Random Fractals and Markov Processes

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Abstract. This is a survey on the sample path properties of Markov processes, especially fractal properties of the random sets and measures determined by their sample paths. The class of Markov processes considered in this paper include Lévy processes in $\mathbb{R}^d$, diffusions on fractals and on $\mathbb{R}^d$, Feller processes determined by pseudo-differential operators and so on. We summarize recent results for Lévy processes such as the Hausdorff and packing dimensions of their ranges, level sets, and multiple points; regularity properties of local times and self-intersection local times; multifractal analysis of the occupation measures and sample paths. Our emphasis is on general Markovian techniques that are applicable to other Markov processes.

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1. Introduction

The study of sample path properties of Brownian motion, and more generally of stable Lévy processes on $\mathbb{R}^d$ has been one of the most interesting subjects in probability theory. Hausdorff dimension and Hausdorff measure have been very useful tools for such studies since the pioneering work of Lévy (1953) and Taylor (1953, 1955, 1967). There have been several excellent comprehensive survey papers on sample path properties of Lévy processes, e.g., Taylor (1973), Fristedt (1974), Pruitt (1975), Taylor (1986a), as well as the books of Bertoin (1996, 1999) and Sato (1999), from which I have benefited greatly.

The birth of fractal geometry, due in great measure to the work of Benoit Mandelbrot, has brought many new ideas and new geometric tools [such as packing dimension and packing dimension profiles, average densities, multifractals] into the studies of fine properties of stochastic processes. In the past decade, not only many new delicate results have been discovered for Brownian motion and stable Lévy processes [see Lawler (1999) for a nice survey on the fractal properties of Brownian motion], but there has also been tremendous interest in studying other Markov processes such as diffusions on fractals, and Feller processes related to pseudo-differential operators; see the monographs of Barlow (1998), Jacob (1996) and the references therein for more information.

The object of this paper is to give an expository account of fractal properties of Markov processes. In the historical development of the studies of sample path properties of Markov processes, results have usually been obtained for Brownian motion first, then for symmetric stable processes of index α $(0 < \alpha < 2)$, and then for general Lévy processes or Markov processes. At each stage of generalization, some special properties of the processes are used. Due to its importance in the general theory on Markov processes, in most parts of this paper, we will concentrate on recent results for the sample paths of Lévy processes, with an emphasis on methods that are applicable to more general Markov processes. Whenever possible, we will give a unified treatment for several classes of Markov processes.

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Markov process with values in a metric space (S, ρ). Throughout this paper, we are interested in the sample path properties of X, that is, properties of the function $X(t) = X(t, \omega)$ for fixed $\omega \in \Omega$. When we say that sample paths of the process X have property P almost surely (with positive probability, resp.), we mean that the set $\{\omega \in \Omega : X(\cdot, \omega) \text{ has property } P\}$ is an event and has probability 1 (positive probability, respectively). The following are some examples of random sets generated by X:

Range (Image): $X([0,1]) = \big\{x \in S : x = X(t) \text{ for some } t \in [0,1]\big\}$;
Graph set: $\mathrm{Gr}X([0,1]) = \big\{(t, X(t)) \in [0,1] \times S : t \in [0,1]\big\}$;
Level set: $X^{-1}(x) = \big\{t \in \mathbb{R}_+ : X(t) = x\big\}$, $\quad x \in S$; or more generally,
Inverse image: $X^{-1}(F) = \big\{t \in \mathbb{R}_+ : X(t) \in F\big\}$, where $F \subset S$.

This paper is organized as follows. In Section 2, we collect several classes of Markov processes whose sample path properties will be discussed. We recall their definitions and some basic properties that will be used in the sequel.

In Section 3, we recall the definitions and properties of various tools from fractal geometry; these include Hausdorff measure and dimension, packing measure and dimension, packing dimension profile, capacity, average densities and multifractals. While most of the materials can be found in the books of Falconer (1990, 1997), Mattila (1995), some of them such as packing dimension profiles and multifractal spectrum for functions are more recent, see Falconer and Howroyd (1997) and Jaffard (1999, 2001).

Section 4 studies the Hausdorff dimension and packing dimension of the range of Markov processes. We prove some general formulae for $\dim_{\mathrm{H}} X([0,1])$, $\dim_{\mathrm{P}} X([0,1])$ and $\dim_{\mathrm{H}} X(E)$ in terms of the transition function of X, which extend the well-known results of Pruitt (1969), Taylor (1986b) and so on.

Section 5 is about the exact Hausdorff measure and packing measure of the range $X([0,1])$. Some useful techniques for evaluating Hausdorff and packing measures are discussed.

Sections 6 and 7 concern the fractal properties of level sets and inverse images. Potential theory plays an important role in this section. The existence and regularity of local times are also discussed.

Section 8 concerns the uniform Hausdorff and packing dimension results for the range and inverse image of a Markov process. These results [when they exist] are stronger than those described in Sections 4–7 and they can be applied to derive fractal dimension or measure results involving random index sets. For example, the Hausdorff dimension of the set of multiple points or collision points of a Markov process can be obtained from the Hausdorff dimension of the level set of certain related processes.

Section 9 is on the existence of multiple points of a Markov process or the intersections of independent Markov processes, and on the fractal properties of the set of multiple points and multiple times when they are not empty. Several different approaches for the intersection problem are discussed.

Some exact capacity estimates for the range and the inverse image are given in Section 10. Capacities are also natural tools in studying self-intersections of $X(t)$ when t is restricted to compact sets.

Section 11 summarizes recent results on average densities and tangent measure distributions of the occupation measures of Brownian motion.

Finally, Section 12 discusses the multifractal structure of the sample paths of X as well as the random measures induced by X, where X is either a Brownian motion or a more general Lévy process. Limsup type random fractals play important roles in these studies.

Throughout this paper, we will use K to denote unspecified positive and finite constants which may differ from line to line. Some specific constants are denoted by $K_1, K_2, \ldots$. The Euclidean metric and the ordinary scalar product in $\mathbb{R}^d$ are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. The Lebesgue measure in $\mathbb{R}^d$ is denoted by λ_d. Given two functions g and h on $\mathbb{R}^d$, $g \asymp h$ means that there exists a positive and finite constant $K \geq 1$ such that $K^{-1}h(x) \leq g(x) \leq Kh(x)$ for all $x \in \mathbb{R}^d$. We use $A \hat{=} B$ to indicate that A is defined by B.

2. Markov processes

In this section, we first briefly recall the definition of some basic notions about Markov processes and related properties. Then we will describe several classes of Markov processes whose sample functions will be studied later in the paper. For the general theory of Markov processes, we refer to Blumenthal and Getoor (1968), Sharpe (1988) and Khoshnevisan (2002).

Let (S, ρ) be a locally compact separable complete metric space with Borel σ-algebra $\mathcal{S}$. We assume that there is a Radon measure μ on S which plays the role of a reference measure. Recall that a Radon measure is a σ-finite Borel regular measure on $(S, \mathcal{S})$ which is finite on compact sets.

A family of functions $\{P_{s,t}(x, A) : 0 \le s < t\}$, where $P_{s,t}(x, A) : S \times \mathcal{S} \to \mathbb{R}_+$, is called a *transition function system* on S if the following conditions are satisfied:

(i) for all $0 \le s < t$ and for each fixed $x \in S$, $P_{s,t}(x, \cdot)$ is a probability measure on $(S, \mathcal{S})$;

(ii) for all $0 \le s < t$ and for all $A \in \mathcal{S}$, $P_{s,t}(x, A)$ is a measurable function of x;

(iii) for all $0 \le s < t < u$, $x \in S$ and all $A \in \mathcal{S}$,

$$P_{s,u}(x, A) = \int_S P_{s,t}(x, dy) P_{t,u}(y, A). \tag{2.1}$$

The relationship (2.1) is the *Chapman–Kolmogorov equation.* If, in addition, the transition function $P_{s,t}(x, A)$ satisfies

$$P_{s,t}(x, A) = P_{s,t}(0, A - x)$$

for all $0 \le s < t < u$, $x \in S$ and all $A \in \mathcal{S}$, then we say that the transition function $P_{s,t}(x, A)$ is *spatially homogeneous* (or translation invariant). In the above, $A - x = \{y - x : y \in A\}$.

A transition function $P_{s,t}(x, A)$ is said to be *temporally homogeneous* if there exists a family of functions $\{P_t(x, A), t > 0\}$ such that $P_{s,t}(x, A) = P_{t-s}(x, A)$ for all $0 \le s < t$. In this case, the Chapman–Kolmogorov equation can be written as

$$P_{s+t}(x, A) = \int_S P_s(x, dy) P_t(y, A). \tag{2.2}$$

Later, we will also write $P(t, x, A)$ for $P_t(x, A)$.

A stochastic process $X = \{X(t), \mathcal{M}, \mathcal{M}_t, \theta_t, \mathbb{P}^x\}$ with values in $(S, \mathcal{S})$ is called a Markov process with respect to a filtration $\{\mathcal{M}_t : t \ge 0\}$ [i.e., $\mathcal{M}_t$ is a σ-algebra for each $t \ge 0$ and $\mathcal{M}_s \subseteq \mathcal{M}_t$ for all $0 \le s < t$] having $P_{s,t}(x, A)$ as transition function provided

(i) X is adapted to $\{\mathcal{M}_t\}$, i.e., $X(t)$ is measurable with respect to $\mathcal{M}_t$ for all $t \ge 0$;

(ii) for all $0 \le s < t$ and all bounded measurable function f on $(S, \mathcal{S})$,

$$\mathbb{E}\big\{f(X(t)) \mid \mathcal{M}_s\big\} = P_{s,t} f(X(s)), \tag{2.3}$$

where

$$P_{s,t} f(x) = \int_S f(y) P_{s,t}(x, dy).$$

By letting $f = \mathbb{1}_A$ and taking conditional expectation with respect to $X(s)$ in (2.3), we see that $P_{s,t}(x, \cdot)$ is the conditional distribution of $X(t)$, given $X(s) = x$.

If the transition function $P_{s,t}(x, A)$ is temporally homogeneous, then X is called a temporally homogeneous Markov process with respect to $\{\mathcal{M}_t\}$. When $\mathcal{M}_t \hat{=} \sigma\{X(s) : s \le t\}$ for all $t \ge 0$, we will write $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in S\}$ or simply $X = \{X(t), t \in \mathbb{R}_+\}$.

We assume throughout that X is a Hunt process so that its sample functions $X(\cdot, \omega)$ are right continuous and have finite left limit [or cadlag], the augmented filtration $\{\mathcal{F}_t, t \ge 0\}$ is right continuous [i.e., $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$] and that X

has the strong Markov property. Except in Section 2.2, we will always consider temporally homogeneous Markov processes.

Let $\mathcal{B}_b(S)$ be the space of all bounded measurable functions from S to $\mathbb{R}$. Corresponding to the (temporally homogeneous) transition function $P_t(x, A)$, we define the transition operator T_t on $\mathcal{B}_b(S)$ by

$$T_t f(x) = \int_S f(y) P_t(x, dy) \quad \text{for } t > 0 \text{ and } f \in \mathcal{B}_b(S)$$

and $T_0 f(x) = f(x)$. Then the Chapman–Kolmogorov equation (2.2) implies that $\{T_t, t \geq 0\}$ is a semigroup of bounded linear operators on $\mathcal{B}_b(S)$, i.e., $T_s \circ T_t = T_{s+t}$ for all $s, t \geq 0$.

We say that a Markov process X is *symmetric* if for all f, $g \in \mathcal{C}_c(S)$,

$$\int_S f(x) T_t g(x) \mu(dx) = \int_S g(x) T_t f(x) \mu(dx), \tag{2.4}$$

where $\mathcal{C}_c(S)$ denotes the space of all continuous functions on S with compact support.

Now, for simplicity, we assume $S \subseteq \mathbb{R}^d$ and let $\mathcal{C}_0(\mathbb{R}^d)$ be the Banach space of continuous functions on $\mathbb{R}^d$ that tend to 0 at infinity, equipped with the uniform norm $\|\cdot\|$.

Definition 2.1. A Markov process $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in S\}$ is called a *Feller process* if $\{T_t, t \geq 0\}$ is a Feller semigroup, i.e., for every $f \in \mathcal{C}_0(\mathbb{R}^d)$,

(i) $T_t f \in \mathcal{C}_0(\mathbb{R}^d)$ for every $t \geq 0$;
(ii) $\lim_{t \to 0} \|T_t f - f\| = 0$.

Furthermore, if for every $t \geq 0$, T_t maps $\mathcal{B}_b(S)$ into $\mathcal{C}_0(\mathbb{R}^d)$, then $\{T_t, t \geq 0\}$ is called a strong Feller semigroup and the process X is called a *strong Feller process.*

The *infinitesimal generator* A of the semigroup $\{T_t, t \geq 0\}$ is defined by

$$Au = \lim_{t \to 0} \frac{T_t u - u}{t}, \quad \forall\, u \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) = \Big\{ u \in \mathcal{C}_0(\mathbb{R}^d) : \lim_{t \to 0} \frac{T_t u - u}{t} \text{ exists in } \|\cdot\| \Big\}$$

is called the *domain* of A. The operator $(A, \mathcal{D}(A))$ is a densely defined closed operator on $\mathcal{C}_0(\mathbb{R}^d)$ and determines $\{T_t, t \geq 0\}$ uniquely.

When the transition function $P_t(x, \cdot)$ is absolutely continuous with respective to the measure μ on $(S, \mathcal{S})$, X has a transition density which will be denoted by $p_t(x, y)$. Hence for all $x \in S$ and $A \in \mathcal{S}$,

$$P_t(x, A) = \int_A p_t(x, y) d\mu(y).$$

A sufficient condition for the existence of a transition density is that X has the strong Feller property; see Hawkes (1979, Lemma 2.1).

Next we recall the definition of self-similar Markov processes, which was first introduced and studied by Lamperti (1972) for Markov processes on $[0, \infty)$ under the name "semi-stable". Later, Graversen and Vuolle–Apiala (1986) considered self-similar Markov processes on $\mathbb{R}^d$ or $\mathbb{R}^d \setminus \{0\}$ and investigated the connections between the multi-dimensional self-similar Markov processes and Lévy processes.

We assume that $(S, \mathcal{S})$ is $\mathbb{R}^d$, $\mathbb{R}^d\backslash\{0\}$ or $\mathbb{R}^d_+$ with the usual Borel σ-algebra. Let Δ be a point attached to S as an isolated point and let $H > 0$ be a given constant. A temporally homogeneous Markov process $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in S\}$ with state space $S \cup \{\Delta\}$ is called an *H-self-similar Markov process* if its transition function $P(t, x, A)$ satisfies:

$$P(0, x, A) = \mathbb{1}_A(x) \quad \text{for all} \quad x \in S,\ A \in \mathcal{S} \tag{2.5}$$

and

$$P(t, x, A) = P(at, a^H x, a^H A) \ \text{ for all } \ t > 0,\ a > 0,\ x \in S,\ A \in \mathcal{S}, \tag{2.6}$$

where for any $c \in \mathbb{R}$, $cA \hat{=} \{cx : x \in A\}$. The constant H is called the *self-similarity index* of X. The conditions (2.5) and (2.6) are equivalent to

$$\big\{X(t),\, t \in \mathbb{R}_+,\, \mathbb{P}^x,\, x \in S\big\} \stackrel{d}{=} \big\{a^{-H} X(at),\, t \in \mathbb{R}_+,\, \mathbb{P}^{a^H x},\, x \in S\big\}, \tag{2.7}$$

where $X \stackrel{d}{=} Y$ denotes that the two processes X and Y have the same finite dimensional distributions. If (2.6) only holds for some constant $a > 1$, then X is called *semi-self-similar with index H*. Such a constant $a > 1$ is called an *epoch* of the process X [cf. Sato (1999, p.74)] and it is often useful in proving limiting theorems for X. See, e.g., Fukushima et al. (1999), Bass and Kumagai (2000), Wu and Xiao (2002b).

In Sections 2.1 and 2.2 below, we will discuss several important classes of self-similar Markov processes including Brownian motion, strictly stable Lévy processes and processes of *Class L*. More examples of self-similar Markov processes and related references can be found in Xiao (1998), Liu and Xiao (1998). Examples of semi-self-similar Markov processes include Brownian motion on nested fractals, see Section 2.5.

2.1. Lévy processes. Lévy processes form a very important class of Markov processes. Besides Brownian motion, there has been tremendous interest in studying general Lévy processes, both in theory and in applications. For more information, we refer to the recent books of Bertoin (1996) and Sato (1999) for the general theory and to Bertoin (1999) for the study of the subordinators. Moreover, many properties of more general Markov processes can be obtained by comparing them with appropriate Lévy processes. See, for example, Schilling (1996, 1998a, b).

A stochastic process $X = \{X(t),\ t \geq 0\}$ on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$, with values in $\mathbb{R}^d$, is called a *Lévy process*, if for every $s, t \geq 0$, the increment $X(t + s) - X(t)$ is independent of the process $\{X(r),\, 0 \leq r \leq t\}$ and has the same distribution as $X(s)$ [i.e., X has stationary and independent increments], and such that $t \mapsto X(t)$ is continuous in probability. In particular, $\mathbb{P}\{X(0) = 0\} = 1$.

For every $x \in \mathbb{R}^d$, the law of the process $x + X$ under $\mathbb{P}$ is denoted by $\mathbb{P}^x$. We will write indifferently $\mathbb{P}$ or $\mathbb{P}^0$. Note that $\mathbb{P}^x\{X(0) = x\} = 1$. That is, under $\mathbb{P}^x$, the Lévy process X starts from x.

Under $\mathbb{P}$, the finite dimensional distributions of a Lévy process X are completely determined by the distribution of $X(1)$. It is well-known that the class of possible distributions for $X(1)$ is precisely the class of infinitely divisible laws. This implies that for every $t > 0$ the characteristic function of $X(t)$ is given by

$$\mathbb{E}\big[e^{i\langle \xi, X(t)\rangle}\big] = e^{-t\psi(\xi)},$$

where, by the Lévy–Khintchine formula,

$$\psi(\xi) = i\langle \mathsf{a}, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma\xi'\rangle + \int_{\mathbb{R}^d} \Big[1 - e^{i\langle x,\xi\rangle} + \frac{i\langle x,\xi\rangle}{1+|x|^2}\Big]\, \mathsf{L}(dx), \qquad \forall \xi \in \mathbb{R}^d, \tag{2.8}$$

and $\mathsf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and L is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2}\, \mathsf{L}(dx) < \infty.$$

The function ψ is called the *Lévy exponent* of X, and L is the corresponding *Lévy measure*. There are several different characterizations for the exponent ψ. Note that $\psi(0) = 0$ and that by Bochner's theorem the function $\xi \mapsto e^{-t\psi(\xi)}$ is continuous and *positive definite* for each $t \geq 0$ since it is the Fourier transform of a probability measure. Hence, by a theorem of Schoenberg (1938) [see also Theorems 7.8 and 8.4 in Berg and Frost (1975)], the Lévy exponent ψ is a continuous *negative definite function.* Such functions appeared in Schoenberg (1938) in connection with isometric imbedding in Hilbert spaces. It seems that this concept had also appeared in the (unpublished) work of Beurling. We refer to Chapter II of Berg and Frost (1975) for a systematic account on negative definite functions. We will see that the Lévy exponent ψ plays very important roles in studying the Lévy process X and many sample path properties of X can be described in terms of ψ. In this regard, we also note that

$$\mathrm{Re}\ \psi(\xi) \geq 0, \ \text{ and } \quad \mathrm{Re}\ \psi(-\xi) = \mathrm{Re}\ \psi(\xi), \qquad \forall\, \xi \in \mathbb{R}^d.$$

A Lévy process X in $\mathbb{R}^d$ is called *symmetric* if $-X$ and X have the same finite-dimensional distributions under $\mathbb{P}$ [note that this is consistent with (2.4)]. It is clear that X is symmetric if and only if $\psi(\xi) \geq 0$, for all $\xi \in \mathbb{R}^d$.

