $h \to 0$ and $\sigma^2(h)$ is concave on $(0, \delta)$ for some $\delta > 0$, then X is one-sided strongly locally ϕ-nondeterministic for $\phi(r) = \sigma^2(r)$. $\square$

The most important example of SLND Gaussian random field is the N-parameter fractional Brownian motion $B_H = \{B_H(t), t \in \mathbb{R}^N\}$ of index H $(0 < H < 1)$. This is a centered, real-valued Gaussian random field with covariance function

$$\mathbb{E}\big(B_H(t)B_H(s)\big) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).$$

The strong local ϕ-nondeterminism of B_H with $\phi(r) = r^{2H}$ follows from Lemma 7.1 of Pitt (1978), where the self-similarity of B^H has played an essential role. For a stationary Gaussian process $X = \{X(t), t \in \mathbb{R}\}$, Cuzick and DuPreez (1982) have given a sufficient condition for X to be strongly locally ϕ-nondeterministic in terms of its spectral measure F. More precisely, they have proved that if the absolutely continuous part of $dF(\lambda)$ has the property that

$$\frac{dF(\lambda/r)}{\phi(r)} \ge h(\lambda)d\lambda \qquad \forall\, 0 < r \le r_0 \tag{2.10}$$

and

$$\int_0^\infty \frac{\log h(\lambda)}{1+\lambda^2}\, d\lambda > -\infty, \tag{2.11}$$

then X is SLϕND. Their proof uses the ideas from Cuzick (1977) and relies on the special properties of stationary Gaussian processes. Note that when $N = 1$, the strong local r^{2H}-nondeterminism of B_H can also be derived from the above result of Cuzick and DuPreez (1982) by using the Lamperti transformation. This approach can be applied to study self-similar Gaussian processes in general.

In Section 3 we will give a sufficient condition for Gaussian random fields with stationary increments to be strongly locally nondeterministic.

2.3 Sectorial local nondeterminism for anisotropic Gaussian random fields

In Definition 2.2, (2.7) measures the prediction error in terms of the distance between t and the region where the information is known. This works if the Gaussian random field X has certain approximately isotropic property, but can not be expected to hold for general anisotropic random fields. In fact, it has been well-known that the Brownian sheet does not have this type of strong local nondeterminism. This accounts for the significant difference between the existing methods for studying the fractional Brownian motion and the Brownian sheet.

Recently, Khoshnevisan and Xiao (2004b) have shown that the Brownian sheet possesses the so-called *sectorial local-nondeterminism*. This property leads to a unification of many of the methods developed for fractional Brownian motion and those for the Brownian sheet and to solutions of several problems on the image and multiple points of the Brownian sheet. See Khoshnevisan and Xiao (2004b), Khoshnevisan, Wu and Xiao (2005) for further information.

In the following, we will discuss sectorial local nondeterminism for fractional Brownian sheets. Recall that, for a given vector $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$, a real-valued fractional Brownian sheet $B_0^{\vec{H}} = \{B_0^{\vec{H}}(t), t \in \mathbb{R}_+^N\}$ with Hurst index $\vec{H}$ is a centered Gaussian random field with covariance function given by

$$\mathbb{E}\Big[B_0^{\vec{H}}(s)B_0^{\vec{H}}(t)\Big] = \prod_{\ell=1}^{N} \frac{1}{2}\Big(s_\ell^{2H_\ell} + t_\ell^{2H_\ell} - |s_\ell - t_\ell|^{2H_\ell}\Big), \quad s, t \in \mathbb{R}_+^N. \tag{2.12}$$

It follows from (2.12) that $B_0^{\vec{H}}(t) = 0$ a.s. for every $t \in \partial\mathbb{R}_+^N$, where $\partial\mathbb{R}_+^N$ denotes the boundary of $\mathbb{R}_+^N$.

Let $B_1^{\vec{H}}, \ldots, B_d^{\vec{H}}$ be d independent copies of $B_0^{\vec{H}}$. Then the Gaussian random field $B^{\vec{H}} = \{B^{\vec{H}}(t),\ t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}^d$ defined by

$$B^{\vec{H}}(t) = (B_1^{\vec{H}}(t), \ldots, B_d^{\vec{H}}(t)), \qquad \forall\ t \in \mathbb{R}_+^N \tag{2.13}$$

is called an (N, d)-fractional Brownian sheet with Hurst index $\vec{H} = (H_1, \ldots, H_N)$. It follows from (2.12) that $B^{\vec{H}}$ has the following operator-self-similarity: for any $N \times N$ diagonal matrix $A = (a_{ij})$ with $a_{ii} = a_i > 0$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \ne j$, we have

$$\big\{B^{\vec{H}}(At),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \Big\{\prod_{j=1}^{N} a_j^{H_j} B^{\vec{H}}(t),\ t \in \mathbb{R}^N\Big\}, \tag{2.14}$$

where $X \stackrel{d}{=} Y$ means that the two processes have the same finite dimensional distributions. Moreover, for every $\ell = 1, \ldots, N$, $B^{\vec{H}}$ is a fractional Brownian motion in $\mathbb{R}^d$ of Hurst index H_ℓ along the direction of the ℓth axis.

If $N > 1$ and $H_1 = \cdots = H_N = 1/2$, then B^H is the (N,d)-Brownian sheet. See Orey and Pruitt (1973) and Khoshnevisan (2002) for systematic accounts on the Brownian sheets.

Fractional Brownian sheets arise naturally in many areas such as in stochastic partial differential equations [cf. Øksendal and Zhang (2000), Hu, Øksendal and Zhang (2000)] and in the studies of most visited sites of symmetric Markov processes [cf. Eisenbaum and Khoshnevisan (2002)]. One of the important features of $B^{\vec{H}}$ is that, when $H_1, \ldots, H_N$ are different, it has different probabilistic and analytic behaviors along different directions and thus is highly anisotropic. Recently, there have been interest in using anisotropic Gaussian random fields to model bone structure [Bonami and Estrade (2003)] and aquifer structure in hydrology [Benson et al. (2004)]. We believe that the results and techniques for characterizing the anisotropic properties of the fractional Brownian sheet in terms of $\vec{H}$ will also be helpful for studying other types of anisotropic Gaussian random fields.

The main tools for analyzing the dependence structure of $B_0^{\vec{H}}$ are the following stochastic integral representations. They can be proved by verifying the covariance functions.

- **Moving average representation**

$$B_0^{\vec{H}}(t) = c_{2,4}^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} \prod_{\ell=1}^{N} g_{H_\ell}(t_\ell, s_\ell) W(ds), \tag{2.15}$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard real-valued Brownian sheet and for $H \in (0,1)$ and $s, t \in \mathbb{R}$,

$$g_H(t,s) = \big((t-s)_+\big)^{H-1/2} - \big((-s)_+\big)^{H-1/2},$$

with $s_+ = \max\{s, 0\}$, and where $c_{2,4}$ is the normalizing constant given by

$$c_{2,4}^2 = \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} \Big[\prod_{\ell=1}^{N} g_{H_\ell}(1, s_\ell)\Big]^2 ds.$$

- **Harmonizable representation**

$$B_0^{\vec{H}}(t) = c_{2,5}^{-1} \int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}} \widehat{W}(d\lambda), \tag{2.16}$$

where $\widehat{W}$ is the Fourier transform of white noise in $\mathbb{R}^N$ and $c_{2,5} > 0$ is the normalizing constant so that $\mathrm{Var}\big(B_0^{\vec{H}}(\langle 1 \rangle)\big) = 1$. This representation for $B_0^{\vec{H}}$ is proved by Herbin (2004).

The following sectorial LND of fractional Brownian sheet is proved by Wu and Xiao (2005), extending a results of Khoshnevisan and Xiao (2004b) on the Brownian sheet.