In the following, we list some special cases of Lévy processes:

(a). *Stable Lévy processes.* A Lévy process X in $\mathbb{R}^d$ is called a *stable* Lévy process with index $\alpha \in (0, 2]$ if its Lévy measure L is of the form

$$\mathsf{L}(dx) = \frac{dr}{r^{1+\alpha}}\, \nu(dy), \quad \forall\, x = ry, \quad (r, y) \in \mathbb{R}_+ \times \mathbb{S}_d, \tag{2.9}$$

where $\mathbb{S}_d = \{y \in \mathbb{R}^d : |y| = 1\}$ is the unit sphere in $\mathbb{R}^d$ and $\nu(dy)$ is an arbitrary finite Borel measure on $\mathbb{S}_d$. In the literature, stable Lévy processes in $\mathbb{R}^d$ of index $\alpha = 1$ are also called *Cauchy processes.* It follows from (2.8) and (2.9) that the Lévy exponent ψ_α of a stable Lévy process of index $\alpha \in (0, 2]$ can be written as

$$\begin{aligned}
\psi_\alpha(\xi) &= \int_{\mathbb{S}_d} |\langle \xi, y\rangle|^\alpha \Big[1 - i\ \mathrm{sgn}(\langle \xi, y\rangle) \tan\big(\frac{\pi\alpha}{2}\big)\Big] \mathsf{M}(dy) + i\langle \xi, \mu_0\rangle \quad \text{if } \ \alpha \neq 1, \\
\psi_1(\xi) &= \int_{\mathbb{S}_d} |\langle \xi, y\rangle| \Big[1 + i\ \frac{\pi}{2} \mathrm{sgn}(\langle \xi, y\rangle) \log|\langle \xi, y\rangle|\Big] \mathsf{M}(dy) + i\langle \xi, \mu_0\rangle,
\end{aligned}$$

where the pair (M, μ_0) is unique, and the measure M is called the *spectral measure* of X. See Samorodnitsky and Taqqu (1994, pp.65–66). When $d = 1$, ψ_α can be conveniently expressed as

$$\begin{aligned}
\psi_\alpha(\xi) &= \sigma^\alpha |\xi|^\alpha \big[1 - i\beta\, \mathrm{sgn}(\xi) \tan\frac{\pi\alpha}{2}\big] + i\xi\mu_0 \quad \text{if } \ \alpha \neq 1, \\
\psi_1(\xi) &= \sigma|\xi| \big[1 + i\frac{\pi}{2}\beta\, \mathrm{sgn}(\xi) \log|\xi|\big] + i\xi\mu_0,
\end{aligned} \tag{2.10}$$

where the constants $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu_0 \in \mathbb{R}$ are called the scale, skewness and shift parameters, respectively. Throughout, we will tacitly assume that all stable distributions are *non-degenerate*; that is, the measure M is not supported by any diametral plane of $\mathbb{S}_d$. Then, it is possible to see that there exists a positive and finite constant K, such that

$$\mathrm{Re}\ \psi_\alpha(\xi) \geq K|\xi|^\alpha, \qquad \forall \xi \in \mathbb{R}^d. \tag{2.11}$$

A stable Lévy process X on $\mathbb{R}^d$ with index $\alpha \in (0, 2]$ is said to be *strictly stable* if its Lévy exponent ψ_α has the form

$$\psi_\alpha(\xi) = |\xi|^\alpha \int_{\mathbb{S}_d} w_\alpha(\xi, y)\, \mathsf{M}(dy), \tag{2.12}$$

where

$$\begin{aligned} w_\alpha(\xi, y) &= \Big[1 - i\ \mathrm{sgn}(\langle \xi, y\rangle) \tan\big(\frac{\pi\alpha}{2}\big)\Big] \cdot \Big|\Big\langle \frac{\xi}{|\xi|}, y\Big\rangle\Big|^\alpha, \qquad \text{if } \alpha \neq 1; \\ w_1(\xi, y) &= \Big|\Big\langle \frac{\xi}{|\xi|}, y\Big\rangle\Big| + \frac{2i}{\pi}\langle \xi, y\rangle \log\big|\langle \xi, y\rangle\big| \end{aligned}$$

and, in addition, when $\alpha = 1$, M must also have the origin as its center of mass, i.e.,

$$\int_{\mathbb{S}_d} y\, \mathsf{M}(dy) = 0. \tag{2.13}$$

See, for example, Samorodnitsky and Taqqu (1994, p.73). We remark that the asymmetric Cauchy processes [i.e., the Cauchy processes whose spectral measures M do not satisfy (2.13)] are not strictly stable. The presence of the logarithmic term is the source of many difficulties associated with the studies of sample path properties of the asymmetric Cauchy processes, which have to be treated separately.

It follows from (2.12) that strictly stable Lévy processes of index α are $(1/\alpha)$-self-similar [under $\mathbb{P}^x$ for all $x \in \mathbb{R}^d$]. Conversely, a self-similar Lévy process must be a strictly stable Lévy process, see Sato (1999, p.71). A particularly interesting class arises when we let M be the uniform distribution on $\mathbb{S}_d$. In this case, $\psi(\xi) = \sigma^\alpha|\xi|^\alpha$ for some constant $\sigma > 0$, and X is called the *isotropic stable Lévy process* with index α. Note that isotropic processes are sometimes called symmetric processes in the literature. It is well-known that when $\alpha = 2$, $2^{-1/2}\sigma^{-1}X$ is a Brownian motion. This is a Gaussian process with continuous sample paths. All other stable Lévy processes have discontinuous sample paths.

As discovered in Taylor (1967), it is natural to distinguish between two types of strictly stable processes: those of *Type A*, and those of *Type B*. A strictly stable Lévy process X is of *Type A*, if

$$p(t, y) > 0, \qquad \forall t > 0,\ y \in \mathbb{R}^d,$$

where $p(t, y)$ is the density function of $X(t)$; all other stable Lévy processes are of *Type B*. Taylor (1967) has shown that if $\alpha \in (0, 1)$, and if the measure M is concentrated on a hemisphere, then X is of *Type B*, while all other strictly stable Lévy processes of index $\alpha \neq 1$ are of *Type A*.

(b). *Subordinators.* A subordinator X is a Lévy process in $\mathbb{R}$ with increasing sample paths. Equivalently, a real-valued Lévy process X is a subordinator if and only if $\Sigma = 0$ in (2.8) [i.e., X has no Gaussian part], its Lévy measure L is concentrated on $[0, \infty)$ and satisfies $\int_0^1 x\mathsf{L}(dx) < \infty$. In studying a subordinator X it is more convenient to use its Laplace transform

$$\mathbb{E}\Big[\exp\big(-uX(1)\big)\Big] = \exp\big(-g(u)\big),$$

where

$$g(u) = cu + \int_0^\infty \big[1 - \exp(-ur)\big]\mathsf{L}(dr),$$

and $c \geq 0$ is a constant and L is the (same) Lévy measure. The function g is called the *Laplace exponent* of X. It follows from Theorems 21.2 and 21.3 in Sato (1999) or Bertoin (1999, p.9) that if $c = 0$ and $\mathsf{L}(\mathbb{R}_+) < \infty$, then X is a compound Poisson process and its sample path is a step function; otherwise, the sample function of X is strictly increasing.

Besides being of great interest in their own right, subordinators are an important tool in studying the fractal properties of Lévy processes (e.g., co-dimension arguments, zero set of Lévy processes, etc.), They can also be used to generate new Lévy processes. That is, if $\tau = \{\tau_t, t \geq 0\}$ is a subordinator with $\tau_0 = 0$ and is independent of a Lévy process X, then the process Y defined by $Y(t) = X(\tau_t)$ is also a Lévy process (this is called *subordination* in the sense of S. Bochner). The transition function of Y can be expressed explicitly as

$$\mathbb{P}\big\{Y(t) \in B\big\} = \int_0^\infty \mathbb{P}\big\{X(s) \in B\big\}\mathbb{P}(\tau_t \in ds).$$

For an extensive account of subordinators and their properties, see Bertoin (1996, Chapter III; 1999).

(c). *Operator stable Lévy processes.* A Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}^d$ $(d > 1)$ is called *operator stable* if the distribution ν of $X(1)$ is *full* [i.e., not supported on any $(d-1)$-dimensional hyperplane] and ν is *strictly operator stable*, i.e., there exists a linear operator A on $\mathbb{R}^d$ such that

$$\nu^t = t^A\nu \qquad \text{for all } \ t > 0,$$

where ν^t denotes the t-fold convolution power of the infinitely divisible law ν and $t^A\nu$ is the image measure of ν under the linear operator t^A which is defined by

$$t^A = \sum_{n=0}^\infty \frac{(\log t)^n}{n!}\, A^n.$$

The linear operator A is called an exponent of X. The set of all possible exponents of an operator stable law is characterized in Theorem 7.2.11 of Meerschaert and Scheffler (2001).

On the other hand, a stochastic process $X = \{X(t), t \in \mathbb{R}\}$ is said to be *operator self-similar* if there exists a linear operator B on $\mathbb{R}^d$ such that for every $c > 0$,

$$\{X(ct), t \geq 0\} \stackrel{d}{=} \{c^B X(t), t \geq 0\},$$

where B is called a self-similarity exponent of X.

Hudson and Mason (1982) proved that if X is a Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1)$ is full, then X is operator self-similar if and only if $X(1)$ is strictly operator stable. In this case, every exponent of $X(1)$ is also a self-similarity exponent of X.

Operator stable Lévy processes are scaling limits of random walks on $\mathbb{R}^d$, normalized by linear operators; see Meerschaert and Scheffler (2001, Chapter 11).

Clearly, all strictly stable Lévy processes in $\mathbb{R}^d$ of index α are operator stable with exponent $A = \alpha^{-1}I$, where I is the identity operator in $\mathbb{R}^d$. The Lévy process $X = \{X(t), t \geq 0\}$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)),$$

where $X_1 \ldots, X_d$ are independent stable Lévy processes in $\mathbb{R}$ with indices $\alpha_1, \ldots, \alpha_d \in (0, 2]$ respectively, is called a *Lévy process with stable components.* This type of Lévy processes was first studied by Pruitt and Taylor (1969), and it is sometimes useful in constructing counterexamples [see Hendricks (1972)]. It is easy to verify that X is an operator stable Lévy process with exponent A which has $\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}$ on the diagonal and 0 elsewhere. Examples of operator stable Lévy process with dependent components have been considered by Shieh (1998) and recently by Becker–Kern et al. (2002). For systematic information about operator stable laws and operator stable Lévy processes, see Meerschaert and Scheffler (2001).

Now we return to general Lévy processes. In order to extend results on the sample paths of Brownian motion and stable Lévy processes to general Lévy processes in $\mathbb{R}^d$, Blumenthal and Getoor (1961) introduced the following indices β, β', β'' and obtained certain sample path properties of X in terms of these indices. Later Pruitt (1969) and Hendricks (1983) defined the indices γ and γ', respectively, and showed their relevance to the Hausdorff dimension of the range of Lévy processes. These indices have played important roles in studying the sample path properties of Lévy processes. See the survey papers of Taylor (1973, 1986a), Fristedt (1974) and Pruitt (1975). It is an interesting problem to understand the relationship among these indices. Related results and open questions can be found in Pruitt and Taylor (1996).

To be more specific, the upper index β of X is defined in terms of its Lévy measure L as

$$\begin{aligned}\beta &= \inf\Big\{\alpha > 0 : \int_{|y|<1} |y|^\alpha \, \mathsf{L}(dy) < \infty\Big\}\\ &= \inf\Big\{\alpha > 0 : \ r^\alpha \mathsf{L}\big\{y : |y| > r\big\} \to \infty \ \text{ as } r \to 0\Big\}. \qquad (2.14)\end{aligned}$$

When X is a Lévy process without a Gaussian part and a in (2.8) is appropriately chosen, Blumenthal and Getoor (1961, Theorem 3.2) showed that the upper index β can be expressed in terms of the Lévy exponent ψ:

$$\begin{aligned}\beta &= \inf\Big\{\alpha > 0 : \lim_{\xi\to\infty} |\xi|^{-\alpha} \operatorname{Re}\psi(\xi) = 0\Big\}\\ &= \inf\Big\{\alpha > 0 : \lim_{\xi\to\infty} |\xi|^{-\alpha}\, \psi(\xi) = 0\Big\}. \qquad (2.15)\end{aligned}$$

We mention that, under the same conditions on ψ, Millar (1971, Theorem 3.3) has provided a characterization of the index β by using a class of subordinators called "jump processes" associated to X.

The parameters $\beta^{''}$ and $\beta^{'}$ depend on the behavior of $\operatorname{Re}\psi$ at ∞:

$$\beta^{''} = \sup\Big\{\alpha \geq 0 : \lim_{|\xi|\to\infty} |\xi|^{-\alpha} \text{Re } \psi(\xi) = \infty\Big\}. \qquad (2.16)$$

$$\beta^{'} = \sup\Big\{\alpha \geq 0 : \int_{\mathbb{R}^d} |\xi|^{\alpha-d}\, \frac{1 - \exp(-\text{Re } \psi(\xi))}{\text{Re } \psi(\xi)} d\xi < \infty\Big\}. \qquad (2.17)$$

Blumenthal and Getoor (1961) showed that $0 \le \beta'' \le \beta' \le \beta \le 2$ and these indices could be distinct. However, when the process X is strictly stable with index $\alpha \in (0, 2]$, all these indices equal α.

When X is a subordinator with Laplace exponent g, Blumenthal and Getoor (1961) defined the index

$$\sigma = \sup\Big\{\alpha \le 1 : \int_1^\infty \frac{u^{\alpha-1}}{g(u)}\,du < \infty\Big\} \tag{2.18}$$

and showed that both σ and the upper index β can be expressed in terms of the Laplace exponent g:

$$\sigma = \sup\Big\{\alpha \ge 0 : \lim_{u\to\infty} u^{-\alpha} g(u) = \infty\Big\},$$

$$\beta = \inf\Big\{\alpha \ge 0 : \lim_{u\to\infty} u^{-\alpha} g(u) = 0\Big\}.$$

They also proved that $0 \le \beta' \le \sigma \le \beta \le 1$. Later, Horowitz (1968) found another representation for σ in terms of the Lévy measure:

$$\sigma = \sup\Big\{\alpha :\ x^{\alpha-1}\int_0^x \mathsf{L}(y,\infty)dy \to \infty \ \text{ as } x \to 0\Big\};$$

moreover, he showed that $\dim_{\mathrm{H}} X([0,1]) = \sigma$ a.s., where $\dim_{\mathrm{H}} E$ denotes the Hausdorff dimension of E. See Section 3.1 for its definition.

In studying the Hausdorff dimension of the range of a general Lévy process X in $\mathbb{R}^d$, Pruitt (1969) defined the index γ by means of the behavior of the expected time spent by X in a small ball:

$$\gamma = \sup\Big\{\alpha \ge 0 : \limsup_{r\to 0}\ r^{-\alpha}\int_0^1 \mathbb{P}\big\{|X(t)| \le r\big\}dt < \infty\Big\}. \tag{2.19}$$

Pruitt (1969) showed that for a subordinator $\gamma = \sigma$ [this is related to the above result of Horowitz (1968)] and for a symmetric Lévy process $\gamma = \min\{\beta', d\}$, but in general β' and γ can be different.

Pruitt's definition of γ is hard to calculate. The question of expressing the index γ in terms of the Lévy exponent ψ was raised in Pruitt (1969, 1975) and he obtained some partial results. This problem has recently been solved by Khoshnevisan, Xiao and Zhong (2003) who have shown that

$$\gamma = \sup\Big\{\alpha < d : \int_{\xi\in\mathbb{R}^d:\ |\xi|>1} \mathrm{Re}\Big(\frac{1}{1+\psi(\xi)}\Big)\frac{d\xi}{|\xi|^{d-\alpha}} < +\infty\Big\}. \tag{2.20}$$

The parameter γ' was due to Hendricks (1983),

$$\gamma' = \sup\Big\{\alpha \ge 0 : \liminf_{r\to 0}\ r^{-\alpha}\int_0^1 \mathbb{P}\big\{|X(t)| \le r\big\}dt < \infty\Big\}. \tag{2.21}$$

Taylor (1986b) proved that γ' equals the packing dimension of the range of X. For a subordinator X, it follows from the results of Fristedt and Taylor (1992) on the packing measure of the range $X([0,1])$ that $\gamma' = \beta$; see also Bertoin (1999, Theorem 5.1 and Lemma 5.2). Since (2.21) is not easy to evaluate for a general Lévy process, it would be useful to represent γ' in terms of the Lévy exponent ψ, as (2.20) for γ.

Using the results of Blumenthal and Getoor (1961), Pruitt (1969), Taylor (1986) and Bertoin (1999) on the Hausdorff and packing dimensions of $X([0,1]$, we have the following relationship among the indices γ, γ', β' and β

$$0 \le \beta' \wedge d \le \gamma \le \gamma' \le \beta \wedge d$$

and for a subordinator, $\gamma = \sigma \le \gamma' = \beta$. Further results on relations among these indices for general Lévy processes and some open problems can be found in Pruitt and Taylor (1996).

It is clear that both indices γ and γ' can be defined for any (Markov) process. In Sections 4.1 and 4.2, we will prove that $\dim_{\rm H} X([0,1]) = \gamma$ and $\dim_{\rm P} X([0,1]) = \gamma'$ for a very large class of Markov processes, where $\dim_{\rm P} E$ denotes the packing dimension of E. See Section 3.2 for its definition.

Due to the spatial homogeneity of Lévy processes, their transition operators are convolution operators. One can see that both the transition operator and infinitesimal generator of a Lévy process are pseudo-differential operators. See subsection 2.3 below for the definition of pseudo-differential operators and related references. This serves as a starting point for studying Feller processes generated by pseudo-differential operators.

2.2. Additive processes. A stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$, with values in $\mathbb{R}^d$, is called an *additive process* if, for every $s, t \ge 0$, the increment $X(t+s) - X(t)$ is independent of the process $\{X(r),\, 0 \le r \le t\}$, $X(0) = 0$ a.s. and such that $t \mapsto X(t)$ is continuous in probability.

Note that an additive process X has independent increments, but the increments may not be stationary. Hence, it is, in general, not temporally homogeneous. The class of additive process is very large. For example, if X is a Lévy process in $\mathbb{R}^d$ and $\tau(s)$ is any deterministic function that is increasing and right continuous, then $Y(s) = X(\tau(s))$ defines an additive process.

Of special interest is the class of self-similar additive processes. $X = \{X(t), t \in \mathbb{R}_+\}$ is called *broad-sense self-similar* if, for every $a > 0$, $a \ne 1$, there exist $b = b(a) > 0$ and a function $c(t) : \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\big\{X(at),\ t \in \mathbb{R}_+\big\} \stackrel{d}{=} \big\{bX(t) + c(t),\ t \in \mathbb{R}_+\big\}.$$

By Theorem 13.11 in Sato (1999), we know that if an additive process X is broad-sense self-similar, then there exists a constant $H > 0$ such that $b(a) = a^H$ for all $a > 0$. The constant H is called the self-similarity index of X. If $c(t) \equiv 0$, then X is self-similar as defined before.

Recall that a probability measure ν on $\mathbb{R}^d$ is said to be *self-decomposable* or *of Class L* if, for any $a > 1$, there is a probability measure ρ_a on $\mathbb{R}^d$ such that

$$\widehat{\nu}(\xi) = \widehat{\nu}(a^{-1}\xi)\, \widehat{\rho_a}(\xi), \qquad \forall \xi \in \mathbb{R}^d,$$

where $\widehat{\nu}$ is the Fourier transform of ν. It is easy to see that any stable distribution on $\mathbb{R}^d$ is self-decomposable [cf. Sato (1999, p.91)], but the *Class L* is much larger than the class of stable distributions.

It is well-known that a Lévy process X is self-similar if and only if it is strictly stable. For additive processes, there is an analogous result. This is due to Sato (1991), who showed that (i). If $X = \{X(t), t \in \mathbb{R}_+\}$ is broad-sense self-similar, then, for every $t \in \mathbb{R}_+$, the distribution of $X(t)$ is self-decomposable.

(ii). For every non-trivial self-decomposable distribution ν on $\mathbb{R}^d$ and any $H > 0$, there exists an additive process X such that X is self-similar with index H and the distribution of $X(1)$ is ν.

Hence, self-similar additive processes are also called processes of *class L*. We refer to Sato (1999) for systematic information on additive processes.

In contrast to the rich theory of Lévy processes, much less work on the sample path properties of additive processes has been carried out. Note that each distribution ν of *Class L* induces two kinds of processes of independent increments; one is a Lévy process and the other is a process of *Class L*. Both of them have ν as their distributions at $t = 1$. It is of interest to compare their probabilistic properties. Some results along this line have been established recently. For example, Sato (1991) and Watanabe (1996) have compared the asymptotic behavior of increasing self-similar additive processes $X = \{X(t), t \geq 0\}$ as $t \to 0$ and $t \to \infty$ with those of stable subordinators. Yamamuro (2000a, b) has obtained a criterion for the recurrence and transience of processes of *Class L*, which are different from those for Lévy processes. However, as far as I know, few results on fractal properties of their sample paths have been established for general processes of *Class L*.

2.3. Lévy-type Markov processes and pseudo-differential operators. In recent years, many authors have investigated Markov processes that are comparable in some sense to Lévy processes. In this subsection, we briefly discuss the Feller processes related to pseudo-differential operators and refer to Jacob (1996), Schilling (1998a, b), Jacob and Schilling (2001), Kolokoltsov (2000) for more information.

For simplicity, we take $S = \mathbb{R}^d$. Let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support and let $\mathcal{C}_0^\infty(\mathbb{R}^d)$ be the Banach space of continuous functions on $\mathbb{R}^d$ that tend to 0 at infinity equipped with the uniform norm $\|\cdot\|$. A *pseudo-differential operator* is an operator $q(x, D)$ on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of the form

$$q(x, D)u(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} q(x, \xi)\widehat{u}(\xi) d\xi,$$

where the function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, called the *symbol* of the operator $q(x, D)$, is assumed to be measurable in (x, ξ) and of polynomial growth in ξ, and where $\widehat{u}$ is the Fourier transform of u, i.e.,

$$\widehat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle \xi,x\rangle} u(x) dx.$$

Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in \mathbb{R}^d\}$ be a Feller process with values in $\mathbb{R}^d$. We denote its semigroup and infinitesimal generator by $\{T_t, t \geq 0\}$ and $(A, \mathcal{D}(A))$, respectively. Define the function $\lambda_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ by

$$\lambda_t(x, \xi) = \mathbb{E}^x \Big[e^{-i\langle \xi, X(t)-x\rangle} \Big], \tag{2.22}$$

which is the characteristic function of the random variable $X(t) - x$ on the probability space $(\Omega, \mathcal{M}, \mathbb{P}^x)$. Under some mild regularity conditions on X, Jacob (1998) proves that the restriction of T_t on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator with symbol $\lambda_t(x,\xi)$, that is, for every $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$T_t\, u(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \lambda_t(x,\xi)\widehat{u}(\xi)d\xi,$$

and, if the space of test functions $\mathcal{C}_c^\infty(R^d) \subset \mathcal{D}(A)$, then the infinitesimal generator $(A, \mathcal{D}(A))$ can be expressed as

$$Au(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} q(x,\xi)\widehat{u}(\xi)d\xi, \quad \forall\ u \in \mathcal{C}_0(\mathbb{R}^d), \tag{2.23}$$

where

$$-q(x,\xi) = \lim_{t\to 0} \frac{\lambda_t(x,\xi) - 1}{t}$$

and $\lambda_t(x,\xi)$ is defined as in (2.22). In other words, A is a pseudo-differential operator with symbol $q(x,\xi)$.

It is easy to see that if X is a Lévy process in $\mathbb{R}^d$ with exponent ψ, then its transition operator and generator are pseudo-differential operators with the symbols $\lambda_t(x,\xi) = e^{-t\psi(\xi)}$ and $q(x,\xi) = \psi(\xi)$, respectively. More precisely,

$$Au(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \psi(\xi)\widehat{u}(\xi)d\xi, \qquad \forall\ u \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

Note that both symbols above are constant in x and the corresponding pseudo-differential operators are said to have "constant coefficients".

More generally, a theorem of Courrége (1965) [see Jacob (1996)] implies that, if $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is contained in $\mathcal{D}(A)$, then the symbol $q(x,\xi)$ of A is locally bounded and, for every fixed x, is given by the Lévy–Khintchine formula

$$q(x,\xi) = i\langle \mathsf{a}(x),\xi\rangle + \frac{1}{2}\langle \xi, \Sigma(x)\xi^{'}\rangle + \int_{\mathbb{R}^d}\Big[1 - e^{i\langle y,\xi\rangle} + \frac{i\langle y,\xi\rangle}{1+|y|^2}\Big]\,\mathsf{L}(x,dy), \quad \forall \xi \in \mathbb{R}^d, \tag{2.24}$$

where $\mathsf{a}(x)$, $\Sigma(x)$ and $\mathsf{L}(x,dy)$ satisfy the same conditions as in (2.8).

However, unlike the one-to-one correspondence between Lévy processes and continuous negative definite functions given by the Lévy–Khintchine formula, condition (2.24) is only necessary for $q(x,\xi)$ to be the symbol of the generator of a Feller process. Additional sufficient conditions that ensure the existence of a Feller process for a given symbol $q(x,\xi)$ have recently been obtained. In fact, given a function $q(x,\xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that $\xi \mapsto q(x,\xi)$ is continuous and negative definite, there are several probabilistic and analytic ways to construct a Markov process X having $q(x,\xi)$ as its symbol. See Jacob (1996, Chapter 4) or Jacob and Schilling (2001, p.149) and references therein for more details.

To give some examples of such Feller processes, we mention the stable jump diffusions considered in Kolokoltsov (2000). Roughly speaking, these are the processes corresponding to stable Lévy motions in the same way as the normal diffusions corresponding to Brownian motion. A stable jump-diffusion is a Feller process whose generator has the same form as that of a stable Lévy process with coefficients depending on the position x. Locally, it resembles a stable Lévy motion, hence it is expected that a stable jump diffusion has fractal properties similar to those of a stable Lévy process. Some of these properties can be derived from results in Kolokoltsov (2000) and Sections 4.1 and 4.2.

Another class of Feller processes determined by pseudo-differential operators is the so-called stable-like process, that is, we allow the index α to depend on the position x. Its symbol is of the form

$$q(x,\xi) = |\xi|^{\alpha(x)} \quad \text{or} \quad q(x,\xi) \asymp 1 + |\xi|^{\alpha(x)},$$

where the function $\alpha(x)$ satisfies $0 < \alpha_0 \leq \alpha(x) \leq \alpha_\infty < 2$ and has modulus of continuity of order $o(|\log h|^{-1})$ as $h \to 0$; cf. Bass (1988a, b), Hoh (2000), Kikuchi and Negoro (1997).

Similar to the studies of Lévy processes, Fourier analytic methods are very useful in investigating probabilistic properties of a Feller process related to a pseudo-differential operator. Given such a process X, an interesting question is to characterize the properties of X by using the symbol $q(x,\xi)$. One of the approaches is to compare the symbol $q(x,\xi)$ with a fixed continuous negative definite function $\psi(\xi)$. For example, Schilling (1998a, b) has shown that, under suitable conditions, $q(x,\xi) \asymp \psi(\xi)$ implies estimates on the semigroup $\{T_t, t \geq 0\}$ of X and that of the Lévy process with exponent $\psi(\xi)$; from which asymptotic and Hausdorff dimension properties can be derived. The behavior of X is in some sense similar to the behavior of the Lévy process with exponent ψ. Under more restrictive conditions, it is even possible to obtain estimates on the transition functions similar to those of a Lévy process with exponent $\psi(\xi)$; see Negoro (1994).

More generally, sample path properties of the Feller process X can be described through asymptotic properties of its symbol $q(x,\xi)$. Schilling (1998a, b) introduced several indices using $q(x,\xi)$, similar to those for Lévy processes based on ψ, and studied the growth and Hausdorff dimension properties of X.

2.4. Ornstein–Uhlenbeck type Markov processes. Another class of Markov processes that are related to Lévy processes is formed by the Ornstein–Uhlenbeck type Markov processes. Their sample path properties may also be investigated by using the geometric and analytic tools described in this paper.

The notion of Ornstein–Uhlenbeck type Markov processes was introduced by Sato and Yamazato (1984). Such a process $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a Feller process with infinitesimal generator

$$A = G - \sum_{j=1}^{d} \sum_{k=1}^{d} Q_{jk} x_k \frac{\partial}{\partial x_j},$$

where G is the infinitesimal generator of a Lévy process $Z = \{Z(t), t \geq 0\}$ in $\mathbb{R}^d$ and Q is a real $d \times d$ matrix of which all eigenvalues have positive real parts. An equivalent definition of the process X is given by the unique solution of the equation

$$X(t) = x - \int_0^t QX(s)ds + Z(t),$$

which can be expressed as

$$X(t) = e^{-tQ}x + \int_0^t e^{(s-t)Q} dZ(s),$$

where the stochastic integral with respect to the Lévy process Z is defined by convergence in probability from integrals of simple functions; see e.g., Samorodnitsky and Taqqu (1994). The name for the process X comes from the fact that if Z is Brownian motion in $\mathbb{R}^d$ and $Q = I$, the $d \times d$ identity matrix, then X is the ordinary Ornstein–Uhlenbeck process.

Since an Ornstein–Uhlenbeck type Markov process X is determined by the Lévy process $\{Z(t), t \in \mathbb{R}_+\}$ and the matrix Q, it is natural to ask how the properties of X are related to those of Z and Q. Several authors have studied the sample path properties of X. For example, Shiga (1990), Sato et al. (1994), Watanabe (1998), Yamamuro (1998) have established criteria for recurrence and transience of Ornstein–Uhlenbeck type Markov processes. There have also been some partial results on the lower and upper bounds for the Hausdorff dimension of the range of X, see Wang (1997), Deng and Liu (1999). However, for a general Ornstein–Uhlenbeck type Markov process X, even $\dim_{\rm H} X([0, 1])$ is not known.

2.5. Fractional diffusions. Initial interest in the properties of diffusion processes and random walks on fractals came from mathematical physicists working in the theory of disordered media. Their studies raised the natural question of how to define analytic objects such as the "Laplacian" on fractal sets. Goldstein (1987) and Kusuoka (1987) [see Barlow (1998) or Kigami (2001) for these references] were the first to construct mathematically a Brownian motion $X = \{X(t), t \in \mathbb{R}_+\}$ on the Sierpinski gasket G, a connected fractal subset of $\mathbb{R}^2$. By defining the Laplacian on G as the infinitesimal generator of X, their results suggested a *probabilistic approach* [following the terminology of Kigami (2001)] to the problem of defining the Laplacian on fractals. On the other hand, Kigami (1989) gives a direct definition of the Laplacian on the Sierpinski gasket G. This *analytical approach* has been extended to construct the Laplacians on more general finitely ramified fractals by Kigami (1993). We refer to Kigami (2001) for a systematic treatment of this subject.

Barlow and Perkins (1988) have investigated the properties of Brownian motion X on the Sierpinski gasket G systematically. They show that the process X, like the standard Brownian motion, is a strong Markov process having a continuous symmetric transition density $p(t, x, y)$ with respect to the normalized Hausdorff measure on G. Barlow and Perkins (1988) have also studied the existence and joint continuity of the local times of X and proved a result for the modulus of continuity in the space variable for the local time process. Since then, many authors have investigated the existence and various properties of diffusions on more general fractals, and there has been a rapid development in probability and analysis on fractals; see Barlow (1998) and Kigami (2001) for additional historical background and further information.

In order to give a unified treatment of diffusions on various fractals, Barlow (1998) defines the class of fractional diffusions. Even though the assumptions there are a little too restrictive for us, these Markov processes are well suited to be analyzed by using general Markovian methods [or even Gaussian principles] and fractal geometric techniques.

The following definitions of a fractional metric space and a fractional diffusion are taken from Barlow (1998, Section 3).

Definition 2.2. Let (S, ρ) be a locally compact separable complete metric space and let μ be a Radon measure on $(S, \mathcal{S})$. The triple (S, ρ, μ) is called a *fractional metric space* (FMS for short) if the following conditions are satisfied:

(a). (S, ρ) has the midpoint property, i.e., for every $x, y \in S$, there exists $z \in S$ such that
$$\rho(x, z) = \rho(z, y) = \frac{1}{2}\rho(x, y);$$

(b). There exist $d_f > 0$ and positive constants c_1 and c_2 such that

$$c_1\, r^{d_f} \le \mu\big(B(x,r)\big) \le c_2\, r^{d_f} \quad \text{for all } \ x \in S,\ 0 \le r \le r_0 = \mathrm{diam} S. \tag{2.25}$$

Examples of FMS include $\mathbb{R}^d$ with the Euclidean distance $|\cdot|$ and Lebesgue measure λ_d, the Sierpinski gasket G equipped with the geodesic metric (which is equivalent to $|\cdot|$) and the self-similar measure μ with equal weights.

DEFINITION 2.3. Let (S, ρ, μ) be a fractional metric space. A Markov process $X = \{X(t), t \in \mathbb{R}_+; \mathbb{P}^x, x \in S\}$ is called a *fractional diffusion* on S if

(a). X is a conservative Feller diffusion with state space S.
(b). X has a symmetric transition density $p(t,x,y) = p(t,y,x)$ ($\forall t > 0, x, y \in S$), which is, for each $t > 0$, continuous in (x,y).
(c). There exist positive constants α, β, γ, $c_3, \ldots, c_6$ such that

$$c_3 t^{-\alpha} \exp\Big\{-c_4 \rho(x,y)^{\beta\gamma} t^{-\gamma}\Big\} \le p(t,x,y) \le c_5 t^{-\alpha} \exp\Big\{-c_6 \rho(x,y)^{\beta\gamma} t^{-\gamma}\Big\}, \tag{2.26}$$

for all $x, y \in S$ and $0 < t \le r_0^{\beta}$.

The above conditions are a little too restrictive. For studying the sample path properties of X, the important conditions are (2.25) and (2.26). The proof of Lemma 3.8 in Barlow (1998) shows that under these two conditions, $\alpha = d_f/\beta$.

The following are some examples of fractional diffusions.

EXAMPLE 2.4. [Diffusion processes on $\mathbb{R}^d$] For any given number $\lambda \in (0,1]$, let $\mathcal{A}(\lambda)$ denote the class of all measurable, symmetric matrix-valued functions $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ which satisfy the ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \frac{1}{\lambda}|\xi|^2 \quad \text{for all } \ x,\ \xi \in \mathbb{R}^d.$$

For each $a \in \mathcal{A}(\lambda)$, let $\mathcal{L} = \nabla \cdot (a\nabla)$ be the corresponding second order partial differential operator. By Theorem II.3.1 of Stroock (1988), we know that $\mathcal{L}$ is the infinitesimal generator of a d-dimensional diffusion process $X = \{X(t),\ t \ge 0\}$, which is strongly Feller continuous. Moreover, its transition density function $p(t,x,y) \in \mathcal{C}\big((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\big)$ satisfies the following inequality

$$\frac{1}{K t^{d/2}} \exp\Big(-\frac{K|y-x|^2}{t}\Big) \le p(t,x,y) \le \frac{K}{t^{d/2}} \exp\Big(-\frac{|y-x|^2}{Kt}\Big)$$

for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, where $K = K(a,d) \ge 1$ is a constant. The above estimate is due to Aronson (1967) [see Stroock (1988)]. Thus, X is a fractional diffusion with $\alpha = d/2$, $\beta = 2$ and $\gamma = 1$.

EXAMPLE 2.5. [Diffusions on the Sierpinski gasket and affine nested fractals] Let G be the Sierpinski gasket and let X be the Brownian motion on G. Barlow and Perkins (1988) have proved that X is a fractional diffusion with $\alpha = \log 3/\log 5$, $\beta = \log 5/\log 2$ and $\gamma = 1/(\beta - 1)$.

More generally, Fitzsimmons et al. (1994) have defined a class of *finitely ramified* self-similar fractals in $\mathbb{R}^d$, which they call the *affine nested fractals.* Roughly speaking, S is called an affine nested fractal if it is generated by a family $\{\psi_1, \ldots, \psi_N\}$ of contracting similitudes satisfying the open set condition and certain symmetry, connectivity and nesting properties. In addition, S is called a *nested fractal* if all the similitudes have the same contraction ratio. For any affine nested fractal S, Fitzsimmons et al. (1994) construct a Brownian motion X on S and prove that its transition density $p(t,x,y)$ satisfies (2.26) for an intrinsic metric ρ with

$$\alpha = \frac{d_s}{2}, \quad \beta = d_w, \quad \gamma = \frac{1}{\beta - 1}, \tag{2.27}$$

where d_s is the *spectral dimension* of the affine nested fractal S which describes the asymptotic frequency of the eigenvalues of the infinitesimal generator A of X, and d_w is the *walk dimension* of S. The relationship among d_f, d_s and d_w is $d_s = 2d_f/d_w$. The diffusions on affine nested fractals defined in Fitzsimmons et al. (1994) extend those on the nested fractals considered by Lindstrøm (1990) and Kumagai (1993). It is worthwhile to mention that the class of nested fractals has two advantages: (i). For a nested fractal S, the intrinsic metric ρ on S is related to the Euclidean metric $|\cdot|$ by

$$\rho(x,y) \asymp |x-y|^{d_c},$$

where d_c is the chemical exponent of the nested fractal S, see Fitzsimmons et al. (1994, p.608). Hence one can use the ordinary Hausdorff and packing measure [i.e., in the Euclidean metric] to characterize the fractal properties of Brownian motion X on S. (ii). The Brownian motion X on a nested fractal S is semi-self-similar with $a = N/(1-c)$ [cf. (2.6) or (2.7)], where N is the number of similitudes that generate S and $c \in (0,1)$ is a constant related to the return probability of the approximating random walk. See Fukushima et al. (1999, Lemma 2.1), Bass and Kumagai (2000) and Lindstrøm (1990) for details. In particular, for the Brownian motion X on the Sierpinski gasket G, it is semi-self-similar with $a = 5$.