Lemma 2.6 *Let $B_0^{\vec{H}} = \{B_0^{\vec{H}}(t),\ t \in \mathbb{R}_+^N\}$ be a fractional Brownian sheet in $\mathbb{R}$ with Hurst index $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$. Then for any $\varepsilon > 0$, there is a constant $c_{2,6} > 0$ such that for all integers $n \geq 2$, $t^1, \ldots, t^n \in [\varepsilon, \infty)^N$,*

$$\mathrm{Var}\Big(B_0^{\vec{H}}(t^n)\Big|B_0^{\vec{H}}(t^j), 1 \leq j \leq n-1\Big) \geq c_{2,6} \sum_{\ell=1}^{N} \min_{0 \leq j \leq n-1} |t_\ell^n - t_\ell^j|^{2H_\ell}, \quad (2.17)$$

where $t_\ell^0 = 0$ for every $\ell = 1, \ldots, N$.

The proof of Lemma 2.6 makes use of the harmonizable representation of $B_0^{\vec{H}}$ and a Fourier analytic argument. This lemma plays key roles in Ayache, Wu and Xiao (2005) who verify a conjecture of Xiao and Zhang (2002) on the joint continuity of local times of a fractional Brownian sheet $B^{\vec{H}}$, and in Wu and Xiao (2005) who study the geometric properties of the sample paths of $B^{\vec{H}}$.

2.4 Local nondeterminism for stable processes

In this subsection, we will discuss briefly the properties of local nondeterminism for stable random fields. First we mention the following papers which are closely related to the topics of this paper, but will not be further addressed because all the random fields considered there possess certain Markovian nature. Ehm (1981) has established many deep results on the sample path properties of the stable sheet and his arguments rely crucially on the property of independent increments of the stable sheet. Khoshnevisan, Xiao and Zhong (2003a, b) have extended several of Ehm's results to additive Lévy processes and have also established some useful connections between hitting probabilities and a class of natural capacities. Mountford and Nualart (2004) and Mountford (2004) determine the exact Hausdorff measure functions for the level sets of an additive Brownian motion and additive stable processes, respectively. The property of independent increments of Lévy processes and a certain type of Markov property have played crucial roles in the work of these authors. We refer to the survey papers of Khoshnevisan and Xiao (2004a) and Xiao (2004) for further information along this line.

The class of symmetric α-stable (SαS) self-similar processes and random fields is very large; see Samorodnitsky and Taqqu (1994) for a systematic account. Of special interest are the linear fractional stable motion and harmonizable fractional stable motion introduced by Taqqu and Wolpert (1983), Maejima (1983), Cambanis and Maejima (1989), respectively. They are natural stable analogues of fractional Brownian motion.

Compared to Gaussian random fields, much less about the probabilistic, analytic and statistical properties of such stable random fields has been known. We believe that an appropriate notion of strong local nondeterminism for stable random fields will be helpful to solve several open problems on local times and self-intersection local times, as well as to investigate other sample path properties.

The notion of local nondeterminism has been extended to SαS processes and random fields by Nolan (1988, 1989), and has proven to be a useful tool in studying the local times and self-intersection local times of certain self-similar stable processes with stationary increments. See, for example, Kôno and Shieh (1993), Shieh (1993) and Xiao (1995).

One of the difficulties of extending LND from Gaussian random fields to α-stable random fields $X = \{X(t), t \in \mathbb{R}^N\}$ is that, when $0 < \alpha < 2$, there is no covariance to measure dependence of $X(t^1), \ldots, X(t^n)$. Nolan (1989) has relied on the L^α-representations of symmetric α-stable random fields [see Hardin (1982) or Samorodnitsky and Taqqu (1994)] and the approximation properties of normed or quasi-normed linear spaces.

We first consider the case $N = 1$. Let $T \subset \mathbb{R}$ be a closed interval. The following definition is due to Nolan (1989, Definition 3.1) which reduces to (2.3) when $\alpha = 2$.

Definition 2.7 *A real-valued SαS process $X = \{X(t), t \in \mathbb{R}\}$ is called locally nondeterministic on T if for every integer $n > 1$, there exists a constant $c_n \geq 1$ depending on n only such that for all sufficiently close $t_1 < t_2 < \ldots < t_n$ in T,*

$$\begin{aligned}
&\Big|\mathbb{E}\Big(e^{ic_n\, u_1 X(t_1)}\Big)\prod_{j=2}^{n}\mathbb{E}\Big(e^{ic_n\, u_j(X(t_j)-X(t_{j-1}))}\Big)\Big| \\
&\qquad \leq \Big|\mathbb{E}\exp\Big\{i\big(u_1X(t_1)+\sum_{j=2}^{n}u_j(X(t_j)-X(t_{j-1}))\big)\Big\}\Big| \qquad (2.18)\\
&\qquad \leq \Big|\mathbb{E}\Big(e^{ic_n^{-1}\, u_1 X(t_1)}\Big)\prod_{j=2}^{n}\mathbb{E}\Big(e^{ic_n^{-1}\, u_j(X(t_j)-X(t_{j-1}))}\Big)\Big|
\end{aligned}$$

for all $u_j \in \mathbb{R}$ ($j = 1, \ldots, n$).

Hardin (1982) proved that for every real-valued, separable in probability, SαS process $X = \{X(t), t \in \mathbb{R}\}$, there exist a measure space $(E, \mathcal{B}, \mu)$ and a collection of real-valued functions $\{\kappa(t, \cdot), t \in \mathbb{R}\} \subseteq L^\alpha(E, \mathcal{B}, \mu)$ such that for all integers $n \geq 1$ the joint distribution of $X(t_1), \ldots, X(t_n)$ is determined by

$$\mathbb{E}\exp\Big(i\sum_{j=1}^{n}u_jX(t_j)\Big) = \exp\Big(-\Big\|\sum_{j=1}^{n}u_j\kappa(t_j)\Big\|_\alpha^\alpha\Big), \qquad (2.19)$$

where $\|\cdot\|_\alpha$ is the quasi-norm in $L^\alpha(E, \mathcal{B}, \mu)$ and $\kappa(t_j) \hat{=} \kappa(t_j, \cdot)$. Based on this fact, Nolan (1989) proves that (2.18) in Definition 2.7 is equivalent to the following: for every integer $n > 1$, there exists a constant $c_{2,7} = c_{2,7}(n) \geq 1$

depending on n only such that

$$\begin{aligned} c_{2,7}^{-1}\Big(\big\|u_1\kappa(t_1)\big\|_\alpha + \sum_{j=2}^n \big\|u_j(\kappa(t_j)-\kappa(t_{j-1}))\big\|_\alpha\Big) \\ \le \Big\|u_1\kappa(t_1) + \sum_{j=2}^n u_j\big(\kappa(t_j)-\kappa(t_{j-1})\big)\Big\|_\alpha \\ \le c_{2,7}\Big(\big\|u_1\kappa(t_1)\big\|_\alpha + \sum_{j=2}^n \big\|u_j(\kappa(t_j)-\kappa(t_{j-1}))\big\|_\alpha\Big) \end{aligned} \tag{2.20}$$

for all $u_j \in \mathbb{R}$ and all $t_1 < t_2 < \ldots < t_n$ in T such that $t_n - t_1$ is sufficiently small. Nolan (1989, Theorem 3.2) also gives some other equivalent definitions of LND for real-valued SαS processes.

An (N,d)-random field $X = \{X(t), t \in \mathbb{R}^N\}$ is called *an (N,d,α)-stable field* if for all integers $n \ge 1$, $t^1,\ldots,t^n \in \mathbb{R}^N$ and $u^1,\ldots,u^n \in \mathbb{R}^d$, the random variables $\sum_{j=1}^n \langle u^j, X(t^j)\rangle$ are SαS random variables. For a given measure space $(E,\mathcal{B},\mu)$, let $L^\alpha(E,\mathcal{B},\mu;\mathbb{R}^d)$ denote the collections of $\mathbb{R}^d$-valued functions $\kappa(\cdot)$ such that $\kappa(\cdot) = (\kappa_1(\cdot),\ldots,\kappa_d(\cdot))$ and $\kappa_j(\cdot) \in L^\alpha(E,\mathcal{B},\mu)$ for every $j = 1,\ldots,d$. It is possible to represent an (N,d,α)-stable field X in some $L^\alpha(E,\mathcal{B},\mu;\mathbb{R}^d)$. That is, there is a family of functions $\{\kappa(t,\cdot), t \in \mathbb{R}^N\}$ in the space $L^\alpha(E,\mathcal{B},\mu;\mathbb{R}^d)$ such that

$$\mathbb{E}\exp\Big(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) = \exp\Big(-\Big\|\sum_{j=1}^n \langle u^j, \kappa(t^j)\rangle\Big\|_\alpha^\alpha\Big). \tag{2.21}$$

Nolan (1989, Definition 3.3) defines LND of an (N,d,α)-stable field X in terms of the family $\{\kappa(t,\cdot), t \in \mathbb{R}^N\}$.