Example 2.6. [Diffusions on the Sierpinski carpets] The Brownian motion X on the Sierpinski carpet defined by Barlow and Bass (1992, 1999) satisfies (2.26) with α, β and γ given by (2.27). The biggest difference between affine nested fractals and the Sierpinski carpets is that the former are finitely ramified while the latter are infinitely ramified. Because of this, diffusions on the Sierpinski carpets are significantly more difficult to construct and study.

If $S \subset \mathbb{R}^d$ and the triple $(S, |\cdot|, \mu)$ satisfies (2.25), then S is called a *d-set*. Recently, Chen and Kumagai (2002) have studied jump diffusions on d-sets and have obtained estimates for their transition densities. Their diffusions can be viewed as analogues of stable Lévy processes on fractals.

3. Tools from fractal geometry

In this section, we bring together definitions and some basic properties of Hausdorff measure and Hausdorff dimension, packing measure and packing dimension, capacity, multifractal analysis and average densities. They will serve as tools for analyzing fine properties of the stochastic processes discussed in this paper. More systematic information on fractal geometry can be found in Falconer (1990, 1997) and Mattila (1995).

3.1. Hausdorff dimension and Hausdorff measure. Let Φ be the class of functions $\varphi : (0,\delta) \to (0,\infty)$ which are right continuous, monotone increasing with

$\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \le K, \quad \text{for } 0 < s < \frac{1}{2}\delta. \tag{3.1}$$

The inequality (3.1) is usually called a doubling property. A function φ in Φ is often called a *measure function* or *gauge function*.

For $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \Big\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \Big\}, \tag{3.2}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (3.2) is called an ε-covering of E. It is well-known that φ-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is φ-m measurable. A function $\varphi \in \Phi$ is called an *exact (or a correct) Hausdorff measure function* for E if $0 < \varphi\text{-}m(E) < \infty$.

Remark 3.1. In (3.2) we only use coverings of E by balls, hence φ-m is usually called a *spherical Hausdorff measure* in the literature. Under (3.1), φ-m is equivalent to the Hausdorff measure defined by using coverings by arbitrary sets.

The *Hausdorff dimension* of E is defined by

$$\dim_{\rm H} E = \inf \big\{ \alpha > 0 :\ s^{\alpha}\text{-}m(E) = 0 \big\}.$$

The following lemma is often useful in determining upper bounds for the Hausdorff dimensions of the range, graph and inverse images. The proofs of the first two inequalities can be found in Kahane (1985a) or Falconer (1990). The last one was proved in Kaufman (1985) and Monrad and Pitt (1987). For the definition of local times, see Section 6.1.

Lemma 3.2. *Let $I \subset \mathbb{R}^N$ be a hyper-cube. If there is a constant $\alpha \in (0, 1)$ such that for every $\varepsilon > 0$, the function $f : I \to \mathbb{R}^d$ satisfies a uniform Hölder condition of order $\alpha - \varepsilon$ on I, then for every Borel set $E \subset I$*

$$\dim_{\rm H} f(E) \le \min \Big\{ d,\ \frac{1}{\alpha} \dim_{\rm H} E \Big\},$$

$$\dim_{\rm H} {\rm Gr} f(E) \le \min \Big\{ \frac{1}{\alpha} \dim_{\rm H} E,\ \dim_{\rm H} E + (1 - \alpha) d \Big\}.$$

If, in addition, f has a bounded local time on I, then for every Borel set $F \subset \mathbb{R}^d$,

$$\dim_{\rm H} X^{-1}(F) \le N - \alpha d + \alpha \dim_{\rm H} F.$$

Hausdorff dimension is closely related to the Bessel–Riesz capacity, as discovered by Frostman (1935). More generally, let S be any metric space equipped with the Borel σ-algebra $\mathcal{S}$. A kernel κ is a measurable function $\kappa : S \times S \to [0, \infty]$. For a Borel measure μ on S, the energy of μ with respect to the kernel κ is defined by

$$I_\kappa(\mu) = \int_S \int_S \kappa(x, y) \mu(dx) \mu(dy).$$

For $\Lambda \subseteq S$, the *capacity of Λ with respect to κ*, denoted by ${\rm Cap}_\kappa(\Lambda)$, is defined by

$${\rm Cap}_\kappa(\Lambda) = \Big[\inf_{\mu \in \mathcal{P}(\Lambda)} I_\kappa(\mu) \Big]^{-1},$$

where $\mathcal{P}(\Lambda)$ is the family of probability measures carried by Λ, and, by convention, $\infty^{-1} = 0$. Note that $\mathrm{Cap}_\kappa(\Lambda) > 0$ if and only if there is a probability measure μ on Λ with finite κ-energy. We will mostly consider the case when $\kappa(x, y) = f(|x - y|)$, where f is a non-negative and non-increasing function. In particular, if $f(r) = r^{-\alpha}$, then the corresponding Cap_κ is called the *Bessel–Riesz capacity of order* α and is denoted by Cap_α. The capacity dimension of Λ is defined by

$$\dim_c(\Lambda) = \sup\{\alpha > 0 : \mathrm{Cap}_\alpha(\Lambda) > 0\}.$$

The well-known Frostman's theorem [cf. Kahane (1985a, p.133)] states that for any compact set Λ in $\mathbb{R}^d$, $\dim_{\mathrm{H}} \Lambda = \dim_c(\Lambda)$. This result gives a very useful analytic way for the lower bound calculation of Hausdorff dimension. Let $\Lambda \subset \mathbb{R}^d$, in order to show $\dim_{\mathrm{H}} \Lambda \geq \alpha$, one only needs to find a measure μ on Λ such that the α-energy of μ is finite. For many deterministic and random sets such as self-similar sets or the range of a stochastic process, there are natural choices of μ.

Given a measure function $\varphi \in \Phi$ and a set $E \subset \mathbb{R}^d$, it is often more complicated to evaluate the Hausdorff measure φ-$m(E)$. From (3.2) we see that, in order to obtain an upper bound for the φ-Hausdorff measure of E, it is sufficient to construct a sequence of ε_n-coverings of E such that $\varepsilon_n \to 0$ and the corresponding sums are bounded. However, it is more difficult to use the above definition directly to obtain a lower bound for φ-$m(E)$ because one needs to consider all possible coverings of E by sets of diameter less than ε. This difficulty can usually be circumvented by applying the following density theorem due to Rogers and Taylor (1961), [see also Taylor and Tricot (1985)], which is a refinement of Frostman's lemma [see e.g., Kahane (1985a)].

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density* of μ at $x \in \mathbb{R}^d$ is defined by

$$\overline{D}^{\varphi}_{\mu}(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\varphi(2r)}.$$

LEMMA 3.3. *Given $\varphi \in \Phi$, there exists a positive constant K such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have*

$$K^{-1}\mu(E) \inf_{x \in E} \left\{\overline{D}^{\varphi}_{\mu}(x)\right\}^{-1} \leq \varphi\text{-}m(E) \leq K\|\mu\| \sup_{x \in E} \left\{\overline{D}^{\varphi}_{\mu}(x)\right\}^{-1}. \tag{3.3}$$

REMARK 3.4. One can define Hausdorff measure on any metric space (S, ρ) by replacing in (3.2) the Euclidean metric by ρ. We will use this remark in Section 7, when we study intersections of the image of a Markov process with a Borel set in the state space. Of course, the second inequality in the above density theorem may not be true in general metric spaces. A sufficient condition for (3.3) to hold is that S has *finite structural dimension*, i.e., for all $0 < a < 1$, there exists a constant M such that every subset of S with sufficiently small diameter δ can be covered by M sets of diameter no greater than $a\delta$. We refer to Howroyd (1994) for further information about Hausdorff measures on a general metric space.

3.2. Packing dimension and packing measure. Packing dimension and packing measure were introduced by Tricot (1982), Taylor and Tricot (1985) as a dual concept to Hausdorff dimension and Hausdorff measure. It is known that only for sets with certain regularities can their packing measure/dimension results be the same as their Hausdorff measure/dimension. For random sets related to the sample paths of Markov processes, the Hausdorff dimension and packing dimension properties may often be different; see Sections 4.1, 4.2 and 12.1. Moreover, even for problems that only concern the Hausdorff dimension, one may still need the packing dimension for their solutions [see Section 7.2]. Hence, in order to characterize the geometric structure of a fractal, it is more desirable to establish both Hausdorff and packing measure/dimension results.

For $\varphi \in \Phi$, define the set function φ-$P(E)$ on $\mathbb{R}^d$ by

$$\varphi\text{-}P(E) = \lim_{\varepsilon\to 0} \sup\Big\{\sum_i \varphi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \varepsilon\Big\}, \tag{3.4}$$

where $\overline{B}$ denotes the closure of B. A sequence of closed balls satisfying the conditions on the right-hand side of (3.4) is called an *ε-packing* of E. Unlike φ-m, the set function φ-P is not an outer measure because it fails to be countably subadditive. However, φ-P is a premeasure, so one can obtain an outer measure φ-p on $\mathbb{R}^d$ by defining

$$\varphi\text{-}p(E) = \inf\Big\{\sum_n \varphi\text{-}P(E_n) : \quad E \subseteq \bigcup_{n=1}^{\infty} E_n\Big\}. \tag{3.5}$$

φ-$p(E)$ is called the *φ-packing measure* of E. Taylor and Tricot (1985) proved that φ-$p(E)$ is a metric outer measure; hence every Borel set in $\mathbb{R}^d$ is φ-p measurable. If $\varphi(s) = s^\alpha$, s^α-$p(E)$ is called the α-dimensional packing measure of E. The *packing dimension* of E is defined by

$$\dim_{\mathrm{P}} E = \inf\big\{\alpha > 0 : \quad s^\alpha\text{-}p(E) = 0\big\}. \tag{3.6}$$

It follows from (3.5) that for any $E \subset \mathbb{R}^d$,

$$\varphi\text{-}p(E) \le \varphi\text{-}P(E). \tag{3.7}$$

Hence we can apply (3.7) to determine an upper bound for φ-$p(E)$. However, it is usually not easy to determine φ-$P(E)$, because we need to consider all the possible packings in (3.4). A lower bound for φ-$p(E)$ can be obtained by using the following density theorem for packing measures [see Taylor and Tricot (1985), Saint-Raymond and Tricot (1988) for a proof].

LEMMA 3.5. *For a given $\varphi \in \Phi$, there exists a finite constant $K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| = \mu(\mathbb{R}^d) < \infty$ and any Borel set $E \subseteq \mathbb{R}^d$,*

$$K^{-1}\mu(E) \inf_{x\in E} \big\{\underline{D}^{\varphi}_{\mu}(x)\big\}^{-1} \le \varphi\text{-}p(E) \le K\|\mu\| \sup_{x\in E} \big\{\underline{D}^{\varphi}_{\mu}(x)\big\}^{-1}, \tag{3.8}$$

where

$$\underline{D}^{\varphi}_{\mu}(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\varphi(2r)}$$

is the lower φ-density of μ at x.

There is an equivalent definition for $\dim_{\mathrm{P}} E$ which is sometimes more convenient to use. For any $\varepsilon > 0$ and any bounded set $E \subseteq \mathbb{R}^d$, let

$$N_1(E, \varepsilon) = \text{smallest number of balls of radius } \varepsilon \text{ needed to cover } E$$

and

$$N_2(E, \varepsilon) = \text{ largest number of disjoint balls of radius } \varepsilon \text{ with centers in } E.$$

Then we have

$$N_2(E,\varepsilon) \le N_1(E,\varepsilon) \le N_2(E,\varepsilon/2).$$

To simplify the notations, we write $N(E,\varepsilon)$ for $N_1(E,\varepsilon)$ or $N_2(E,\varepsilon)$ indifferently. Then the *upper and lower box-counting dimension* of E are defined as

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{\varepsilon\to 0} \frac{\log N(E,\varepsilon)}{-\log \varepsilon}$$

and

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\varepsilon\to 0} \frac{\log N(E,\varepsilon)}{-\log \varepsilon},$$

respectively. If $\overline{\dim}_{\mathrm{B}}(E) = \underline{\dim}_{\mathrm{B}}(E)$, the common value is called the *box-counting dimension* of E. It is easy to verify that

$$0 \le \dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} E \le d \quad \text{and} \quad 0 \le \dim_{\mathrm{P}} E \le \overline{\dim}_{\mathrm{B}} E \le d \tag{3.9}$$

for all bounded sets $E \subseteq \mathbb{R}^d$. Hence $\overline{\dim}_{\mathrm{B}} E$ and $\underline{\dim}_{\mathrm{B}} E$ can be used to determine upper bounds for $\dim_{\mathrm{H}} E$ and $\dim_{\mathrm{P}} E$.

The disadvantage of $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ as dimension is that they are not σ-stable [cf. Tricot (1982), Falconer (1990, p.45)]. One can obtain σ-stable indices $\overline{\dim}_{\mathrm{MB}}$ and $\underline{\dim}_{\mathrm{MB}}$ by letting

$$\overline{\dim}_{\mathrm{MB}} E = \inf\Big\{ \sup_n \overline{\dim}_{\mathrm{B}} E_n : \quad E \subseteq \bigcup_{n=1}^{\infty} E_n \Big\},$$

$$\underline{\dim}_{\mathrm{MB}} E = \inf\Big\{ \sup_n \underline{\dim}_{\mathrm{B}} E_n : \quad E \subseteq \bigcup_{n=1}^{\infty} E_n \Big\}.$$

Following Falconer (1990), we call $\overline{\dim}_{\mathrm{MB}} E$ and $\underline{\dim}_{\mathrm{MB}} E$ the *modified* upper and lower box-counting dimension of E, respectively. Tricot (1982) has proved that $\dim_{\mathrm{P}} E = \overline{\dim}_{\mathrm{MB}}(E)$. Hence, for any set $E \subseteq \mathbb{R}^d$,

$$0 \le \dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{MB}} E \le \overline{\dim}_{\mathrm{MB}} E = \dim_{\mathrm{P}} E \le d. \tag{3.10}$$

Thus, if $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, then all the dimensions in (3.10) coincide.

Since the upper box dimension $\overline{\dim}_{\mathrm{B}}$ of a set is easier to determine, the following lemma from Tricot (1982) is useful in calculating the packing dimension of a set. Recall that $\overline{\dim}_{\mathrm{B}}$ is said to be *uniform* on E if there exists a constant c such that for every $x \in \overline{E}$,

$$\lim_{r\to 0} \overline{\dim}_{\mathrm{B}}(E \cap B(x,r)) = c.$$

LEMMA 3.6. *If E is compact and $\overline{\dim}_{\mathrm{B}}$ is uniform on E, then $\overline{\dim}_{\mathrm{B}}(E) = \dim_{\mathrm{P}} E$.*

It is easy to see that the analogous upper bounds in Lemma 3.2 remain true if one replaces $\dim_{\mathrm{H}}$ by $\dim_{\mathrm{P}}$. However, unlike the Hausdorff dimension cases, if f is a projection from $\mathbb{R}^N$ to $\mathbb{R}^d$ ($d < N$) or f is the Brownian motion in $\mathbb{R}$, these upper bounds are not sharp anymore. In fact, Talagrand and Xiao (1996) have shown that for any function $f : \mathbb{R}^N \to \mathbb{R}^d$ satisfying a uniform Hölder condition of order α on, say, $[0,1]^N$, there are compact sets $E \subset [0,1]^N$ such that

$$\dim_{\mathrm{P}} f(E) < \frac{1}{\alpha} \dim_{\mathrm{P}} E.$$

For more information about the packing dimension of projections, see Järvenpää (1994), Falconer and Howroyd (1997) and the references therein. It turns out that in these cases, the upper bound for the packing dimension of $f(E)$ is determined by the packing dimension profile of E, see Lemma 3.8 below.

REMARK 3.7. In order to study multifractals, Olsen (1995) has introduced multifractal Hausdorff measure (dimension) and multifractal packing measure (dimension) with respect to a Radon measure. They are natural generalizations of the Hausdorff measure and the packing measure discussed above and are more appropriate for multifractal analysis.

3.3. Packing dimension profile. In this subsection, we recall briefly the definitions of packing dimensions of measures and packing dimension profiles introduced by Falconer and Howroyd (1997), and state some of their basic properties. See also Howroyd (2001) for recent developments.

The *packing dimension* of a Borel measure μ on $\mathbb{R}^d$ (or lower packing dimension as it is sometimes called) is defined by

$$\dim_{\mathrm{P}} \mu = \inf \left\{ \dim_{\mathrm{P}} E : \ \mu(E) > 0 \text{ and } E \subseteq \mathbb{R}^d \text{ is a Borel set} \right\}. \tag{3.11}$$

The *upper packing dimension* of μ is defined by

$$\dim_{\mathrm{P}}^* \mu = \inf \left\{ \dim_{\mathrm{P}} E : \ \mu(\mathbb{R}^d \backslash E) = 0 \text{ and } E \subseteq \mathbb{R}^d \text{ is a Borel set} \right\}. \tag{3.12}$$

The *lower and upper Hausdorff dimension* of μ can be defined in a similar way. They are denoted by $\dim_{\mathrm{H}} \mu$ and $\dim_{\mathrm{H}}^* \mu$, respectively. More information on Hausdorff, packing and other dimensions of measures can be found in Hu and Taylor (1994), Falconer (1997).