Definition 2.8 *An (N,d,α)-stable field X is called locally nondeterministic on an interval $T \in \mathcal{A}$ if its representation $\{\kappa(t,\cdot), t \in \mathbb{R}^N\}$ satisfies the following conditions:*

(a) $\|\kappa_j(t)\|_\alpha > 0$ *for all* $t \in T$ *and* $j = 1,\ldots,d$.

(b) $\|\kappa_j(s) - \kappa_j(t)\|_\alpha > 0$ *for all* $s,t \in T$ *with* $|s-t|$ *sufficiently small and* $j = 1,\ldots,d$.

(c) For all integers $n \ge 1$, *arbitrary* $t^1,\ldots,t^n \in T$ *and all* $j = 1,\ldots,d$, *define* M_j^n *to be the subspace of* $L^\alpha(E,\mathcal{B},\mu)$ *spanned by* $\{\kappa_l(t^k) : 1 \le l \le d, 1 \le k \le n$ *and* $(l,k) \ne (j,n)\}$. *Then for all* $j = 1,\ldots,d$,

$$\inf_{t^1\in T} \frac{\|\kappa_j(t^1) - M_j^1\|_\alpha}{\|\kappa_j(t^1)\|_\alpha} > 0 \tag{2.22}$$

and

$$\liminf \frac{\|\kappa_j(t^n) - M_j^n\|_\alpha}{\|\kappa_j(t^n) - \kappa_j(t^{n-1})\|_\alpha} > 0, \tag{2.23}$$

where the liminf is taken over all $t^1, \ldots, t^n \in T$ satisfying (2.5) with $|t^n - t^1| \to 0$, and $\|\kappa_j(t^n) - M_j^n\|_\alpha$ denotes the "L^α-distance" between $\kappa_j(t^n)$ and M_j^n.

Nolan (1989) shows that Condition (c) in Definition 2.8 is equivalent to the assumption that X has, in certain sense, approximately independent components and approximately independent increments. This is useful for establishing the joint continuity of local times of several classes of stable processes or stable random fields; see Nolan (1989), Kôno and Shieh (1993), Shieh (1993) and Xiao (1995). However, as in the Gaussian case, this LND property is not useful for obtaining sharp uniform and/or local growth properties of the local times or self-intersection local times of SαS processes and (N, d, α)-stable fields; see Dozzi and Soltani (1999, section 4) for related remarks. One needs to have a notion of strong local nondeterminism.

When $N = 1$, we recall Remark 2.4 and may define conveniently that an SαS process X is strongly locally nondeterministic on T if there exists a constant $c_{2,8} > 0$ such that the following hold: for all integers $n \geq 2$, we can find a nonsingular $n \times n$ matrix A such that for all $t_1 < t_2 < \ldots < t_n$ in T sufficiently close and all $u_j \in \mathbb{R}$ $(j = 1, \ldots, n)$,

$$\begin{aligned} &\Big|\mathbb{E}\exp\Big\{i\big(u_1 X(t_1) + \sum_{j=2}^n u_j(X(t_j) - X(t_{j-1}))\big)\Big\}\Big| \\ &\qquad \leq \Big|\mathbb{E}\Big(e^{ic_{2,8}\, v_1 X(t_1)}\Big) \prod_{j=2}^n \mathbb{E}\Big(e^{ic_{2,8}\, v_j (X(t_j) - X(t_{j-1}))}\Big)\Big|, \end{aligned} \tag{2.24}$$

where $(v_1, \ldots, v_n) = (u_1, \ldots, u_n)A$.

Dozzi and Soltani (1999) have studied a class of moving average (MA) SαS processes X of the form $X = X_1 + X_2$, where X_1 and X_2 are two independent MA-stable processes, X_1 is strongly locally nondeterministic in the above sense and X_2 is arbitrary. They showed that the arguments of Berman (1973) and Ehm (1981) can be modified to prove uniform and local Hölder conditions for the local times of X.

In light of the theory on Gaussian random fields, there should be several different senses of strong local nondeterminism for (N, d, α)-stable fields. For simplicity, we start by considering only isotropic $(N, 1, \alpha)$-stable fields with stationary increments. It seems natural to define the strong local ϕ-nondeterminism for such SαS random fields as follows.

Definition 2.9 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $(N, 1, \alpha)$-stable field with stationary increments and $X(0) = 0$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$ and let $T \in \mathcal{A}$. Then X is said to be* strongly locally ϕ-nondeterministic *(SLϕND) on T if, in addition to (a) and (b) in Definition 2.8, there exists a constant $c_{2,9} > 0$ such that for all integers $n \geq 1$, all $t, s^1, \ldots, s^n \in T$ sufficiently close,*

$$\|\kappa(t) - M^n\|_\alpha^\alpha \geq c_{2,9}\, \phi\big(\min_{0 \leq j \leq n} |t - s^j|\big), \tag{2.25}$$

where M^n denotes the subspace of $L^\alpha(E,\mathcal{B},\mu)$ spanned by $\{\kappa(s^1),\dots,\kappa(s^n)\}$ and $s^0=0$.

It would be very interesting to determine which classes of self-similar stable random fields are strongly locally nondeterministic in the above sense and to explore various applications of SLND in the studies of stable random fields. We believe that the linear fractional stable motions (or fields) and harmonizable fractional stable motions (or fields) with Hurst index $H\in(0,1)$ [cf. Samorodnitsky and Taqqu (1994), Kokoszka and Taqqu (1994) and Nolan (1989)] are strongly locally ϕ-nondeterministic with $\phi(r)=r^H$. So far, I have only been able to prove this for harmonizable fractional $(N,1,\alpha)$-stable fields when $\alpha\in[1,2]$ and a *one-sided* SLND for $(1,1,\alpha)$-linear fractional stable motion when $\alpha\in(0,2]$.

On the other hand, the $(N,1,\alpha)$-stable sheet $Z_\alpha=\{Z(t),\,t\in\mathbb{R}_+^N\}$ defined in Ehm (1981), which contains the Brownian sheet as a special case, is not strongly locally ϕ-nondeterministic in the sense of Definition 2.9. Moreover, similar to (2.15) and (2.16), we can define two classes of (real-valued) anisotropic fractional stable sheets using stochastic integration with respect to an $(N,1,\alpha)$-stable sheet Z_α or a complex-valued SαS random measure $\widetilde{Z}_\alpha$. They are natural extensions of fractional Brownian sheets to stable random fields.

• **Moving average fractional stable sheets**: for any given $0<\alpha<2$ and $\vec{H}=(H_1,\dots,H_N)\in(0,1)^N$, we define a stable random field $Z^{\vec{H}}=\{Z^{\vec{H}}(t),t\in\mathbb{R}_+^N\}$ with values in $\mathbb{R}$ by

$$Z^{\vec{H}}(t)=\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_N}\prod_{\ell=1}^N h_{H_\ell}(t_\ell,s_\ell)Z_\alpha(ds), \tag{2.26}$$

where $Z_\alpha=\{Z_\alpha(s),s\in\mathbb{R}^N\}$ is a symmetric $(N,1,\alpha)$-stable sheet and for $H\in(0,1)$ and $s,t\in\mathbb{R}$,

$$\begin{aligned} h_H(t,s)=a\Big\{\big((t-s)_+\big)^{H-1/\alpha}-\big((-s)_+\big)^{H-1/\alpha}\Big\}\\ +b\Big\{\big((t-s)_-\big)^{H-1/\alpha}-\big((-s)_-\big)^{H-1/\alpha}\Big\}, \end{aligned} \tag{2.27}$$

where $a,b\in\mathbb{R}$ are constants and $t_-=\max\{-t,0\}$. Using (2.26) and the self-similarity of Z_α, we can verify that the $(N,1,\alpha)$-stable field $Z^{\vec{H}}$ is operator self-similar in the sense of (2.14), and along each direction of $\mathbb{R}_+^N$, $Z^{\vec{H}}$ becomes a real-valued linear fractional stable motion. We will call $Z^{\vec{H}}=\{Z^{\vec{H}}(t),t\in\mathbb{R}_+^N\}$ an $(N,1,\alpha)$-moving average fractional stable sheet.

• **Harmonizable fractional stable sheets**: for any given $0<\alpha<2$ and $\vec{H}=(H_1,\dots,H_N)\in(0,1)^N$, we define the harmonizable fractional stable sheet $\widetilde{Z}^{\vec{H}}=\{\widetilde{Z}^{\vec{H}}(t),t\in\mathbb{R}_+^N\}$ with values in $\mathbb{R}$ by

$$\widetilde{Z}^{\vec{H}}(t)=\mathrm{Re}\int_{\mathbb{R}^N}\prod_{j=1}^N\frac{e^{it_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}}\,\widetilde{Z}_\alpha(d\lambda), \tag{2.28}$$

where $\widetilde{Z}_\alpha$ is a complex-valued SαS random measure. We refer to Samorodnitsky and Taqqu (1994, Chapter 6) for definition and basic properties of complex-valued SαS random measure and the corresponding stochastic integrals.

Similar to the moving average fractional stable sheet, we can verify that $\widetilde{Z}^{\vec{H}}$ is operator self-similar in the sense of (2.14). Along each direction of $\mathbb{R}^N_+$, $\widetilde{Z}^{\vec{H}}$ becomes a real-valued harmonizable fractional stable motion.

Note that, unlike the Gaussian case where both (2.15) and (2.16) determine (up to a constant) the same fractional Brownian sheet, the moving average and harmonizable fractional stable sheets with the same $\alpha \in (0, 2)$ and Hurst index $\vec{H}$ are different random fields. This is true even for $N = 1$; see Samorodnitsky and Taqqu (1994, page 358).

Based on the studies of fractional Brownian sheets, we believe that an appropriate definition of sectorial local nondeterminism should be introduced and it will be useful for studying various sample path properties of such anisotropic stable random fields. This problem will be studied elsewhere, and the rest of the paper deals with Gaussian random fields only.

3 Spectral conditions for strong local nondeterminism of Gaussian random fields

As pointed out by Cuzick and DuPreez (1982, p. 811), it appears to be difficult to establish conditions under which general Gaussian processes possess the various forms of strong local nondeterminism. In this section we provide sufficient conditions for a real-valued Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with stationary increments to be strongly locally ϕ-nondeterministic. In particular, we show that a spectral condition similar to that of Berman (1988) for ordinary LND of Gaussian processes actually implies that X is strongly locally ϕ-nondeterministic and, importantly, $\phi(r)$ is comparable to the variance function $\sigma^2(h)$ with $|h| = r$.

Similar methods, combined with the arguments in Wu and Xiao (2005), can be modified to study the sectorial local nondeterminism of anisotropic Gaussian random fields with stationary increments or Gaussian random fields of fractional Brownian sheet type. By the latter, I mean their covariance functions are defined as tensor products of covariance functions of Gaussian processes with stationary increments. There are many interesting questions on such anisotropic Gaussian random fields due to their various applications; see Bonami and Estrade (2003), Cheridito (2004), Mannersalo and Norros (2002), Mueller and Tribe (2002) and the references therein.

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $X(0) = 0$. We assume that X has stationary increments and continuous covariance function $R(s, t) = \mathbb{E}\big[X(s)X(t)\big]$. According to Yaglom (1957), $R(s, t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda) + \langle s, Qt\rangle, \tag{3.1}$$

where Q is an $N \times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\ \Delta(d\lambda) < \infty. \tag{3.2}$$

The measure Δ is called the *spectral measure* of X.

It follows from (3.1) that X has the following stochastic integral representation:

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1) W(d\lambda) + \langle \mathbf{Y}, t\rangle, \tag{3.3}$$

where $\mathbf{Y}$ is an N-dimensional Gaussian random vector with mean 0 and $W(d\lambda)$ is a centered complex-valued Gaussian random measure which is independent of $\mathbf{Y}$ and satisfies

$$\mathbb{E}\left(W(A)\overline{W(B)}\right) = \Delta(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}$$

for all Borel sets $A,\ B \subseteq \mathbb{R}^N$. From now on, we will assume $\mathbf{Y} = 0$. Consequently, we have

$$\sigma^2(h) = \mathbb{E}\big[\big(X(t+h) - X(t)\big)^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\ \Delta(d\lambda). \tag{3.4}$$

If the function $\sigma^2(h)$ only depends on $|h|$, then X is called an *isotropic* random field. It is important to note that $\sigma^2(h)$ is a negative definite function and can be viewed as the characteristic exponent of a symmetric infinitely divisible distribution; see Berg and Forst (1975) for more information on negative definite functions.

The main results of this section are Theorems 3.1 and 3.4. They give verifiable conditions for a Gaussian random field to be strongly locally nondeterministic in terms of its spectral measure.

Theorem 3.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a mean zero, real-valued Gaussian random field with stationary increments and $X(0) = 0$, and let f be the density function of the absolutely continuous part of the spectral measure Δ of X. Assume that there exist two non-decreasing functions $\phi(r)$ and $q(r) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:*

$$\frac{f(\lambda/r)}{\phi(r)} \geq \frac{r^N}{q(|\lambda|)}, \qquad \forall\, r \in (0,1] \ \text{ and } \ \lambda \in \mathbb{R}^N \tag{3.5}$$

and there exists a positive and finite constant η such that

$$q(r) \leq r^\eta \qquad \text{for all } r > 0 \text{ large enough.} \tag{3.6}$$

Then for every interval $T \in \mathcal{A}$, there exists a constant $0 < c_{3,1} < \infty$ such that for all $t \in T\backslash\{0\}$ and all $0 < r \leq \min\{1, |t|\}$,

$$\mathrm{Var}\big(X(t)|X(s) :\ s \in T,\ |s-t| \geq r\big) \geq c_{3,1}\ \phi(r). \tag{3.7}$$

In particular, X is strongly locally ϕ-nondeterministic on T.

To prove Theorem 3.1, we note that, in the Hilbert space setting, the conditional variance in (3.7) is the square of the $L^2(\mathbb{P})$-distance of $X(t)$ from the subspace generated by $\{X(s):\ s\in T,\ |s-t|\ge r\}$. Hence it is sufficient to show that there exists a positive constant $c_{3,2}$ such that for all integers $n\ge 1$, $a_k\in\mathbb{R}$ and $s_k\in T$ satisfying $|s_k-t|\ge r,\ (k=1,2,\ldots,n)$,

$$\mathbb{E}\Big(X(t)-\sum_{k=1}^n a_k X(s_k)\Big)^2 \ge c_{3,2}\,\phi(r). \tag{3.8}$$

It follows from (3.1) or (3.3) that

$$\begin{aligned}\mathbb{E}\Big(X(t)-\sum_{k=1}^n a_k X(s_k)\Big)^2 &= \int_{\mathbb{R}^N}\Big|e^{i\langle t,\lambda\rangle}-1-\sum_{k=1}^n a_k\big(e^{i\langle s_k,\lambda\rangle}-1\big)\Big|^2\Delta(d\lambda)\\ &\ge \int_{\mathbb{R}^N}\Big|e^{i\langle t,\lambda\rangle}-\sum_{k=0}^n a_k\,e^{i\langle s_k,\lambda\rangle}\Big|^2 f(\lambda)d\lambda, \end{aligned}\tag{3.9}$$

where $a_0=-1+\sum_{k=1}^n a_k$ and $s_0=0$. This part of the proof goes back to Kahane (1985). The last integral can be estimated using the ideas in Pitt (1975, 1978) and Kahane (1985); see Xiao (2005) for a complete proof.