For a finite Borel measure μ on $\mathbb{R}^d$ and for any $s > 0$, define the potential

$$F_s^{\mu}(x,r) = \int_{\mathbb{R}^d} \min \left\{ 1, \ r^s |y-x|^{-s} \right\} d\mu(y).$$

The following equivalent definitions of $\dim_{\mathrm{P}} \mu$ and $\dim_{\mathrm{P}}^* \mu$ in terms of the potential $F_d^{\mu}(x,r)$ are given by Falconer and Howroyd (1997):

$$\dim_{\mathrm{P}} \mu = \sup \left\{ t \geq 0 : \ \liminf_{r \to 0} \ r^{-t} F_d^{\mu}(x,r) = 0 \ \text{ for } \mu\text{-a.e. } \ x \in \mathbb{R}^d \right\} \tag{3.13}$$

and

$$\dim_{\mathrm{P}}^* \mu = \inf \left\{ t > 0 : \ \liminf_{r \to 0} \ r^{-t} F_d^{\mu}(x,r) > 0 \ \text{ for } \mu\text{-a.e. } \ x \in \mathbb{R}^d \right\}. \tag{3.14}$$

Extending the above, Falconer and Howroyd (1997) use the s-dimensional potential $F_s^{\mu}(x,r)$ to define the *packing dimension profile* of μ by

$$\mathrm{Dim}_s \mu = \sup \left\{ t \geq 0 : \ \liminf_{r \to 0} \ r^{-t} F_s^{\mu}(x,r) = 0 \ \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d \right\} \tag{3.15}$$

and the *upper packing dimension profile* of μ by

$$\mathrm{Dim}_s^* \mu = \inf \left\{ t > 0 : \ \liminf_{r \to 0} \ r^{-t} F_s^{\mu}(x,r) > 0 \ \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d \right\}, \tag{3.16}$$

respectively. It is easy to see that

$$0 \leq \mathrm{Dim}_s \mu \leq \mathrm{Dim}_s^* \mu \leq s$$

and, if $s \geq d$, that

$$\mathrm{Dim}_s \mu = \dim_{\mathrm{P}} \mu, \qquad \mathrm{Dim}_s^* \mu = \dim_{\mathrm{P}}^* \mu;$$

see Falconer and Howroyd (1997) for details.

For any analytic set $E \subseteq \mathbb{R}^d$, let $\mathcal{M}_c^+(E)$ be the family of finite Borel measures with compact support contained in E. Then $\dim_{\rm P} E$ can be characterized by the packing dimension of the measures carried by E,

$$\dim_{\rm P} E = \sup\big\{\dim_{\rm P}\mu : \ \mu \in \mathcal{M}_c^+(E)\big\};$$

see Hu and Taylor (1994) for a proof. Motivated by this, Falconer and Howroyd (1997) define the *packing dimension profile* of $E \subseteq \mathbb{R}^d$ by

$$\mathrm{Dim}_s E = \sup\big\{\mathrm{Dim}_s \mu : \ \mu \in \mathcal{M}_c^+(E)\big\}. \tag{3.17}$$

It is easy to show that for every analytic set $E \subseteq \mathbb{R}^d$, $0 \le \mathrm{Dim}_s E \le s$ and for any $s \ge d$, $\mathrm{Dim}_s E = \dim_{\rm P} E$.

The following lemma from Xiao (1997a) gives upper bounds for $\mathrm{Dim}_s \mu$, $\mathrm{Dim}_s^* \mu$ and $\mathrm{Dim}_s E$; see Falconer and Howroyd (1997) for a special case.

LEMMA 3.8. *Let I be a hyper-cube in $\mathbb{R}^N$ and let $f : I \to \mathbb{R}^d$ be a continuous function satisfying a uniform Hölder condition of all orders smaller than α. For any finite Borel measure μ on $\mathbb{R}^N$ with support contained in I and any Borel set $E \subset I$, we have*

$$\dim_{\rm P} \mu_f \le \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d}\, \mu, \qquad \dim_{\rm P}^* \mu_f \le \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d}^*\, \mu, \tag{3.18}$$

where μ_f is the image measure of μ under f, and

$$\dim_{\rm P} f(E) \le \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d}\, E.$$

3.4. Multifractal analysis. The term "multifractal" and its connection with thermodynamics first appeared in the works of the physicists Frisch and Parisi (1985), Halsey et al. (1986). But a constructive and rigorous approach to multifractals as physical models was developed by Mandelbrot (1972, 1974) [see e.g., Falconer (1997), Olsen (2000) for the references mentioned above]. Since then, multifractals have been extensively applied to model various phenomena in many fields. Examples include the growth rate along a DLA-cluster, the distribution of a percolation cluster, the distribution of galaxies in the universe, the time in a model for price variation, and so on. In fractal geometry, multifractals were originally used to analyze the mass concentration of measures and, in particular, to quantify their singularity structure. Nowadays they have become one of the most basic tools that can also be used to study the fine properties of functions or stochastic processes.

Let $(S, \mathcal{S})$ be a measurable space and let $h : S \to \mathbb{R}$ be a measurable function. Let D be a real-valued function defined on $\mathcal{S}$. Then the spectrum with respect to the functions h and D is defined by

$$H(\theta) = D\big(\{x \in S : \ h(x) = \theta\}\big). \tag{3.19}$$

In multifractal analysis, the set function $D(\cdot)$ is usually taken to be the Hausdorff or packing dimensions. Then the basic problem is to calculate the function values $H(\theta)$.

REMARK 3.9. The equality sign in the right-hand side of (3.19) can be replaced by $\le$ or $\ge$ to define more general multifractal spectra.

In the following, we give several examples of multifractal spectra.

EXAMPLE 3.10. Let μ be a locally finite Borel measure on $(S, \mathcal{S})$, and let $h(x)$ be the *local dimension* of μ at x, that is,

$$h(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \tag{3.20}$$

if the limit exists. For any $\theta > 0$, let

$$E_\theta = \big\{x \in S : \ h(x) = \theta\big\}. \tag{3.21}$$

Note that, for most measures μ of interest, E_θ is dense in $\mathrm{supp}(\mu)$, the support of μ, for values of θ for which E_θ is nontrivial. Hence the box-counting dimensions $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ are of little use in differentiating the sizes of E_θ. It is more natural to let $D(\cdot)$ be the Hausdorff dimension or the packing dimension, then $H(\theta) = D(E_\theta)$ is the usual multifractal spectrum $f_\mu(\theta)$ and $F_\mu(\theta)$ of μ, respectively.

If μ is a self-similar measure on $\mathbb{R}^d$ defined by the probabilistic iterated function system $(\{\Psi_i\}_{i=1}^n, \{p_i\}_{i=1}^n)$, where $\Psi_1, \ldots, \Psi_n$ are similarity transforms on $\mathbb{R}^d$ with ratios $r_1, \ldots, r_n \in (0,1)$ and $\{p_i\}_{i=1}^n$ is a probability vector. Then μ satisfies the equation $\mu = \sum_{i=1}^n p_i \mu \circ \Psi_i^{-1}$. For each $q \in \mathbb{R}$, there is a unique number, $\tau(q)$, such that

$$\sum_{i=1}^n p_i^q \ r_i^{\tau(q)} = 1.$$

It can be shown that $\tau(q)$ is a strictly decreasing and convex function of q. Cawley and Mauldin (1992) [see also Falconer (1997, Chapter 11)], under the strong separation condition, have shown that

$$f_\mu(\theta) = F_\mu(\theta) = \inf_{-\infty<q<\infty} \big(\tau(q) + \theta q\big) \quad \text{for all } \ \theta \in [\theta_{\min}, \theta_{\max}], \tag{3.22}$$

where $\theta_{\min} = \min_{1\le i\le n} \log p_i / \log r_i$ and $\theta_{\max} = \max_{1\le i\le n} \log p_i / \log r_i$. That is, the Hausdorff and packing multifractal spectra of μ are given by the Legendre transform of τ.

This type of multifractal formalism has also been proven to hold for several more general classes of measures such as self-similar measures satisfying the weak separation condition, certain self-affine measures, self-conformal measures, statistically self-similar measures, and so on. See Olsen (2000) for a list of references.

If the limit in (3.20) fails to exist, then one can consider lower and upper local dimensions of μ defined by

$$\underline{h}(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and

$$\overline{h}(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively. When μ is the occupation measure of a subordinator, this type of multifractal spectrum for μ has been studied by Hu and Taylor (1997, 2000).

For random measures associated to stochastic processes such as their occupation measures or local times, the following setting will be useful.

EXAMPLE 3.11. Let μ be a Borel measure on $(S, \mathcal{S})$ with local (lower) dimension α everywhere. Define

$$h(t) = \limsup_{r\to 0} \frac{\mu\big(B(t,r)\big)}{r^\alpha(|\log r|)^\beta}$$

and let D be the Hausdorff dimension function or the packing dimension. Then the corresponding function $H(\theta)$ is called a *logarithmic multifractal spectrum* of μ, or more specifically, *multifractal spectrum of the thick points* of μ. In Section 12.3, we will discuss some recent results of Dembo et al. (1999, 2000a, b, 2001), and Shieh and Taylor (1998) on multifractal spectra on thick and thin points for the occupation measures of Brownian motion, Lévy stable processes and subordinators.

EXAMPLE 3.12. Let $f : \mathbb{R}^N \to \mathbb{R}^d$ be a (continuous) function. Define its local Hölder exponent at $t_0 \in \mathbb{R}^N$ by

$$h_f(t_0) = \sup\big\{\alpha > 0 : f \in C^\alpha(t_0)\big\},$$

where $f \in C^\alpha(t_0)$ means that there exists a constant $K > 0$ and a polynomial P_{t_0} of degree at most $\lfloor\alpha\rfloor$ (i.e., the largest integer $\le \alpha$) such that in a neighborhood of t_0,

$$|f(t) - P_{t_0}(t)| \le C|t - t_0|^\alpha. \tag{3.23}$$

Note that if f is continuously differentiable of order $\lfloor\alpha\rfloor$ in a neighborhood of t_0, then the polynomial $P_{t_0}(t)$ is exactly the Taylor expansion of f at t_0 of order $\lfloor\alpha\rfloor$. Nevertheless, (3.23) can hold for a large α even though f is not differentiable in a neighborhood of t_0.

For $S_\theta = \{t : h_f(t) = \theta\}$, the function $d(\theta) = \dim_{\rm H}(S_\theta)$ is called the *spectrum of singularities* of f. It gives geometric information about the distribution of the singularities of f. A function f is called multifractal when its spectrum of singularities is defined at least on a set with non-empty interior.

Multifractal functions have been studied extensively in recent years by several authors using wavelet techniques. Instead of listing the references, we refer to Jaffard (2001) for an expository treatment on this topic and for a list of references. In Section 12.4, we will describe a result of Jaffard (1998) on the sample functions of Lévy processes.

EXAMPLE 3.13. Let $W = \{W(t), t \in \mathbb{R}_+\}$ be the standard Brownian motion in $\mathbb{R}$. It is easy to see that the local Hölder exponent of W is $1/2$ everywhere on its sample path. Define

$$h(t) = \limsup_{\epsilon\to 0} \frac{|W(t+\epsilon) - W(t)|}{\sqrt{2\epsilon|\log\varepsilon|}}.$$

Then for $\theta \in (0, 1]$, $F(\theta) = \{t \in [0, 1] : h(t) = \theta\} \ne \emptyset$, and is called the set of *θ-fast points*. Let $D = \dim_{\rm H}$, then $H(\theta)$ is the Hausdorff dimension of the set of θ-fast points first studied by Orey and Taylor (1974). The packing dimension of $F(\theta)$ always equals 1. See Khoshnevisan, Peres and Xiao (2000) for details.

3.5. Average densities and tangent measure distributions. In this section, we recall the definitions and some basic properties of average densities and tangent measure distributions. They are two useful tools in studying the local geometric properties of fractal sets and measures in $\mathbb{R}^d$.

Average densities were first introduced by Bedford and Fisher (1992) for fractal sets and measures to characterize their fine local properties. Whereas the classical densities fail to exist for fractal measures, the average densities of order n have been shown to exist for a wide range of fractal measures such as self-similar measures, mixing repellers and random measures related to Brownian motion and Lévy stable processes.

Let us recall briefly the definition of average densities. Let μ be a locally finite Borel measure on $\mathbb{R}^d$ and let $\varphi \in \Phi$ be a gauge function. The lower and upper average φ-densities of order two of μ at x are defined by

$$\underline{D}_2^{\varphi}(\mu, x) = \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\varepsilon}^{1} \frac{\mu(B(x,r))}{\varphi(r)} \frac{dr}{r}$$

and

$$\overline{D}_2^{\varphi}(\mu, x) = \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\varepsilon}^{1} \frac{\mu(B(x,r))}{\varphi(r)} \frac{dr}{r},$$

respectively. When $\underline{D}_2^{\varphi}(\mu, x) = \overline{D}_2^{\varphi}(\mu, x)$, the common value is called the *average φ-density of order two of* μ at x and is denoted by $D_2^{\varphi}(\mu, x)$. When $\varphi(s) = s^{\alpha}$, we simply write it as $D_2^{\alpha}(\mu, x)$.

Similarly, the lower and upper average φ-densities of order three of μ at x are defined by

$$\underline{D}_3^{\varphi}(\mu, x) = \liminf_{\varepsilon \to 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \frac{\mu(B(x,r))}{\varphi(r)} \frac{dr}{r|\log r|}$$

and

$$\overline{D}_3^{\varphi}(\mu, x) = \limsup_{\varepsilon \to 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \frac{\mu(B(x,r))}{\varphi(r)} \frac{dr}{r|\log r|}.$$

If $\underline{D}_3^{\varphi}(\mu, x) = \overline{D}_3^{\varphi}(\mu, x)$, the common value is called the *average φ-density of order three of* μ at x. Average densities of higher orders can also be defined using the corresponding Hardy–Riesz log averages. Details and some basic properties of average densities are given in Bedford and Fisher (1992). Among the latter is the hierarchy relationship between the lower and upper average φ-densities and the usual lower and upper φ-densities defined in Sections 3.1 and 3.2: for all $x \in \mathbb{R}^d$,

$$\underline{D}_{\mu}^{\varphi}(x) \le \underline{D}_2^{\varphi}(\mu, x) \le \underline{D}_3^{\varphi}(\mu) \le \overline{D}_3^{\varphi}(\mu) \le \overline{D}_2^{\varphi}(\mu) \le \overline{D}_{\mu}^{\varphi}(x).$$

Average densities are closely related to Mandelbrot's concept of fractal lacunarity. In particular, they can be used to compare the lacunarity (or mass density) of different fractals with the same fractal dimensions. On the other hand, average densities can also be used to characterize the geometric regularity of sets or the symmetry properties of measures. We refer to Falconer (1997), Mörters (1998a) and Mörters and Shieh (1999) for more information and the latest references.

We mention that similar techniques have been applied by Patzschke and Zähle (1992, 1993, 1994) to study the local asymptotic properties of fractal functions and stochastic processes.

Another useful tool to study the local geometry of fractal sets and measures in $\mathbb{R}^d$ is the notion of tangent measure distributions, which appeared in a weak form in Bedford and Fisher (1992) and then in its full strength in Bandt (1992) and Graf (1995). The concept of a tangent measure distribution is an extension of two ideas. One is the idea of introducing tangent measures to characterize the regularity of measures by means of their local behavior [cf. Mattila (1995, Chapter 14), Falconer (1997, Chapter 9)]; the other idea is to use an averaging procedure on the set of scales to define the local characteristics of fractal sets or measures [e.g., the average densities above]. Roughly speaking, tangent measure distributions describe the structure of a set or a measure in the neighborhood of a point by magnifying smaller and smaller neighborhoods of x.

In the following, we recall the definition of tangent measure distributions given by Mörters and Preiss (1998) and Mörters (1998b). Let $\mathcal{M}(\mathbb{R}^d)$ be the Polish space of all non-negative, locally finite Borel measures on $\mathbb{R}^d$ endowed with the vague topology. This is the smallest topology that makes the functionals $\nu \mapsto \int f(x)\nu(dx)$ continuous, where $f : \mathbb{R}^d \to \mathbb{R}$ are arbitrary continuous functions with compact support.

A finite Radon measure P on $\mathcal{M}(\mathbb{R}^d)$ will be called a *measure distribution* on $\mathbb{R}^d$ or, a *random measure* on $\mathbb{R}^d$ if $P(\mathcal{M}(\mathbb{R}^d)) = 1$ [since it can be regarded as the distribution of a random measure]. A sequence $\{P_n\}$ of measure distributions on $\mathbb{R}^d$ is said to *converge vaguely* to a measure distribution P if the following two conditions are satisfied:

(i). $P(C) \geq \limsup_{n\to\infty} P_n(C)$ for every compact set $C \subset \mathcal{M}(\mathbb{R}^d)$, and
(ii). $P(O) \leq \liminf_{n\to\infty} P_n(O)$ for every open set $O \subset \mathcal{M}(\mathbb{R}^d)$.

If, in addition to (ii), (i) holds for every closed set $C \subset \mathcal{M}(\mathbb{R}^d)$, then we say that $\{P_n\}$ converges weakly to P. Note that we allow $\sup_n \|P_n\| = \infty$, so vague convergence is weaker than weak convergence. Of course, if P_n and P are probability measures on $\mathcal{M}(\mathbb{R}^d)$, the two senses of convergence are equivalent.

Let μ be a locally finite Borel measure on $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$ we define a family of Borel measures $\{\mu_{x,r}, r > 0\} \subset \mathcal{M}(\mathbb{R}^d)$ by

$$\mu_{x,r}(B) = \mu(x + rB) \quad \text{for all } \ B \in \mathcal{B}(\mathbb{R}^d).$$

These measures are called the *enlargements* of μ at x. Given a gauge function $\varphi \in \Phi$, define the probability distributions $P^{\varphi}_{2,x,\delta}$ on $\mathcal{M}(\mathbb{R}^d)$ by

$$P^{\varphi}_{2,x,\delta}(M) = \frac{1}{|\log \delta|} \int_{\delta}^{1} \mathbb{1}_M\Big(\frac{\mu_{x,r}}{\varphi(r)}\Big)\frac{dr}{r} \quad \text{for Borel sets } \ M \subset \mathcal{M}(\mathbb{R}^d). \tag{3.24}$$

Let $\mathcal{P}^{\varphi}_2(x,\mu)$ be the family of all limit points of $\{P^{\varphi}_{2,x,\delta}, \delta > 0\}$ as $\delta \downarrow 0$ in the vague convergence. The elements of $\mathcal{P}^{\varphi}_2(x,\mu)$ are called the *φ-tangent measure distributions of order two* of μ at x.

The *φ-tangent measure distributions of order three* of μ at x are defined to be the limit points of $\{P^{\varphi}_{3,x,\delta}, \delta > 0\}$ as $\delta \downarrow 0$ in the vague convergence, where

$$P^{\varphi}_{3,x,\delta}(M) = \frac{1}{\log|\log \delta|} \int_{\delta}^{1/e} \mathbb{1}_M\Big(\frac{\mu_{x,r}}{\varphi(r)}\Big)\frac{dr}{r|\log r|} \quad \text{for Borel sets } \ M \subset \mathcal{M}(\mathbb{R}^d).$$

The tangent measure distributions usually have good geometric regularity even if μ is highly irregular. For a self-similar measure μ on $\mathbb{R}^d$, Bandt (1992) and Graf (1995) proved that, for μ-almost all points $x \in \mathbb{R}^d$, μ has a unique tangent measure distribution at x which is equal to a fixed probability distribution on $\mathcal{M}(\mathbb{R}^d)$ independent of x. Recently, Bandt (2001) has extended the earlier work of Bandt (1992) and Graf (1995) and has constructed explicitly the tangent measure distribution of a self-similar measure μ with respect to another appropriate measure ν, using $\mu(B(x,r))$ as the normalizing function instead of $\varphi(r)$ in (3.24).