In order to apply Theorem 3.1 to investigate the sample path properties of the Gaussian random field X, we need to study the relationship between $\phi(|h|)$ and the function $\sigma^2(h)$. In the following, we assume that the spectral measure Δ is absolutely continuous and its density function $f(\lambda)$ satisfies the following condition [when $N=1$, this is due to Berman (1988)]:

$$0<\underline{\alpha}=\frac{1}{2}\liminf_{\lambda\to\infty}\frac{\beta_N\,|\lambda|^N f(\lambda)}{\Delta\{\xi:|\xi|\ge|\lambda|\}}\le\frac{1}{2}\limsup_{\lambda\to\infty}\frac{\beta_N\,|\lambda|^N f(\lambda)}{\Delta\{\xi:|\xi|\ge|\lambda|\}}=\overline{\alpha}<1, \tag{3.10}$$

where $\beta_1=2$ and for $N\ge 2$, $\beta_N=\mu(S^{N-1})$ is the surface area [i.e., the $(N-1)$-dimensional Lebesgue measure) of S^{N-1}. At the end of this section, we will give some examples of Gaussian random fields satisfying (3.10).

In the rest of this section, we define $\phi(r)=\Delta\{\xi:|\xi|\ge r^{-1}\}$ and $\phi(0)=0$. Then the function ϕ is non-decreasing and left continuous on $[0,\infty)$. The following lemma lists some useful properties of ϕ.

Lemma 3.2 *Assume the condition (3.10) holds. Then for every $0<\varepsilon<2\min\{\underline{\alpha},1-\overline{\alpha}\})$, there exists a constant $r_0>0$ such that for all $0<x\le y\le r_0$,*

$$\Big(\frac{x}{y}\Big)^{2\overline{\alpha}+\varepsilon}\le\frac{\phi(x)}{\phi(y)}\le\Big(\frac{x}{y}\Big)^{2\underline{\alpha}-\varepsilon}. \tag{3.11}$$

Consequently, we have

(i). $lim_{r\to 0}\phi(r)/r^2=\infty$.

(ii). The function ϕ has the following doubling property: there exists a constant $c_{3,3}>0$ such that $\phi(2r)\le c_{3,3}\,\phi(r)$ for all $0<r<r_0/2$.

Remark 3.3 Under the assumption that Δ is absolutely continuous with density $f(\lambda)$, Condition (3.10) is more general than assuming ϕ is regularly varying at 0. Using the terminology of Bingham et al. (1987, pp.65-67), (3.11) implies that ϕ is extended regularly varying at 0 with upper and lower Karamata indices $2\overline{\alpha}$ and $2\underline{\alpha}$, respectively. A necessary and sufficient condition for $\phi(r)$ to be regularly varying at 0 of index 2α is that the limit

$$\alpha = \frac{1}{2} \lim_{r\to\infty} \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{\Delta\{\xi : |\xi| \geq r\}}$$

exists; see Xiao (2005) for details. □

The following theorem shows that the assumption (3.10) implies that X is SLϕND and $\phi(r)$ is comparable with $\sigma^2(h)$ with $|h| = r$ near 0. In Section 4, we will show that it is often more convenient to use the function ϕ to characterize the properties of X.

Theorem 3.4 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a mean zero, real-valued Gaussian random field with stationary increments and $X(0) = 0$. Assume that the spectral measure Δ of X has a density function f that satisfies (3.10). Then*

$$0 < \liminf_{h\to 0} \frac{\sigma^2(h)}{\phi(|h|)} \leq \limsup_{h\to 0} \frac{\sigma^2(h)}{\phi(|h|)} < \infty. \tag{3.12}$$

Moreover, for every interval $T \in \mathcal{A}$, X is strongly locally ϕ-nondeterministic on T.

The first part of Theorem 3.4 is proved using the ideas of Berman (1988, 1991) and the second part follows from (3.12) and Theorem 3.1; see Xiao (2005) for details.

Applying Theorems 3.1 and 3.4 to stationary Gaussian random fields, we have the following partial extension of the result of Cuzick and DuPreez (1982) mentioned in section 2.2. It is not known to me whether (3.6) can be replaced by the weaker condition (2.11).

Corollary 3.5 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a stationary Gaussian random field with mean 0 and variance 1.*

(i). If the spectral measure Δ of X has an absolutely continuous part with density f satisfying (3.5) and (3.6), then for every interval $T \in \mathcal{A}$, X is strongly locally ϕ-nondeterministic on T.

(ii). If the spectral density of X satisfies (3.10), then (3.12) holds and X is SLϕND on T.

We end this section with some more examples of Gaussian random fields whose SLND can be determined.

Example 3.6 Consider the mean zero Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with stationary increments and spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{c(\gamma,\beta,N)}{|\lambda|^{2\gamma}(1+|\lambda|^2)^{\beta}},$$

where γ and β are constants satisfying $\beta+\gamma > \frac{N}{2}$, $0 < \gamma < 1+\frac{N}{2}$ and $c(\gamma,\beta,N) > 0$ is a normalizing constant. Since the spectral density $f_{\gamma,\beta}$ involves both the Fourier transforms of the Riesz kernel and the Bessel kernel, Anh et al. (1999) call the corresponding Gaussian random field the fractional Riesz-Bessel motion with indices β and γ; and they have shown that these Gaussian random fields can be used for modelling simultaneously long range dependence and intermittency.

It is easy to check that Condition (3.10) is satisfied with $\underline{\alpha} = \overline{\alpha} = \gamma+\beta-\frac{N}{2}$. Moreover, since the spectral density $f_{\gamma,\beta}(x)$ is regularly varying at infinity of order $2(\beta+\gamma) > N$, by a result of Pitman (1968) we know that, if $\gamma+\beta-\frac{N}{2} < 1$, then $\sigma(h)$ is regularly varying at 0 of order $\gamma+\beta-N/2$ and

$$\sigma(h) \sim |h|^{\gamma+\beta-N/2} \qquad \text{as} \quad h \to 0.$$

Theorem 3.4 implies that X is SLND with respect to $\sigma^2(h)$. Hence, many sample path properties of the d-dimensional fractional Riesz-Bessel motion X with indices β and γ can be derived from the results in Section 4.

Example 3.7 Let $0 < \alpha < 1$, $0 < c_1 < c_2$ be constants such that $(\alpha c_2)/c_1 < 1$. For any increasing sequence $\{b_n,\, n \geq 0\}$ of real numbers such that $b_0 = 0$ and $b_n \to \infty$, define the function f on $\mathbb{R}^N$ by

$$f(\lambda) = \begin{cases} c_1\,|\lambda|^{-(2\alpha+N)} & \text{if } \ |\lambda| \in (b_{2k}, b_{2k+1}], \\ c_2\,|\lambda|^{-(2\alpha+N)} & \text{if } \ |\lambda| \in (b_{2k+1}, b_{2k+2}]. \end{cases} \tag{3.13}$$

Some elementary calculation shows that, when $\lim_{n\to\infty} b_{n+1}/b_n = \infty$, Condition (3.10) is satisfied with $\underline{\alpha} = (\alpha c_1)/c_2 < \overline{\alpha} = (\alpha c_2)/c_1$. Note that in this case, $c_1\,|h|^{2\alpha} \leq \sigma^2(h) \leq c_2\,|h|^{2\alpha}$ and $c_1\,r^{2\alpha} \leq \phi(r) \leq c_2\,r^{2\alpha}$ for all $h \in \mathbb{R}^N$ and $r > 0$, but both functions are not regularly varying at the origin.

The following is a class of Gaussian random fields for which (3.10) does not hold, but Theorem 3.1 is still applicable.

Example 3.8 For any given constants $0 < \alpha_1 < \alpha_2 < 1$ and any increasing sequence $\{b_n,\, n \geq 0\}$ of real numbers such that $b_0 = 0$ and $b_n \to \infty$, define the function f on $\mathbb{R}^N$ by

$$f(\lambda) = \begin{cases} |\lambda|^{-(2\alpha_1+N)} & \text{if } \ |\lambda| \in (b_{2k}, b_{2k+1}], \\ |\lambda|^{-(2\alpha_2+N)} & \text{if } \ |\lambda| \in (b_{2k+1}, b_{2k+2}]. \end{cases} \tag{3.14}$$

Using such functions f as spectral densities, we obtain a quite large class of Gaussian random fields with stationary increments that are significantly different from the fractional Brownian motion. If X is such a random field, then

there exist positive and finite constants $c_{3,4}$, $c_{3,5} \geq 1$ such that

$$c_{3,4}^{-1}\,|h|^{2\alpha_2} \leq \sigma^2(h) \leq c_{3,4}\,|h|^{2\alpha_1}, \qquad \forall\,|h| \leq 1 \tag{3.15}$$

and

$$c_{3,5}^{-1}\,r^{2\alpha_2} \leq \phi(r) \leq c_{3,5}\,r^{2\alpha_1}, \qquad \forall\,0 < r \leq 1. \tag{3.16}$$

Xiao (2005) shows that we can choose the sequence $\{b_n\}$ appropriately so that the following hold:

(i) $\phi(r) \asymp r^{2\alpha_2}$ for $r \in (0,1)$.