Mörters and Preiss (1998) have investigated the tangent measure distributions of an arbitrary measure μ on $\mathbb{R}^d$ and have shown that they have interesting scaling and shift invariance properties. In order to state their results, we need to recall some definitions. For every $u \in \mathbb{R}^d$, the shift operator T^u on $\mathcal{M}(\mathbb{R}^d)$ is defined by

$$T^u\nu(A) = \nu(u + A) \quad \text{for all } \ A \in \mathcal{B}(\mathbb{R}^d).$$

For every $c > 0$ and $\alpha > 0$, the scaling operator S_c^α on $\mathcal{M}(\mathbb{R}^d)$ is defined by

$$S_c^\alpha \nu(A) = c^{-\alpha}\, \nu(cA) \quad \text{ for all } \; A \in \mathcal{B}(\mathbb{R}^d).$$

A σ-finite measure Q on $\mathcal{M}(\mathbb{R}^d)$ is said to be *stationary* if it is invariant with respect to the shift operators T^u, i.e., $Q \circ (T^u)^{-1} = Q$ for all $u \in \mathbb{R}^d$. The *intensity* of a stationary σ-finite measure Q on $\mathcal{M}(\mathbb{R}^d)$ is defined by

$$\eta = \frac{1}{\lambda_d(B)} \int \nu(B) Q(d\nu),$$

where $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies $\lambda_d(B) > 0$. Note that the stationarity of Q implies that the definition of η is independent of the choice of B. A measure distribution P on $\mathcal{M}(\mathbb{R}^d)$ is called a *Palm distribution* if there is a stationary σ-finite measure Q on $\mathcal{M}(\mathbb{R}^d)$ with finite intensity such that

$$\int_M \nu(B) dQ(\nu) = \int_B P \circ T^u(M) du \quad \text{for all } \; M \subseteq \mathcal{M}(\mathbb{R}^d), \; B \in \mathcal{B}(\mathbb{R}^d).$$

Finally, a measure distribution P on $\mathcal{M}(\mathbb{R}^d)$ is called an *α-self-similar random measure* if P is a Palm distribution that is invariant under the scaling group $\{S_c^\alpha, c > 0\}$, i.e., $P = P \circ \left(S_c^\alpha\right)^{-1}$ for all $c > 0$. This notion of self-similar random measures is due to U. Zähle (1988) who has also studied the Hausdorff dimension of these random measures. Mörters and Preiss (1998, p.64) have extended the above notion of self-similar random measures.

Mörters and Preiss (1998) prove that for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ and every $0 < \alpha \le d$,

(i). at every $x \in \mathbb{R}^d$, every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ is scaling invariant under the scaling group $\{S_c^\alpha, c > 0\}$;

(ii). at μ-almost every $x \in \mathbb{R}^d$ at which the lower α-dimensional density of μ is positive, every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ is a Palm distribution.

Consequently, for μ-almost every $x \in \mathbb{R}^d$, every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ is an α-self-similar random measure [in the more general sense of Mörters and Preiss (1998)]. Related information on tangent measure distributions can be found in Mörters (1998b), Mörters and Shieh (1999).

Motivated by the definition of tangent measure distributions, Falconer (2002a, b) has recently introduced the concept of tangent processes and characterized the tangent processes of Lévy processes and random fields.

4. Hausdorff and packing dimension results for the range

The first result on the Hausdorff dimension of random sets was obtained by Taylor (1953) who determined $\dim_{\mathrm{H}} W([0,1])$ for a Brownian motion W in $\mathbb{R}^d$ ($d \ge 2$); see also Lévy (1953). Since then many authors have investigated the Hausdorff dimension and the exact Hausdorff measure of the range of Brownian motion and Lévy processes. We refer to Taylor (1986a) and the references therein for more information.

In this section, we discuss the Hausdorff and packing dimensions of the range $X([0,1])$ of a Markov process X with values in S. We will see that the expected occupation measure plays a key role in this section. The Hausdorff and packing dimensions of $X(\mathbb{R}_+)$ can be studied similarly. In particular, for any Markov process X satisfying the conditions of Theorem 4.2, the Markov property and the fact that

$\dim_{\rm H} X(\mathbb{R}_+) = \sup_{n\ge 0} \dim_{\rm H} X([n,\ n+1])$ imply that $\dim_{\rm H} X(\mathbb{R}_+) = \dim_{\rm H} X([0,1])$ a.s.

4.1. Hausdorff dimension results for the range. First we summarize some useful techniques for determining upper and lower bounds for the Hausdorff dimension of the range $X([0,1])$ or, more generally, $X(E)$, where $E \subset \mathcal{B}(\mathbb{R}_+)$. Similar arguments also work for calculating the Hausdorff dimension of other random fractals.

In order to obtain an upper bound for $\dim_{\rm H} X(E)$, we can use:

- a covering argument: find a sequence of coverings of $X(E)$, and show that the corresponding sums in (3.2) are bounded. When X is Hölder continuous, e.g., Brownian motion on $\mathbb{R}^d$ or on fractals, then this is given by Lemma 3.2. With the help of Lemma 4.1 below, often a first moment method is sufficient for general Markov processes.
- co-dimension arguments: the potential theory for Lévy processes implies that if $X(E)$ is polar for a symmetric stable process Y in $\mathbb{R}^d$ of index β that is independent of X, then $\dim_{\rm H} X(E) \le d - \beta$; see Proposition 4.11 for a general result. One can also use other random sets such as a random percolation in place of $Y(\mathbb{R}_+)$; see Peres (1999).

To prove lower bounds for the Hausdorff dimension of $X(E)$, one can use the following methods:

- a capacity argument based on the Frostman theorem. In order to show $\dim_{\rm H} X(E) \ge \gamma$, we construct a random Borel measure μ on $X(E)$ and show that μ has finite γ-energy. A natural random measure on $X(E)$ is given by the occupation measure. This argument is also effective for the level sets and self-intersection times, where the random measures are determined by the local times and self-intersection local times, respectively.
- co-dimension arguments: if $X(E)$ is not polar for a strictly stable process Y in $\mathbb{R}^d$ of index β that is independent of X, then $\dim_{\rm H} X(E) \ge d - \beta$, see Proposition 4.11 below.

Let $K_1 > 0$ be a fixed constant. A collection $\Lambda(a)$ of balls (open sets) of radius (diameter) a in metric space (S, ρ) is called K_1-nested if no ball of radius a in S can intersect more than K_1 balls (open sets) of $\Lambda(a)$. Clearly, if $S = \mathbb{R}^d$, then for each integer $n \ge 1$, the collection of dyadic (semi-dyadic) cubes of order n in $\mathbb{R}^d$ is K_1-nested with $K_1 = 3^d$.

The following covering lemma was first proved for Lévy processes in $\mathbb{R}^d$ by Pruitt and Taylor (1969). A similar argument yields an extension to general Markov processes; see Liu and Xiao (1998).

LEMMA 4.1. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogeneous strong Markov process in S with transition function $P(t, x, A)$ and let $\Lambda(a)$ be a fixed K_1-nested collection of balls of radius a $(0 < a \le 1)$ in S. For any $u \ge 0$, we denote by $M_u(a, s)$ the number of balls in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u+s]$. Then for all $x \in S$*

$$\mathbb{E}^x\big[M_u(a,s)\big] \le 2K_1 s \Big[\inf_{y\in S} \mathbb{E}^y\Big(\int_0^s \mathbb{1}_{B(y,a/3)}(X(t))dt\Big)\Big]^{-1},$$

where $\mathbb{1}_B$ is the indicator function of the set B.

For simplicity, we assume that $S = \mathbb{R}^d$.

THEOREM 4.2. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a Markov process in $\mathbb{R}^d$ with transition function $P(t, x, A)$ satisfying the following conditions:*

$$P(t, x, B(x, r_0)) \geq K \quad \text{for all} \ \ t > 0, x \in \mathbb{R}^d \quad \text{and some } r_0 > 0, \tag{4.1}$$

$$P(t, x, B(x, r)) \asymp P(t, 0, B(0, r)) \quad \text{for all} \ \ t > 0, \ x \in \mathbb{R}^d, \ 0 \leq r \leq r_0. \tag{4.2}$$

Then $\dim_{\rm H} X([0, 1]) = \gamma_{\rm low}$ $\mathbb{P}^x$*-a.s. for all* $x \in \mathbb{R}^d$*, where* $\gamma_{\rm low}$ *is defined by*

$$\gamma_{\rm low} = \sup \Big\{ \alpha \geq 0 : \limsup_{r \to 0} \frac{1}{r^\alpha} \int_0^1 P(t, 0, B(0, r)) dt < \infty \Big\}.$$

REMARK 4.3. (i). Similar to Theorem 2 of Pruitt (1969), we can express the index $\gamma_{\rm low}$ in terms of the moments of $X(t)$:

$$\gamma_{\rm low} = \sup \Big\{ \alpha \geq 0 : \ \int_0^1 \mathbb{E}\big(|X(t)|^{-\alpha} \big) dt < \infty \Big\}.$$

(ii). All the spatially homogeneous Markov processes satisfy the condition (4.2) with equality.

The proof of the lower bound in Theorem 4.2 is based on Lemmas 4.4 and 4.5 below. For any $t_0 \in [0, 1]$ and $r > 0$, let

$$T(t_0, r) = \int_{[0,1]} \mathbb{1}_{\{|X(t) - X(t_0)| \leq r\}} dt$$

be the sojourn time of X in the ball $B(X(t_0), r)$.

LEMMA 4.4. *There is a constant $\delta > 0$ such that for all $t_0 \in [0, 1]$, $r > 0$, all $\lambda > 0$ and $x \in \mathbb{R}^d$,*

$$\mathbb{P}^x \Big\{ T(t_0, r) \geq \lambda \tau(2r) \Big\} \leq \exp\big(- \delta \lambda \big), \tag{4.3}$$

where $\tau(r) = \mathbb{E}\big[T(0, r)\big]$.

PROOF. For simplicity, we assume $t_0 = 0$ and write $T(0, r)$ as $T(r)$. First we note that (4.2) implies that for all $x \in \mathbb{R}^d$ and $r > 0$

$$\mathbb{E}^x \big[T(r)\big] = \int_0^1 P\big(t, x, B(x, r)\big) dt \asymp \tau(r).$$

For any integer $n \geq 2$ and $x \in \mathbb{R}^d$, Fubini's theorem and the Markov property of X imply

$$\begin{aligned}
\mathbb{E}^x \Big[T(r)^n \Big] &= \mathbb{E}^x \Big[\int_0^1 \cdots \int_0^1 \prod_{j=1}^n \mathbb{1}_{\{|X(s_j) - X(0)| \leq r\}} \, ds_1 \cdots ds_n \Big] \\
&= n! \int_{0 \leq s_1 \leq \cdots \leq s_n \leq 1} \mathbb{P}^x \Big[\bigcap_{j=1}^n \big\{ |X(s_j) - X(0)| \leq r \big\} \Big] \, ds_1 \cdots ds_n \\
&\leq n! \int_{0 \leq s_1 \leq \cdots \leq s_n \leq 1} \mathbb{P}^x \Big[\bigcap_{j=1}^{n-1} \big\{ |X(s_j) - X(0)| \leq r \big\} \Big] \\
&\qquad\qquad \cdot P\big(s_n - s_{n-1}, x_{n-1}, B(x_{n-1}, 2r)\big) \, ds_1 \cdots ds_n \\
&\leq K^n \, n! \Big[\mathbb{E}(T(2r) \Big]^n,
\end{aligned}$$

where the last step follows by induction and where $K > 0$ is a constant. Hence there exists a positive constant $\delta > 0$, say $\delta = (2K)^{-1}$, such that

$$\mathbb{E}^x \Big[\exp\Big(\delta \frac{T(r)}{\tau(2r)}\Big)\Big] \le 1.$$

Finally, (4.3) follows from the Chebyshev's inequality. □

Using (4.3) and a standard Borel–Cantelli argument, we have

Lemma 4.5. *Assume the conditions of Lemma 4.4. Then for every* $t_0 \in [0,1]$,

$$\limsup_{r\to 0} \frac{T(t_0, r)}{\tau(2r)\log\log 1/r} \le \frac{1}{\delta} \quad \mathbb{P}^x\text{-a.s.} \quad \text{for all} \quad x \in \mathbb{R}^d, \tag{4.4}$$

where $\delta > 0$ *is the constant in Lemma 4.4.*

Proof of Theorem 4.2. To prove the lower bound, we note that Lemma 4.5, Fubini's theorem and Lemma 3.3 together imply

$$\varphi_1\text{-}m\big(X([0,1])\big) \ge K \quad \mathbb{P}^x\text{-a.s.} \quad \text{for all} \quad x \in \mathbb{R}^d, \tag{4.5}$$

where $\varphi_1(r) = \tau(2r)\log\log 1/r$. Hence $\dim_{\rm H} X([0,1]) \ge \gamma_{\rm low}$ $\mathbb{P}^x$-a.s.

Now we prove the upper bound. For any $\beta > \gamma_{\rm low}$, we chose $\alpha \in (\gamma_{\rm low}, \beta)$. Then, by the definition of $\gamma_{\rm low}$, there exists a sequence $\{r_n\}$ of positive numbers such that $r_n \downarrow 0$ and $\tau(r_n) \ge r_n^\alpha$ for all $n \ge 1$. By Lemma 4.1, we see that for each $n \ge 1$, $X([0,1])$ can be covered by $M_0(r_n, 1)$ cubes in $\Lambda(r_n)$ and

$$\mathbb{E}^x\big[M_0(r_n,1)\big] \le \frac{K}{\tau(r_n)} \le K\, r_n^{-\alpha}.$$

Hence we have $s^\beta\text{-}m\big(X([0,1])\big) < \infty$ $\mathbb{P}^x$-a.s., and therefore, Theorem 4.2 is proved. □

Remark 4.6. From the proof, we see that Theorem 4.2 can also be applied to a Markov process that is not temporally homogeneous, provided its transition function $p_{s,t}(x, A)$ is comparable to a function $P(t-s, x, A)$ satisfying conditions (4.1) and (4.2).

Remark 4.7. Theorem 4.2 implies that if the transition functions of two Markov processes are comparable, then the Hausdorff dimension of their ranges are the same. This is related to the results of Schilling (1996). A natural question is that, if a Markov process X with values in $\mathbb{R}^d$ is comparable with a Lévy stable process Y, do the uniform dimension and Hausdorff measure results for Y also hold for X? See Sections 5 and 8 for related results.

Theorem 4.2 can be conveniently applied to Markov processes for which transition functions can be estimated. Examples include Lévy stable processes, Brownian motion on fractals [Barlow (1998)], stable-like processes on fractals [Chen and Kumagai (2002)], stable jump diffusions [Kolokoltsov (2000)], Feller processes determined by pseudo-differential operators [Schilling (1996, 1998)], and so on. In many cases, $\gamma_{\rm low}$ can be calculated in terms of more explicit characteristics of the Markov process X. The following are some corollaries.

Corollary 4.8. [Barlow (1998, p.39)] Let X be a fractional diffusion in Definition 2.3, then

$$\dim_{\rm H} X([0,1]) = \min\{d_f, \beta\} \quad \text{a.s.}$$

When $X = \{X(t), t \in \mathbb{R}_+\}$ is a general Lévy process in $\mathbb{R}^d$, Pruitt (1969) has proved that $\dim_{\mathrm{H}} X([0,1]) = \gamma$ a.s., where the index γ is defined by (2.19). Since (2.19) may be difficult to calculate, it is more desirable to represent γ in terms of the Lévy exponent ψ of X. Pruitt (1969, Theorem 5) addresses this issue by verifying the following estimate for γ:

$$\gamma \geq \sup\Big\{\alpha < d : \int_{|\xi|\geq 1} \frac{1}{|\psi(\xi)|} \frac{d\xi}{|\xi|^{d-\alpha}} < +\infty\Big\}.$$

Moreover, it is shown there that if, in addition, $\operatorname{Re}\Psi(\xi) \geq 2\log|\xi|$ (for all $|\xi|$ large), then

$$\gamma = \sup\Big\{\alpha < d : \int_{\mathbb{R}^d} \operatorname{Re}\Big(\frac{1-e^{-\psi(\xi)}}{\psi(\xi)}\Big) \frac{d\xi}{|\xi|^{d-\alpha}} < +\infty\Big\}.$$

See Fristedt (1974, 377–378) for further discussions on Pruitt's work in this area. Recently, Khoshnevisan, Xiao and Zhong (2003) have settled the problem completely.

Theorem 4.9. *If X denotes a Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ, then*

$$\gamma = \sup\Big\{\alpha < d : \int_{\xi\in\mathbb{R}^d:\ |\xi|>1} \operatorname{Re}\Big(\frac{1}{1+\psi(\xi)}\Big) \frac{d\xi}{|\xi|^{d-\alpha}} < +\infty\Big\}.$$

Remark 4.10. Recently, by using the result of Pruitt (1969), Becker–Kern, Meerschaert and Scheffler (2002) have calculated $\dim_{\mathrm{H}} X([0,1])$ for a class of operator stable Lévy processes X in $\mathbb{R}^d$. Their arguments involve several technical probability estimates of operator stable Lévy processes and require some restrictions on the transition densities of the processes. Theorem 4.9 gives a different, analytic way to attack the problem. We expect that this method will work for the cases that have left unsolved by Becker–Kern, Meerschaert and Scheffler (2002).

The proof of Theorem 4.9 in Khoshnevisan, Xiao and Zhong (2003) relies on potential theory of a class of multi-parameter Lévy random fields, called additive Lévy processes (this should not be confused with additive processes in Section 2.2) and a co-dimension argument, which we explain below.

Let $X_\alpha = \{X_\alpha(t), t \geq 0\}$ be an isotropic stable Lévy process in $\mathbb{R}^d$ of index $\alpha \in (0,2]$. If $\alpha < d$, it is well-known that a compact set $F \subset \mathbb{R}^d$ is polar for X_α, i.e.,

$$\mathbb{P}\big\{X_\alpha(t) \in F \ \text{ for some } t > 0\big\} = 0$$

if and only if the Riesz–Bessel capacity $\mathrm{Cap}_{d-\alpha}(F) = 0$. Kanda (1976) proved that this is true for all strictly stable Lévy processes in $\mathbb{R}^d$. More information on the potential theory of Lévy processes can be found in Sato (1999, Chapter 8) and Bertoin (1996, Chapter III).

Since the Riesz–Bessel capacity is related to the Hausdorff dimension, Taylor (1966) proposed to use the range $X_\alpha(\mathbb{R}_+)$ of a stable Lévy process as a "gauge" to measure the Hausdorff dimension of any Borel set F in $\mathbb{R}^d$. More precisely, Taylor (1966) pointed out that for any Borel set $F \subset \mathbb{R}^d$ with $\dim_{\mathrm{H}} F \geq d-2$,

$$\dim_{\mathrm{H}} F = d - \inf\{\alpha > 0 : \ F \text{ is not polar for } X_\alpha\}, \tag{4.6}$$

and he applied this method to derive the Hausdorff dimension of the set of multiple points of strictly stable Lévy processes. See also Fristedt (1967) for a refinement.