(ii) $\sigma^2(h) \asymp |h|^{2\alpha_2}$ for $|h| \leq 1$.

(iii) Condition (3.10) is not satisfied, but the corresponding Gaussian random field X still has the property of SLϕND.

So far, we have not considered Gaussian random fields with stationary increments and discrete spectral measures. A systematic treatment for such Gaussian random fields will be done elsewhere. In the following, we only give an example of stationary Gaussian processes with discrete spectrum that is strongly locally nondeterministic.

Example 3.9 Let $\{X_n, Y_n, n \geq 0\}$ be a sequence of independent standard normal random variables. Then for each $t \in \mathbb{R}$, the random Fourier series

$$Y(t) = \frac{\sqrt{8}}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n-1}\Big\{X_n\cos\big((2n-1)t\big) + Y_n\sin\big((2n-1)t\big)\Big\} \tag{3.17}$$

converges almost surely [see, e.g., Kahane (1985)], and $Y = \{Y(t),\ t \in \mathbb{R}\}$ is a centered, periodic and stationary Gaussian process with mean 0 and covariance function

$$R(s,t) = 1 - \frac{2}{\pi}|s-t| \quad \text{for} \quad -\pi \leq s-t \leq \pi. \tag{3.18}$$

It is easy to see that the spectrum measure Δ of Y is discrete with $\Delta(\{2n-1\}) = (2n-1)^{-2}$ for all $n \in \mathbb{N}$. Using a result in Berman (1978), it can be shown that for any interval $T \subset [-\pi,\pi]$ with length $|T| \leq \pi/2$ there exists a constant $0 < c_{3,6} < \infty$ such that for all $t \in T$ and all $0 < r \leq \min\{|t|, \pi/2\}$,

$$\mathrm{Var}\big(Y(t)\big|\,Y(s) : s \in T,\ |s-t| \geq r\big) \geq c_{3,6}\,r. \tag{3.19}$$

That is, Y is SLϕND on T with $\phi(r) = r$ and $\sigma^2(h) \asymp \phi(|h|)$; see Shieh and Xiao (2004) for a proof.

4 Applications of strong local nondeterminism

Many authors have applied LND or SLϕND to study various properties of Gaussian and stable random fields. We refer to Geman and Horowitz (1980), Geman et al. (1984), Berman (1991), Dozzi (2002), and the references therein for more information.

In the studies of Gaussian random fields with stationary increments, the variance function $\sigma^2(h)$ has played a significant role and it is typically assumed to be regularly varying at 0 and/or monotone increasing. See Marcus (1968b), Kôno (1970, 1996), Cuzick (1982b), Csörgő et al. (1995), Kasahara et al. (1999), Monrad and Rootzén (1995), Talagrand (1995, 1998), Xiao (1996, 1997a, b, 2003), and so on. Using the results in Section 3, we can prove that, in almost all cases, the regularly varying assumption on $\sigma^2(h)$ can be significantly weakened and the monotonicity assumption can be removed.

In the rest of this section, we show that SLϕND can be applied to extend the small ball probability estimates of Monrad and Rootzén (1995), Shao and Wang (1995) and Stoltz (1996), the results on the exact Hausdorff measure functions of Talagrand (1995) and Xiao (1996, 1997a, b), the Hölder conditions and tail probability of the local times of Xiao (1997a) and Kasahara et al. (1999), to more general Gaussian random fields. For proofs of these results, see Xiao (2005).

We will consider a Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \quad \forall t \in \mathbb{R}^N, \tag{4.1}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$, which satisfies the following Condition (C):

(C1) there exist positive constants δ_0, $c_{4,1}$, $c_{4,2}$ and a non-decreasing, right continuous function $\phi : [0, \delta_0) \to [0, \infty)$ such that

$$\frac{\phi(2r)}{\phi(r)} \le c_{4,1}, \qquad \forall\, r \in [0, \delta_0/2) \tag{4.2}$$

and for all $t \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$ with $|h| \le \delta_0$,

$$c_{4,2}^{-1}\phi(|h|) \le \mathbb{E}\big[\big(Y(t+h) - Y(t)\big)^2\big] \le c_{4,2}\phi(|h|). \tag{4.3}$$

(C2) Y is strongly locally ϕ-nondeterministic on an interval $T \in \mathcal{A}$, say, $T = [0,1]^N$.

4.1 Small ball probability and Chung's law of the iterated logarithm

In recent years, there has been much interest in studying the small ball probability of Gaussian processes. We refer to Li and Shao (2001) and Lifshits (1999) for extensive surveys on small ball probabilities, their applications and open problems.

The next theorem gives estimates on the small ball probability of Gaussian random fields satisfying the condition (C). In particular, the upper bound in (4.4) confirms a conjecture of Shao and Wang (1995), under a much weaker condition.

Theorem 4.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ satisfying the condition (C). Then there exist positive constants $c_{4,3}$ and $c_{4,4}$ such that for all $x \in (0,1)$,*

$$\exp\Big(-\frac{c_{4,3}}{[\phi^{-1}(x^2)]^N}\Big) \le \mathbb{P}\Big\{\max_{t\in[0,1]^N}|X(t)| \le x\Big\} \le \exp\Big(-\frac{c_{4,4}}{[\phi^{-1}(x^2)]^N}\Big), \quad (4.4)$$

where $\phi^{-1}(x) = \inf\{y : \phi(y) > x\}$ is the right-continuous inverse function of ϕ.

The lower bound in (4.4) follows from a general result of Talagrand (1993) and the upper bound is proved in a way similar to that of Monrad and Rootzén (1995). This is where SLϕND of X is applied.

The probability estimate in Theorem 4.1 has many applications. We mention the following Chung's law of the iterated logarithm. When σ is assumed to be regularly varying at 0, this is also obtained in Xiao (1997a).

Corollary 4.2 *If, in addition to the conditions of Theorem 4.1, we assume that X has stationary increments and the spectral measure Δ of X satisfies $\liminf_{\lambda\to\infty}|\lambda|^{N+2}\Delta\big(B(\lambda,r)\big) > 0$, where $B(\lambda,r) = \{x \in \mathbb{R}^N : |x-\lambda| \le r\}$. Then there exists a positive and finite constant $c_{4,5}$ such that*

$$\liminf_{h\to 0}\frac{\sup_{t\in[0,h]^N}|X(t)|}{\phi^{1/2}\big(h/(\log\log(1/h))^{1/N}\big)} = c_{4,5} \qquad a.s. \quad (4.5)$$

The proof of Corollary 4.2 contains two steps: first we apply Theorem 4.1 and slightly modify the proof of Theorem 7.1 in Li and Shao (2001) to show the above liminf is bounded from below and above by positive constants, then we apply the zero-one law of Pitt and Tran (1979) to derive (4.5).

We can also consider the small ball probability of Gaussian random fields under the Hölder-type norm. Let κ be a continuous and non-decreasing function such that $\kappa(r) > 0$ for all $r > 0$. For any function $y \in C_0([0,1]^N)$, we consider the functional

$$\|y\|_\kappa = \sup_{s,t\in[0,1]^N, s\ne t}\frac{|y(s)-y(t)|}{\kappa(|s-t|)}. \quad (4.6)$$

When $\kappa(r) = r^\alpha$, $\|\cdot\|_\kappa$ is the α-Hölder norm on $C_0([0,1]^N)$ and is denoted by $\|\cdot\|_\alpha$.

The following theorem uses SLϕND to improve the results of Stolz (1996). We mention that the conditions of Theorem 2.1 of Kuelbs, Li and Shao (1995) can be weakened in a similar way.

Theorem 4.3 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ satisfying the condition (C). If for some constant $\beta > 0$,*

$$\frac{\phi^{1/2}(r)}{\kappa(r)} \asymp r^{\beta}, \quad \forall r \in (0,1). \tag{4.7}$$

Then there exist positive constants $c_{4,6}$ and $c_{4,7}$ such that for all $\varepsilon \in (0,1)$,

$$\exp\Big(-c_{4,6}\,\varepsilon^{-N/\beta}\Big) \le \mathbb{P}\Big\{\|X\|_{\kappa} \le \varepsilon\Big\} \le \exp\Big(-c_{4,7}\,\varepsilon^{-N/\beta}\Big). \tag{4.8}$$

4.2 Hausdorff dimension and Hausdorff measure of the sample paths

In this section we consider the fractal properties of the range and graph of the Gaussian random field in $\mathbb{R}^d$ defined by (4.1) and fractional Brownian sheets.

When $X = \{X(t), t \in \mathbb{R}^N\}$ is a fractional Brownian motion in $\mathbb{R}^d$, the exact Hausdorff measure functions for the image $X([0,1]^N)$ and graph $\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\}$ were determined by Talagrand (1995) and Xiao (1997c). Their results were extended by Xiao (1996, 1997a) to strongly locally nondeterministic Gaussian random fields with stationary increments and regularly varying variance function $\sigma^2(h)$. Using the results in Section 3 we can prove the following more general result.

Theorem 4.4 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field defined in (4.1). We assume that Condition (C) is satisfied and, in addition, there exists a constant $c_{4,8} > 0$ such that*

$$\int_1^{\infty} \Big(\frac{\phi(a)}{\phi(ax)}\Big)^{d/2} x^{N-1} dx \le c_{4,8} \quad \textit{for all} \;\; a \in (0,1), \tag{4.9}$$

then

$$0 < \varphi_1\text{-}m\big(X([0,1]^N)\big) < \infty \quad a.s., \tag{4.10}$$

where $\varphi_1(r) = \big[\phi^{-1}(r^2)\big]^N \log\log 1/r$.

Remark 4.5 Note that condition (4.9) suggests that X is transient and does not hit points. When X hits points, the Hausdorff dimension of $\mathrm{Gr}X([0,1]^N)$ may be bigger. The results on the exact Hausdorff measure of the graph set $\mathrm{Gr}X([0,1]^N)$ in Xiao (1997a, c) can be extended to Gaussian random fields satisfying Condition (C) in a similar way. □

Remark 4.6 More generally, Kahane (1985) has studied geometric and arithmetic properties of the image $X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}^N$ when X is an (N,d)-fractional Brownian motion. His results have recently be extended and improved by Shieh and Xiao (2004). □

Now we give a brief discussion of the relevance of sectorial LND to the study of fractal properties of fractional Brownian sheets. Further information can be found in Ayache, Wu and Xiao (2005), Khoshnevisan, Wu and Xiao (2005), Wu and Xiao (2005).

Recently, Ayache and Xiao (2004) have obtained the Hausdorff and packing dimensions of the range $B^{\vec{H}}([0,1]^N)$, graph $\mathrm{Gr}B^{\vec{H}}([0,1]^N)$ and the level set for a general (N,d)-fractional Brownian sheet $B^{\vec{H}}$. It would be interesting to find the exact Hausdorff and packing measure functions for these random sets (if they exist). The existing methods for the Brownian sheet [cf. Ehm (1981)] or fractional Brownian motion [cf. Talagrand (1995), Xiao (1997c)] can not be applied directly. I believe that the sectorial local nondeterminism of $B^{\vec{H}}$ will be useful in solving these problems.

Wu and Xiao (2005) have applied the sectorial local nondeterminism of $B^{\vec{H}}$ to study geometric properties of the image set $B^{\vec{H}}(E)$, where $E \subset (0,\infty)^N$ is an arbitrary Borel set, of fractional Brownian sheets. In particular, they have proved the following "uniform" Hausdorff and packing dimension result.

Theorem 4.7 *Let $H \in (0,1)$ be a constant and let $B^{\vec{H}} = \{B^{\vec{H}}(t), t \in \mathbb{R}^N_+\}$ be an (N,d)-fractional Brownian sheet with index $\vec{H} = \langle H \rangle$. If $N \le Hd$, then almost surely*

$$\dim_{\mathrm{H}} B^H(E) = \frac{1}{H} \dim_{\mathrm{H}} E \quad \textit{for all Borel sets} \;\; E \subset (0,\infty)^N. \tag{4.11}$$

and

$$\dim_{\mathrm{P}} B^H(E) = \frac{1}{H} \dim_{\mathrm{P}} E \quad \textit{for all Borel sets} \;\; E \subset (0,\infty)^N. \tag{4.12}$$

Note that when $\vec{H} = \langle \frac{1}{2} \rangle$, (4.11) of Theorem 4.7 recovers the result for the (N,d)-Brownian sheet $W = \{W(t), t \in \mathbb{R}^N_+\}$ proved by Mountford (1989) and Lin (1999); see also Khoshnevisan, Wu and Xiao (2005) for a different, relatively more elementary proof. Both the proofs of Mountford (1989) and Lin (1999) are quite involved and their arguments rely on the special properties of the Brownian sheet such as the independence of the increments, which can not be applied to fractional Brownian sheets. Our proof of Theorem 4.7 is, similar to that in Khoshnevisan, Wu and Xiao (2005), based on the sectorial local nondeterminism of fractional Brownian sheets.

Finally we mention that when $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$ and $H_1, \ldots, H_N$ are different, the Hausdorff and packing dimension of E alone is not enough for determining $\dim_{\mathrm{H}} B^{\vec{H}}(E)$ or $\dim_{\mathrm{P}} B^{\vec{H}}(E)$. Explicit formulas for $\dim_{\mathrm{H}} B^{\vec{H}}(E)$ and $\dim_{\mathrm{P}} B^{\vec{H}}(E)$ are not known.

4.3 Local times and level sets of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with stationary increments in $\mathbb{R}^d$ defined by (4.1). If the real-valued random field Y satisfies Condition (C)

and for some $\varepsilon > 0$,

$$\int_{[0,1]^N} \frac{dh}{\sigma^{d+\varepsilon}(h)} < \infty, \tag{4.13}$$

then it follows from Theorem 26.1 in Geman and Horowitz (1980) [see also Berman (1973) and Pitt (1978)] that X has a jointly continuous local time $L(x,t)$ for $(x,t) \in \mathbb{R}^d \times T$ which satisfies certain Hölder conditions in the time and space variables, respectively.

When X is strongly locally nondeterministic and satisfies certain regularity assumptions, Xiao (1997a) has established sharp local and uniform Hölder conditions for the local time $L(x,t)$ in the time variable t. Besides of their own interest, these Hölder conditions are also useful in studying the fractal properties of the sample paths of X. In the following, we show that the results in Xiao (1997a) and Kasahara et al. (1999) still hold under Condition (C). For simplicity, we will only consider the case $N = 1$.

Theorem 4.8 *Let $X = \{X(t), t \in \mathbb{R}\}$ be a centered Gaussian process in $\mathbb{R}^d$ defined by (4.1). We assume that the associated Gaussian process Y satisfies Condition (C) and there exist constants $0 < \gamma_0 < 1$ and $c_{4,9} > 0$ such that*

$$\int_0^1 \Big(\frac{\phi(a)}{\phi(as)}\Big)^{\frac{d}{2}+\gamma_0} ds \le c_{4,9} \qquad \text{for all} \quad a \in (0, \delta_0). \tag{4.14}$$

For any $B \in \mathcal{B}(\mathbb{R})$ define $L^*(B) = \sup_{x\in\mathbb{R}^d} L(x,B)$. Then there exists a positive and finite constant $c_{4,10}$ such that for all $t \in \mathbb{R}$,

$$\limsup_{r\to 0} \frac{L^*(B(t,r))}{\varphi_2(r)} \le c_{4,10} \qquad a.s. \tag{4.15}$$

and for all intervals $T \subseteq \mathbb{R}$, there exists a positive and finite constant $c_{4,11}$ such that

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B(t,r))}{\varphi_3(r)} \le c_{4,11} \qquad a.s., \tag{4.16}$$

where $B(t,r) = (t-r, t+r)$,

$$\varphi_2(r) = \frac{r}{\phi(r(\log\log 1/r)^{-1})^{d/2}} \qquad \text{and} \qquad \varphi_3(r) = \frac{r}{\phi(r(\log 1/r)^{-1})^{d/2}}.$$

Remark 4.9 If X has stationary increments and its spectral measure satisfies (3.10) and $1 > \overline{\alpha}\, d$. Then Lemma 3.2 implies that (4.14) is satisfied for any $\gamma_0 \in (0, (1-\overline{\alpha}d)/(2\overline{\alpha}))$. □

Similar to Xiao (1997a), the proof of Theorem 4.8 is based on the moment estimates for $L(x,B)$ and $L(x+y,B) - L(x,B)$ and a chaining argument. The following lemma provides the key estimates, whose proofs rely on SLϕND of X.