With the help of potential theory of additive stable processes studied in Khoshnevisan and Xiao (2002, 2003a, b) and Khoshnevisan, Xiao and Zhong (2003), the restriction on F can be removed. Recall that an N-parameter additive stable process in $\mathbb{R}^d$ of index $\alpha \in (0,2]$, denoted by $X_{\alpha,N} = \{X_{\alpha,N}(t);\ t \in \mathbb{R}^N_+\}$, is defined by

$$X_{\alpha,N}(t) = X_1(t_1) + \cdots + X_N(t_N),$$

where $X_1, \ldots, X_N$ are independent isotropic stable processes with index α each. The following result can be easily derived from Theorem 4.5 in Khoshnevisan and Xiao (2003b).

PROPOSITION 4.11. For any Borel set $F \subset \mathbb{R}^d$,

$$\dim_{\rm H} F = d - \inf\big\{N\alpha > 0 : \ F \ \text{ is not polar for } X_{\alpha,N}\big\}.$$

Following Khoshnevisan and Shi (2000), this argument of finding $\dim_{\rm H} F$ is called a co-dimension argument. As an application, we consider the following example.

EXAMPLE 4.12. Equip $[0,1]^d$ with the Borel σ-field. Suppose $F = F(\omega)$ is a random set in $[0,1]^d$ (i.e., $\mathbb{1}_{F(\omega)}(x)$ is jointly measurable) such that for any compact $E \subset [0,1]^d$, we have

$$\mathbb{P}\{F \cap E \neq \emptyset\} = \begin{cases} 1 & \text{if } \dim_{\rm H} E > \gamma \\ 0 & \text{if } \dim_{\rm H} E < \gamma. \end{cases} \tag{4.7}$$

Then by taking $E = X_{\alpha,N}(\mathbb{R}^N_+)$ for appropriately chosen α and N and applying Proposition 4.11, we see that $\dim_{\rm H} F = d - \gamma$ almost surely.

REMARK 4.13. Results similar to Example 4.12 were established in Peres (1996) using fractal percolation and a co-dimension argument, and in Khoshnevisan and Shi (2000). Similar arguments can also be found in Hawkes (1971a), Khoshnevisan, Peres and Xiao (2000).

Now we return to the study of the fractal properties of the range of a Markov process X. Once $\dim_{\rm H} X([0,1])$ is known, two natural questions may be asked: (i) Can we determine $\dim_{\rm H} X(E)$ for every Borel set $E \subset \mathbb{R}_+$? (ii) Is there an exact Hausdorff measure function for $X([0,1])$? These two problems for Brownian motion and Lévy processes have been under rigorous investigation by several authors since the pioneering works of Taylor (1953) and Lévy (1953).

We will discuss Question (ii) for Lévy processes and more general Markov processes in Section 5. In the following, we summarize some results about Question (i) for Markov processes. Additional information can be found in Taylor (1986a).

Question (i) for Brownian motion in $\mathbb{R}^d$ was first considered by McKean (1955) [see Taylor (1986a) for the reference]. Blumenthal and Getoor (1960a, b) extended McKean's result first to a symmetric stable Lévy process and then to an arbitrary stable Lévy process X in $\mathbb{R}^d$, including the asymmetric Cauchy process. Their results can be restated as follows: Let X be a stable Lévy process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$. Then for every Borel set $E \subset \mathbb{R}_+$,

$$\dim_{\rm H} X(E) = \min\big\{d,\ \alpha \dim_{\rm H} E\big\} \quad \text{a.s.} \tag{4.8}$$

For a general Lévy process X, Blumenthal and Getoor (1961) established the following upper and lower bounds for $\dim_{\rm H} X(E)$ in terms of the upper index β and lower indices β' and β'' of X: for every $E \subset \mathbb{R}_+$, almost surely

$$\dim_{\rm H} X(E) \le \beta \dim_{\rm H} E \quad \text{if } \ \beta < 1, \tag{4.9}$$

$$\dim_{\mathrm{H}} X(E) \geq \begin{cases} \beta' \dim_{\mathrm{H}} E & \text{if } \beta' \leq d, \\ \min\{1, \beta'' \dim_{\mathrm{H}} E\} & \text{if } \beta' > d = 1; \end{cases}$$

if, in addition, X is a subordinator, then

$$\sigma \dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} X(E) \leq \beta \dim_{\mathrm{H}} E \quad \text{a.s.}$$

The restriction that $\beta < 1$ in (4.9) was removed by Millar (1971). Blumenthal and Getoor (1961, p.512) also conjectured that there is a function $f : [0,1] \to [0,d]$ depending only on X such that

$$\dim_{\mathrm{H}} X(E) = f(\dim_{\mathrm{H}} E) \quad \text{a.s.} \tag{4.10}$$

and they suspected that (4.10) might hold with the simple linear function $f(x) = \dim_{\mathrm{H}} X([0,1])\, x$. However, Hendricks (1972) has given an example of Lévy processes with stable components which shows that (4.10) cannot hold for any linear function f. Hendricks (1973) proved that for any Lévy process with stable components, (4.10) holds for a certain piecewise linear function f; see also Becker–Kern, Meerschaert and Scheffler (2002). Hawkes and Pruitt (1974, p.285) have further shown that linear functions f are not even enough for subordinators. In fact, their result shows that in general, $\dim_{\mathrm{H}} X(E)$ may not be determined by the (ordinary) Hausdorff dimension of E alone; hence the conjecture (4.10) can not be true for any function f. When X is a subordinator, Hawkes (1978b, Theorem 3) proves that $\dim_{\mathrm{H}} X(E)$ is a.s. equal to the Hausdorff-type dimension of E which is defined as $\inf\{\alpha > 0 : \ h^{\alpha}\text{-}m(E) = 0\}$, where h is a function determined by the Laplace exponent of X and is related to the exact Hausdorff measure function of the range $X([0,1])$ obtained by Fristedt and Pruitt (1971). Let X be an arbitrary Lévy process in $\mathbb{R}^d$ with exponent ψ, Khoshnevisan and Xiao (2003b) have recently established a general formula for $\dim_{\mathrm{H}} X(E)$ in terms of ψ for any Borel set $E \subset \mathbb{R}_+$. In particular, if X is symmetric or it has the lower index $\beta'' > 0$, then

$$\dim_{\mathrm{H}} X(E) = \sup\Big\{\gamma \in (0,d) : \ \mathrm{Cap}_{\kappa_\gamma}(E) > 0\Big\},$$

where κ_γ is the kernel defined by

$$\kappa_\gamma(x,y) = \int_{\mathbb{R}^d} e^{-|x-y|\psi(\xi)} |\xi|^{\gamma-d}\, d\xi, \qquad \forall x, y \in \mathbb{R}.$$

Note that, when $\beta'' > 0$, by using the Fourier transform of the function $\xi \mapsto |\xi|^{\gamma-d}$ $(0 < \gamma < d)$ it can be shown that $\kappa_\gamma(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.

For certain Markov processes that are comparable to Lévy processes, Question (i) has been considered by Schilling (1996, 1998b) who has extended the results of Blumenthal and Getoor (1961) and Millar (1971). It would be interesting to see whether the results of Khoshnevisan and Xiao (2003b) can also be extended to such Markov processes.

On the other hand, Liu and Xiao (1998) have studied Question (i) for Markov processes that are approximately self-similar. The following result is an extension of Theorem 3.1 in Liu and Xiao (1998).

THEOREM 4.14. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogeneous strong Markov process in $\mathbb{R}^d$ with transition function $P(t,x,A)$. Assume that there exist positive constants r_0, K_2 and K_3 such that*

$$P\big(t, x, B(x,r)\big) \geq K_2 \min\Big\{1, \Big(\frac{r}{t^{\alpha}}\Big)^d\Big\}, \qquad \forall\, x \in \mathbb{R}^d,\ 0 \leq r \leq r_0 \tag{4.11}$$

and

$$P\big(t,x,B(y,r)\big) \le K_3 \min\Big\{1, \Big(\frac{r}{t^\alpha}\Big)^d\Big\} \tag{4.12}$$

for all $x,\ y \in \mathbb{R}^d$ *with* $|x-y| \le r_0$ *and all* $0 \le r \le r_0$. *Then for every Borel set* $E \subset \mathbb{R}_+$

$$\dim_{\mathrm{H}} X(E) = \min\Big\{d, \frac{1}{\alpha}\dim_{\mathrm{H}} E\Big\} \quad \mathbb{P}^x\text{-}a.s. \tag{4.13}$$

It is clear that the conditions (4.11) and (4.12) in Theorem 4.14 can be satisfied by many Markov processes in Section 2.

4.2. Packing dimension results for the range. Since Hausdorff and packing dimension characterize different aspects of fractals, more information on a random set can be obtained if both of its Hausdorff and packing dimensions are known. In this section, we consider the packing dimension of the range of X. The first result on packing dimension and packing measure of random fractals was obtained by Taylor and Tricot (1985), in which they studied the packing measure of the range of Brownian motion in $\mathbb{R}^d$ with $d \ge 3$. Taylor (1986b) proves that for any Lévy process X, $\dim_{\mathrm{P}} X([0,1]) = \gamma'$ a.s., where γ' is the index defined in (2.21); see also Pruitt and Taylor (1996) for a proof. When X is a general subordinator, Fristedt and Taylor (1992) determine the exact packing measure of the range $X([0,1])$. It follows from their results that if X a subordinator, then $\dim_{\mathrm{P}} X([0,1]) = \beta$ a.s., where β is the upper index of X defined by (2.15). See also Bertoin (1999, Theorem 5.1 and Lemma 5.2) for a direct proof. Therefore, for any subordinator X, $\gamma' = \beta$.

Compared to the tool box for evaluating the Hausdorff dimension of a random set, fewer techniques are available for packing dimension. In order to obtain an upper bound for $\dim_{\mathrm{P}} F$, we can use the inequality $\dim_{\mathrm{P}} F \le \overline{\dim}_{\mathrm{B}} F$ to look for coverings of F by balls of equal radius. We have to be cautious, because upper bounds obtained in this way may not be sharp. The density theorem of Taylor and Tricot (1985) [cf. Lemma 3.5] remains to be the main tool for proving a lower bound for $\dim_{\mathrm{P}} F$. An alternative way is to use the packing dimension profile, which looks promising and is worthy of further study.

The following theorem is an extension of the result of Taylor (1986b) on Lévy processes to general Markov processes and it is an analogue of Theorem 4.2.

THEOREM 4.15. *Let* $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ *be a Markov process in* $\mathbb{R}^d$ *with transition function* $P(t,x,A)$ *satisfying the conditions (4.1) and (4.2) in Theorem 4.2. Then* $\dim_{\mathrm{P}} X([0,1]) = \gamma_{\mathrm{up}}$ $\mathbb{P}^x$*-a.s., where* γ_{up} *is defined by*

$$\gamma_{\mathrm{up}} = \sup\Big\{\alpha \ge 0 : \liminf_{r\to 0} \frac{1}{r^\alpha}\int_0^1 P\big(t,0,B(0,r)\big)dt < \infty\Big\}.$$

PROOF. The proof is similar to that of Taylor (1986b). The lower bound $\dim_{\mathrm{P}} X([0,1]) \ge \gamma_{\mathrm{up}}$ follows from Lemma 3.5 and the definition of γ_{up}. We note that, since we are dealing with liminf, all we need is Fatou's lemma and a first moment argument. On the other hand, in order to prove the upper bound $\dim_{\mathrm{P}} X([0,1]) \le \gamma_{\mathrm{up}}$, it suffices to show $\overline{\dim}_{\mathrm{B}} X([0,1]) \le \gamma_{\mathrm{up}}$ a.s, which follows from Lemma 4.1 and a first moment argument. □

As an example, we mention that Theorem 4.15 can be applied easily to all elliptic diffusion processes in $\mathbb{R}^d$, stable jump diffusions in Kolokoltsov (2000) and stable-like processes on d-sets in Chen and Kumagai (2002). For these processes, the packing and Hausdorff dimensions of the range are equal. Theorem 4.15 can also be applied to fractional diffusions on a fractional metric space S to derive

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \min\{d_f, \beta\} \quad \mathbb{P}^x\text{-a.s.}$$

However, the following natural question for Lévy processes remains open:

QUESTION 4.16. Let X be a Lévy process in $\mathbb{R}^d$ with exponent ψ, can γ' and hence $\dim_{\mathrm{P}} X([0,1])$ be represented in terms of ψ?

Pruitt and Taylor (1996) have proved several interesting results about the relationship among γ, γ' and other indices. They also raise several questions and conjectures regarding γ' and liminf behavior of $T(r)$, the sojourn time of X in $B(0,r)$. As far as I know, the following problem has not been solved.

QUESTION 4.17. For a Lévy process X in $\mathbb{R}^d$, is it true that

$$\gamma' = \inf\left\{\alpha \geq 0 : \lim_{r\to 0} r^{-\alpha} T(r) = \infty \ \text{ a.s.}\right\}?$$

Next we consider the packing dimension of $X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}_+$. First we note that if X satisfies the conditions in Theorem 4.14 and E has the property that $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, then Lemma 4.1 and Theorem 4.14 imply

$$\dim_{\mathrm{P}} X(E) = \dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{1}{\alpha}\dim_{\mathrm{P}} E\right\} \quad \mathbb{P}^x\text{-a.s.} \tag{4.14}$$

When $X = \{X(t),\ t \in \mathbb{R}_+\}$ is a strictly stable Lévy process in $\mathbb{R}^d$ with index $\beta \in (0,2]$ [so that X satisfies the conditions of Theorem 4.14 with $\alpha = 1/\beta$], Perkins and Taylor (1987) prove that, if $\beta \leq d$, then with probability 1

$$\dim_{\mathrm{P}} X(E) = \beta \dim_{\mathrm{P}} E \quad \text{for every Borel set} \quad E \subseteq \mathbb{R}_+. \tag{4.15}$$

This result is stronger than (4.14) since the exceptional null event does not depend on E [hence (4.15) is called a *uniform dimension result*; see Section 8 for more information]. However, when $\beta > d$ [i.e., $d = 1$ and $\beta > 1$], (4.14) does not even hold for Brownian motion W. Talagrand and Xiao (1996) construct a compact set $E \subset \mathbb{R}_+$ such that $\dim_{\mathrm{P}} W(E) < 2\dim_{\mathrm{P}} E$ a.s.; they also obtain the best possible lower bound for $\dim_{\mathrm{P}} W(E)$. Xiao (1997a) solves the problem of finding $\dim_{\mathrm{P}} W(E)$ by proving

$$\dim_{\mathrm{P}} W(E) = 2\mathrm{Dim}_{1/2} E \quad \text{a.s.}, \tag{4.16}$$

where $\mathrm{Dim}_s E$ is the packing dimension profile of E defined in (3.17). The arguments of Xiao (1997a) are still valid for fractional diffusions, hence results similar to (4.16) also hold for such processes. However, for a stable Lévy process X, the method used in Xiao (1997a) for obtaining an upper bound for $\dim_{\mathrm{P}} X(E)$ breaks down due to the existence of jumps. While we believe that this is only a technical difficulty and can be overcome by using special properties of stable Lévy processes, we do not know how to solve the following:

QUESTION 4.18. Let X be the Markov process as in Theorem 4.14. Find a general formula for $\dim_{\mathrm{P}} X(E)$.

We also mention that, if X is a Lévy process with stable components in $\mathbb{R}^d$ or an operator stable Lévy process in $\mathbb{R}^d$, the general formula for the packing dimension of $X(E)$ has not been established [a special case is $\dim_{\rm P} {\rm Gr}W(E)$, since ${\rm Gr}W(E)$ is the image of the space-time Brownian motion $(t, W(t))$]. In this case, the packing dimension profile in (3.17) does not seem to be appropriate for characterizing $\dim_{\rm P} X(E)$. One may need to introduce a corresponding concept of packing dimension profile that can capture different growths in different directions.

It is not clear to us whether Theorems 4.2, 4.14 and 4.15 can be applied to an Ornstein–Uhlenbeck type Markov process X in $\mathbb{R}^d$ associated to a general Lévy process Z and $d \times d$ matrix Q. As we mentioned earlier, both $\dim_{\rm H} X([0,1])$ and $\dim_{\rm P} X([0,1])$ are unknown.

Finally we mention that, besides fractal dimensions, it is often of interest to determine the topological structure of the range of a stochastic process X. Let X be a Lévy process in $\mathbb{R}_+$ and let $R_t = \overline{X([0,t])}$ be its closed range over the interval $[0,t]$. Since the sample functions of X are cadlag, we see that R_t is a perfect set for every $t > 0$ and

$$R_t = \big\{X(s),\ X(s-),\ 0 < s < t\big\} \cup \big\{X(0), X(t-)\big\}$$

[see e.g., Mountford and Port (1991, p.224) for a proof]. Kesten (1976) proves that for a class of Lévy processes that are "close to" a Cauchy process, R_t is a nowhere dense set with positive Lebesgue measure. He also gives a sufficient condition for R_t to contain an open interval surrounding $X(0)$. The latter result is related to the properties of the local times of X, as shown by Kesten (1976): if X has a local time $\ell(x,t)$ that is continuous in x, then R_t contains an open interval around $X(0)$. Barlow (1981) proves an important 0-1 law which asserts that either R_t is nowhere dense for every $t > 0$ a.s. or R_t contains an interval for every $t > 0$ a.s. The problem of classifying Lévy processes according to the structure of R_t has been investigated by several authors [Barlow (1981, 1985), Pruitt and Taylor (1985), Barlow et al. (1986a), Mountford and Port (1991)], but it has not been settled completely. See Section 6.1 for information on local time and its connection to the structure of R_t.[1]

5. Hausdorff and packing measure for the range and graph

There has been a long history of studying the exact Hausdorff measure of random sets related to the sample paths of Brownian motion, Lévy processes and Gaussian random fields, starting with the works of Lévy (1953), Taylor (1953), Ciesielski and Taylor (1962) and Taylor (1964) for Brownian motion in $\mathbb{R}^d$. The Hausdorff measure of the range and graph of Lévy stable processes were evaluated by Taylor (1967), Jain and Pruitt (1968), Pruitt and Taylor (1969), just to mention a few. We refer to Taylor (1986a) for an extensive summary of the related results and techniques for Lévy processes, along with a list of references. We note that the problem of determining the exact Hausdorff measure of the range of subordinators has been completely solved by Fristedt and Pruitt (1971). Their result is useful in studying the Hausdorff measure of the level sets of a Markov process. See Section 6 below.

It is worthwhile to mention that several authors have investigated similar problems for non-Markov processes and random fields. See Ehm (1981), Talagrand

[1] I would like to thank the referee for pointing out the work of Kesten and Barlow on the range of Lévy processes and their connections to local times.

(1995, 1998) and Xiao (1996, 1997b) for related results on the range and graph sets of the Brownian sheet and fractional Brownian motion. In turn, the arguments in their papers can sometimes be applied for studying the fractal properties of Markov processes as well.