Lemma 4.10 *Under the conditions of Theorem 4.8, there exist positive and finite constants $c_{4,12}$ and $c_{4,13}$ such that for all integers $n \geq 1$, $r > 0$ small, $x \in \mathbb{R}^d$ and $0 < \gamma < \gamma_0$, we have*

$$\mathbb{E}\big[L(x,r)^n\big] \leq \frac{c_{4,12}^n\, r^n}{\phi(r/n)^{nd/2}} \tag{4.17}$$

and

$$\mathbb{E}\big[L(x+y,r) - L(x,r)\big]^n \leq \frac{c_{4,13}^n |y|^{n\gamma}\, r^n}{\big[\sigma(r/n)\big]^{(d+2\gamma)n/2}}\ (n!)^\gamma. \tag{4.18}$$

Theorem 4.8 can be applied to determine the Hausdorff dimension and exact Hausdorff measure of the level set $X^{-1}(x) = \{t \in \mathbb{R} : X(t) = x\}$, where $x \in \mathbb{R}^d$. For example, the Hausdorff dimension of $X^{-1}(x)$ has been studied by Berman (1970, 1972), Adler (1981), Monrad and Pitt (1987) for index α-Gaussian processes; and the exact Hausdorff measure of $X^{-1}(x)$ has been studied by Xiao (1997a) for a class of strongly locally nondeterministic Gaussian random fields with stationary increments.

By applying (4.16) of Theorem 4.8, Xiao (2005) has proved the following uniform Hausdorff dimension result for the level sets of the Gaussian process X, extending the previous results of Berman (1972), Monrad and Pitt (1987).

Theorem 4.11 *Let $X = \{X(t), t \in \mathbb{R}\}$ be a Gaussian process in $\mathbb{R}^d$ with stationary increments defined by (4.1). We further assume that Y satisfies the assumptions of Theorem 3.4. Then with probability one,*

$$\dim_{\mathrm{H}} X^{-1}(x) = 1 - \alpha^* d \quad \textit{for all} \ \ x \in \mathcal{O}, \tag{4.19}$$

where α^* is the upper index of σ defined by

$$\alpha^* = \inf\Big\{\gamma \geq 0 : \lim_{h\to 0} \frac{\sigma(h)}{|h|^\gamma} = \infty\Big\}$$

with the convention $\inf \emptyset = \infty$*, and where $\mathcal{O}$ is the (random) open set defined by*

$$\mathcal{O} = \bigcup_{s,t\in\mathbb{Q};\, s<t} \Big\{x \in \mathbb{R}^d :\ L(x,[s,t]) > 0\Big\}.$$

.

Remark 4.12 It is an interesting question to characterize the random open set $\mathcal{O}$. Monrad and Pitt (1987) have given a real-valued *periodic* stationary Gaussian process X for which $\mathcal{O}$ is a proper subset of $\mathbb{R}$ [because the range of X is a.s. bounded]. They have shown a sufficient condition in terms of the spectral measure of a stationary (N,d)-Gaussian random field X so that $\mathcal{O} = \mathbb{R}^d$ holds. Monrad and Pitt (1987) also point out that the self-similarity of an (N,d)-fractional Brownian motion B_α implies that if $N > \alpha d$ then $\mathcal{O} = \mathbb{R}^d$ almost surely. However, we do not know whether $\mathcal{O} = \mathbb{R}^d$ is true for the (N,d)-Gaussian random fields satisfying the conditions of Theorem 3.4. □

The local time $L(0,1)$ [i.e., $L(x,1)$ at $x=0$] of a Gaussian process X sometimes appears as limit in some limit theorems on the occupation measure of X; see, for example, Kôno (1996) and Kasahara and Ogawa (1999). Since there is little knowledge on the explicit distribution of $L(0,1)$, it is of interest in estimating the tail probability $\mathbb{P}\{L(0,1) > x\}$ as $x \to \infty$. This problem has been considered by Kasahara et al. (1999) under some quite restrictive conditions on the Gaussian process X. The next theorem is an extension of the main result in Kasahara et al. (1999).

Theorem 4.13 *Let $X = \{X(t), t \in \mathbb{R}\}$ be a centered Gaussian process in $\mathbb{R}^d$ defined by (4.1). We assume that the associated process Y satisfies Condition (C) and the condition (4.14) with $\gamma_0 = 0$. Then*

$$-\log \mathbb{P}\big\{L(0,1) > x\big\} \asymp \frac{1}{\phi^{-1}(1/x^2)}, \tag{4.20}$$

where ϕ^{-1} is the inverse function of ϕ as defined in Theorem 4.1.

Theorem 4.13 follows from the moment estimates for $L(0,1)$ in Lemma 4.15 and the following lemma on the tail probability of nonnegative random variables. When ψ is a power function or a regularly varying function, Lemma 4.14 is well known.

Lemma 4.14 *Let ξ be a non-negative random variable and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function having the doubling property. If there exist positive constants $c_{4,14}$ and $c_{4,15}$ such that*

$$c_{4,14}^n\, \psi(n)^n \le \mathbb{E}(\xi^n) \le c_{4,15}^n\, \psi(n)^n$$

for all n large enough, then there exist positive constants $c_{4,16} > c_{4,15}$, $c_{4,17}$ and $c_{4,18}$ such that for all $x > 0$ large enough,

$$e^{-c_{4,17}x} \le \mathbb{P}\big\{\xi \ge c_{4,16}\, \psi(x)\big\} \le e^{-c_{4,18}x}. \tag{4.21}$$

Lemma 4.15 *Under the assumptions of Theorem 4.13, there exist positive and finite constants $c_{4,19}$ and $c_{4,20}$ such that for all integers $n \ge 1$,*

$$\frac{c_{4,19}^n}{\phi(1/n)^{nd/2}} \le \mathbb{E}\big[L(0,1)^n\big] \le \frac{c_{4,20}^n}{\phi(1/n)^{nd/2}}. \tag{4.22}$$

One may also consider the existence and continuity of the local times $L(x,E)$ of an (N,d)-Gaussian random field on any Borel set $E \subset \mathbb{R}^N$. These problems are closely related to the questions whether the image $X(E)$ has positive Lebesgue measure and/or interior points; see Pitt (1978), Kahane (1985), Shieh and Xiao (2004), Khoshnevisan and Xiao (2004b). Strong local nondeterminism and sectorial local nondeterminism have proven to be very useful for solving these problems. On the other hand, similar to Theorem 4.13, the distribution of $L(0,E)$ can be studied for a large class of fractal sets, say d-sets.

We end this section with the following remarks and open questions.

Remark 4.16 The problem of establishing sharp uniform and local Hölder conditions for the self-intersections local times of Gaussian random fields satisfying Condition (C) remains to be open. For background and some related results, see Berman (1991). □

Remark 4.17 Using sectorial local nondeterminism, Ayache, Wu and Xiao (2005) have established joint continuity and sharp Hölder conditions for the local times of a fractional Brownian sheet $B^{\vec{H}}$. Their results suggest some interesting questions for general anisotropic Gaussian random fields that may be further investigated. □

Question 4.18 We know that (4.13) is sufficient for the existence of a jointly continuous local time of locally nondeterministic Gaussian random field X. However, this condition is not necessary. When X is an (N, d)-Gaussian random field with stationary increments, is it possible to provide a necessary and sufficient condition for the joint continuity of $L(x, t)$ in terms of ϕ?

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