Taylor and Tricot (1985) have evaluated the exact packing measure of the range of a transient Brownian motion in $\mathbb{R}^d$ (i.e., $d \geq 3$). The corresponding problems for the range of a planar Brownian motion and for the graph have been considered by Le Gall and Taylor (1986), Rezakhanlou and Taylor (1988), respectively. However, compared to the results on exact Hausdorff measure of random sets, fewer on their packing measure counterpart have been established for general Lévy processes or other Markov processes. See Sections 5.2, 6.2 and 9.2 for more details.

5.1. Hausdorff measure of $X([0,1])$. The study of exact Hausdorff measure of $X([0,1])$ or $\mathrm{Gr}X([0,1])$ consists of two parts: lower bound and upper bound. For a Markov process X, it is relatively easy to obtain a lower bound for the Hausdorff measure of $X([0,1])$. It follows from the LIL for the occupation measure of X [cf. Lemma 4.5] and Lemma 3.3 that

$$\varphi_1\text{-}m(X([0,1])) \geq K \quad \mathbb{P}^x\text{-a.s.,} \tag{5.1}$$

where $\varphi_1(r) = \tau(2r)\log\log 1/r,\ \tau(r) = \mathbb{E}[T(0,r)]$ is defined in Lemma 4.4 and $K > 0$ is a constant. In many cases such as when X is a stable Lévy process of *type A* or a Brownian motion on certain nested fractals, the function φ_1 is in fact an exact Hausdorff measure function for $X([0,1])$; see Taylor (1967), Wu and Xiao (2002a, b).

As for obtaining an upper bound, one needs to construct economic coverings for the range of X. This is usually more involved because an economic covering of $X([0,1])$ must reflect the fine structure such as the local oscillation behavior of the sample paths of X. Since the local oscillation of X may change from point to point, the sets (cubes or balls) in an economic covering must be of widely differing sizes. There are two different approaches in the literature, both of them use a "good point" and "bad point" argument. One involves the state space, while the other involves the parameter space.

a. In order to construct an economic covering for the range or graph of X, Taylor (1964, 1967) classified the points in the *state space* $\mathbb{R}^d$ into "good" points and "bad" points, according to the amount of sojourn time of the restarted process spent near these points. Results on hitting probabilities and strong Markov property are needed in order to estimate the number of dyadic cubes that contain bad points.
b. In constructing an economic covering for the range of fractional Brownian motion, Talagrand (1995) classified the points in the *parameter space* into "good" times and "bad" times according to the local asymptotic behavior of fBm at these times. Typically, $t_0 \in [0,1]$ is "good" if the oscillation of X around $X(t_0)$ is small on a sequence of intervals $[t_0 - r_n, t_0 + r_n]$, where $r_n \downarrow 0$, so that $X([t_0 - r_n, t_0 + r_n])$ can be covered by balls with small radius. Such asymptotic behavior is characterized by Chung's law of the iterated logarithm for fBm. That is why small ball probability estimates are useful in calculating the Hausdorff measure of the range and graph of X. See Li and Shao (2001) for an extensive survey on small ball probabilities and their applications. An advantage of Talagrand's approach is that results

on hitting probabilities of X, which are difficult to establish since fBm is not Markovian, are not needed. This method can sometimes be applied to continuous Markov processes such as fractional diffusions as well; see Wu and Xiao (2002a, b) for more details. On the other hand, Talagrand's argument does not apply directly to processes with discontinuous sample paths such as Lévy processes.

The following is a special case of a result from Wu and Xiao (2002b) about the Hausdorff measure of the range and graph of a class of Feller processes including certain fractional diffusion processes. Let S be $\mathbb{R}^d$ or a closed subset equipped with the Euclidean metric $|\cdot|$ and the Borel σ-algebra $\mathcal{S}$. Let μ be a σ-finite positive Radon measure on $(S, \mathcal{S})$ which satisfies Condition (2.25) in Definition 2.2. Recall that for a Markov process X on S satisfying Condition (d) in Definition 2.3, $\alpha = d_f/\beta$.

Theorem 5.1. *Let X be a strong Markov process on S satisfying Condition (d) in Definition 2.3. If $\alpha > 1$, then there exists a constant $K \geq 1$ such that for all $x \in S$, $\mathbb{P}^x$-almost surely*

$$K^{-1}t \leq \varphi_2\text{-}m(X([0,t])) \leq \varphi_2\text{-}m(\mathrm{Gr}X([0,t])) \leq Kt \tag{5.2}$$

for all $t > 0$, where $\varphi_2(r) = r^{d_f/\alpha} \log\log 1/r$. If $\alpha < 1$, then for all $x \in S$, $\mathbb{P}^x$-almost surely

$$K^{-1}t \leq \varphi_3\text{-}m(\mathrm{Gr}X([0,t])) \leq Kt \quad \textit{for all} \ \ t > 0, \tag{5.3}$$

where $\varphi_3(r) = r^{1-\alpha+d_f}(\log\log 1/r)^{\alpha}$.

To end this subsection, we mention that upper and lower bounds for the exact Hausdorff measure of $X(E)$, where $E \subset \mathbb{R}_+$, have been considered by Hawkes (1978b) for subordinators, and by Perkins and Taylor (1987) for stable Lévy processes. However, in their results, the Hausdorff measure functions for the lower bound and upper bound do not match. Recently, Li, Peres and Xiao (2002) have found an exact Hausdorff measure function for the image $W(E)$ of Brownian motion, where $E \subset \mathbb{R}_+$ is a self-similar set. It is of some interest to study the problem for more general processes and/or parameter sets E.

5.2. Packing measure of $X([0,1])$. The study of the exact packing measure of the range of a stochastic process has a more recent history. Taylor and Tricot (1985) proved the following theorem for Brownian motion in $\mathbb{R}^d$ ($d \geq 3$).

Theorem 5.2. *Let $W = \{W(t), t \in \mathbb{R}_+\}$ be Brownian motion in $\mathbb{R}^d$ with $d \geq 3$. Then there exists a positive and finite constant K such that with probability 1,*

$$\varphi_4\text{-}p\big(W([0,t])\big) = Kt \quad \textit{for all} \ \ t > 0, \tag{5.4}$$

where $\varphi_4(r) = r^2/(\log|\log r|)$.

Much as is the case for results on Hausdorff measures, the proof of Theorem 5.2 consists of two parts: lower bound and upper bound. For proving the lower bound, Taylor and Tricot (1985) appeal to the lower density theorem [cf. Lemma 3.5], which leads to proving

$$\liminf_{r\to 0} \frac{T_1(r) + T_2(r)}{\varphi_4(r)} = 2 \quad \text{a.s.,} \tag{5.5}$$

where T_1 and T_2 are independent copies of the sojourn time process $T = \{T(r), r \geq 0\}$. We note that (5.5) relies on the small ball probability of $T(r)$, i.e., $\mathbb{P}\{T(r) \leq x\}$, which, in turn, is related to the large tails of $M_1 = \max_{t \in [0,1]} |W(t)|$. See Taylor and Tricot (1985) for more details.

In order to prove the upper bound in (5.4), Taylor and Tricot (1985) use a "good point" or "bad point" argument that is dual to those in Taylor (1964, 1967), together with the upper inequality in (3.8). A different argument based on the local oscillation of the sample paths can be found in Xiao (1996, 2003).

For the planar Brownian motion, Le Gall and Taylor (1986) prove that for any measure function φ, the packing measure $\varphi\text{-}p\big(W([0,t])\big)$ is either 0 or ∞, and they give the following criterion:

THEOREM 5.3. *Let $W = \{W(t), t \in \mathbb{R}_+\}$ be Brownian motion in $\mathbb{R}^2$. If $\varphi(r) = r^2 \log(1/r)h(r)$, where $h : \ [0,1) \to [0,1)$ is monotone increasing but $\log(1/r)h(r)$ is decreasing, then with probability 1,*

$$\varphi\text{-}p\big(W([0,t])\big) = \begin{cases} 0 \\ \infty \end{cases} \quad \textit{according as} \quad \sum_{n=1}^{\infty} h\big(2^{-2^n}\big) \begin{cases} < \infty \\ = \infty. \end{cases} \tag{5.6}$$

As for the packing measure of the graph $\mathrm{Gr}W([0,1])$ of Brownian motion W in $\mathbb{R}^d$, Rezakhanlou and Taylor (1988) proved that if $d \geq 3$, then φ_4 in Theorem 5.2 is also a correct packing measure function for $\mathrm{Gr}W([0,1])$ [see also Xiao (2003) for a different proof]. However, if $d = 1$ or 2, then similar to (5.6) for any measure function $\varphi \in \Phi$, $\varphi\text{-}p\big(W([0,t])\big)$ is either 0 or ∞.

Taylor (1986b) proved that a result similar to (5.6) holds for the packing measure of the range of a stable Lévy process in $\mathbb{R}^d$ with index $\alpha < d$. The reason for this is that efficient packing of the range comes from using points on the sample path where there are unusually large jumps, or from another less intuitive point of view, that the small tail of the sojourn time $T(r)$ has a power-law decay. Further results on the asymmetric Cauchy process and subordinators have been established by Rezakhanlou and Taylor (1988), and Fristedt and Taylor (1992), respectively.

So far, no packing measure results have been established for Markov processes other than those mentioned above. It would be interesting to study the exact packing measure of the range and graph of fractional diffusions, jump diffusions and other Feller processes.

Finally, it is worthwhile to mention that the φ-Hausdorff measure and the φ-packing measure of the range of a function f are closely related to the weak and strong φ-variation of f, respectively; see Taylor and Tricot (1985). Strong and weak variations have been studied by Taylor (1972) for Brownian motion, Fristedt and Taylor (1973) for stable Lévy processes, Kawada and Kôno (1973) and Xiao (1997c) for certain Gaussian processes, and Marcus and Rosen (1994) for local times of Markov processes. It would be interesting to study the weak and strong variations of more general Feller processes such as diffusion processes on fractals, jump diffusions, and so on. The methods in Xiao (1997c) may be more convenient for studying the case of diffusion processes on fractals.

5.3. Further questions about the range. In the following, we list some unsolved problems about the Hausdorff and packing measure of the range of a Markov process X. Problem 5.4 was raised by Taylor (1986a, p.392) and it is still open in general.

Question 5.4. For a general Lévy process in $\mathbb{R}^d$, show that there is always a correct measure function $\varphi \in \Phi$ which makes φ-$m(X([0,1]))$ finite and positive, and give a method for constructing this φ.

Some partial results on the upper and lower bounds for the Hausdorff measure of the range of a symmetric Lévy process in $\mathbb{R}$ were obtained by Dupuis (1974). It follows from Lemmas 4.5 and 3.3 that, for the function $\varphi_1(r) = \tau(2r) \log\log 1/r$, the φ_1-Hausdorff measure of $X([0,1])$ is always bounded below by a positive constant. However, the Hausdorff measure φ_1-$m\big(X([0,1])\big)$ may not necessarily be finite, as shown by a strictly stable process of *Type B* with index $\alpha \in (0,1)$, for which a correct Hausdorff measure function for $X([0,1])$ is proved by Taylor (1967) to be $\varphi(r) = r^\alpha (\log\log 1/r)^{1-\alpha}$. Hence, the form of a correct Hausdorff measure function may also be sensitive to the asymptotic properties of $h(t,r) = \mathbb{P}\{|X(t)| \le r\}$, not just the average $\tau(r) = \mathbb{E}\{T(r)\}$. Before Problem 5.4 is solved completely, a reasonable question would be to find conditions on the process X to ensure that the function φ_1 (or some other function) is a correct Hausdorff measure function for $X([0,1])$. Another more specific, perhaps easier, problem is to find the exact Hausdorff measure function for the range of an operator stable Lévy process in $\mathbb{R}^d$; see Problem 5.6 below.

It would also be interesting to determine the exact Hausdorff measure functions for the range and other random sets of a Feller process X corresponding to a pseudo-differential operator with symbol $q(x,\xi)$. We have the following questions:

Question 5.5. Under what conditions on $q(x,\xi)$ can one determine an exact Hausdorff measure function φ for $X([0,1])$? How is φ related to $q(x,\xi)$? A natural starting point is to consider a symbol $q(x,\xi)$ that is comparable to that associated with a stable Lévy process.

Question 5.6. Find exact Hausdorff and packing measure functions for the image and graph of operator stable Lévy processes in $\mathbb{R}^d$. The Hausdorff and packing dimensions of $X([0,1])$, under some additional conditions, have been obtained by Becker–Kern et al. (2002).

Fristedt and Taylor (1992, p.494) believe that if a Lévy process X is sufficiently close to Brownian motion, then its range has an exact packing measure function. The following question is motivated by their remarks.

Question 5.7. Find conditions on the Lévy exponent ψ or the Lévy measure L of a symmetric Lévy process X in $\mathbb{R}^d$ ($d \ge 3$) so that $X([0,1])$ has an exact packing measure function.

6. Level sets of Markov processes and local times

Let $X = \{X(t), t \ge 0\}$ be a Markov process with values in S. In this section, we consider the level set

$$X^{-1}(x_0) = \{t \in \mathbb{R}_+ : \ X(t) = x_0\}, \qquad x_0 \in S.$$

Since we are interested in its fractal properties, it is first necessary to ask whether $X^{-1}(x_0)$ is empty or not. For Lévy processes, this problem has been solved completely by Kesten (1969) and Bretagnolle (1971).

To study the fractal properties of the level sets as well as those of the inverse image $X^{-1}(F)$ of a stochastic process X, we need to have a (random) measure supported on $X^{-1}(x_0)$. Such a measure can be extended from the local times $\ell(x_0, t)$ of X [see (6.2) below] when $\ell(x, t)$ has a version that is jointly continuous in (x, t); see Theorem 8.6.1 in Adler (1981). For a Markov process X, local times can give more information about its level sets, see property (LT) below.

6.1. Local times: existence and regularity. The literature on local times is very extensive. Our particular interest is to use the local times of a stochastic process X to study the fractal and other fine properties of the sample paths of X.

The following is a definition of local times of a stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$. For any Borel set $I \subseteq \mathbb{R}_+$, the occupation measure of X on I is defined as:

$$\mu_I(B) = \lambda_1\big\{t \in I : X(t) \in B\big\}, \qquad \forall B \in \mathcal{B}(\mathbb{R}^d). \tag{6.1}$$

If μ_I is absolutely continuous with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$, we say that $X(t)$ has *local times* on I, and define its local times, $\ell(\bullet, I)$, as the Radon–Nikodým derivative of μ_I with respect to λ_d, i.e.,

$$\ell(x, I) = \frac{d\mu_I}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d. \tag{6.2}$$

In the above, x is the so-called *space variable* and I is the *time variable*. Sometimes, we write $\ell(x, t)$ in place of $\ell(x, [0, t])$.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula*: for every Borel set $I \subseteq \mathbb{R}_+$, and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_I f(X(t))\, dt = \int_{\mathbb{R}^d} f(x)\ell(x, I)\, dx. \tag{6.3}$$

Bertoin (1996, Chapter V) gives a nice treatment of local times of a Lévy process as occupation densities. There are several different ways to define local times of a Markov process X, such as continuous additive functional, jumps across the level, etc. See the survey paper by Taylor (1973), Barlow et al. (1986a) and the references therein for more details.

The principal properties of the local times of a Markov process that are of particular interest to us are the following, which will be called (LT) property for easy reference.

(i). The inverse of $\ell(x, \cdot)$ is a (perhaps exponentially killed) subordinator [see Blumenthal and Getoor (1968, p.214) for the precise formulation]. When X is a Lévy process in $\mathbb{R}$ with exponent ψ, then the Laplace exponent g of the corresponding subordinator is

$$g(u) = \left[\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{u + \psi(x)}\right) dx\right]^{-1}. \tag{6.4}$$

(ii). The range of this subordinator differs from $X^{-1}(x)$ by at most a countable number of points.

See Bertoin (1996, pp.130–131) for a proof of (LT) when X is a Lévy process.

For a Lévy process $X = \{X(t), t \ge 0\}$ in $\mathbb{R}$, a necessary and sufficient condition for the existence of local times of X is that

$$\int_{\mathbb{R}} \mathrm{Re}\Big(\frac{1}{1+\psi(\xi)}\Big)\, d\xi < \infty. \tag{6.5}$$

This is proved by Hawkes (1986) and is based on a remarkable result of Kesten (1969, Theorem 2). See also Bertoin (1996, p.126).

In the following, we consider the regularity properties of local times when they exist.

(a). *Joint continuity.* Getoor and Kesten (1972) proved some necessary conditions and (different) sufficient conditions for the local times of a Markov process to have a version, still denoted by $\ell(x,t)$, which is continuous in (t,x) [so $\ell(x,t)$ is said to be jointly continuous]. Their results were improved by Barlow (1985) in the special case of Lévy processes. Later, a sufficient condition in terms of the metric entropy or majorizing measure were obtained by Barlow and Hawkes (1985), which is proven by Barlow (1988) to be necessary as well. The theorem in the form below is taken from Bertoin (1996). Let

$$m(\varepsilon) = \lambda_1\Big\{a \in \mathbb{R} : \frac{1}{\pi}\int_{\mathbb{R}} \big(1-\cos(a\xi)\big)\mathrm{Re}\Big(\frac{1}{\psi(\xi)}\Big) d\xi < \varepsilon\Big\}.$$

THEOREM 6.1. *Let X be a Lévy process in $\mathbb{R}$ with exponent ψ satisfying (6.5). Assume further that 0 is regular for $\{0\}$. Then X has a jointly continuous local time $\ell(x,t)$ if and only if*

$$\int_{0+} \sqrt{\log 1/m(\varepsilon)}\; d\varepsilon < \infty.$$

Marcus and Rosen (1992) give an alternative approach to the joint continuity of local times of symmetric Markov processes. The key of their approach is to connect [via the Dynkin-type isomorphism theorem] the joint continuity of $\ell(x,t)$ with the sample path continuity of a mean zero stationary Gaussian process G having $u^1(x,y)$ as its covariance function, where $u^1(x,y)$ is the 1-potential density of X. Then they apply the Dudley–Fernique theorem or the theorem of Talagrand (1987) on the regularity of Gaussian processes. See Marcus and Rosen (1992, 2001) for more details.

In yet another different approach, based on moment estimates, Berman (1985) has proved some sufficient conditions for the joint continuity of the local times of Markov processes, in terms of their transition density functions. This method is applicable to fractional diffusions. However, for Brownian motions on the Sierpinski gasket and carpets, much more can be done. By applying the bounds on the potential kernel densities of such diffusions, Barlow and Perkins (1988), Barlow and Bass (1990, 1992) have established the joint continuity and modulus of continuity for their local times. See also Barlow (1998, Theorem 3.32).

As we mentioned at the end of Section 4.2, the local time $\ell(x,t)$ of a Markov process X has a close connection with the structure of the closed range $R_t = \overline{X([0,t])}$. It is easy to see that if for fixed $t > 0$, $\ell(x,t)$ is continuous in the space variable x, then the open set $\{x : \ell(x,t) > 0\} \subseteq R_t$. Several authors have investigated the structure of R_t for Lévy processes; see Kesten (1976), Barlow's (1981), Pruitt and Taylor (1985), Barlow et al. (1986a), Mountford and Port (1991). However, the problem of determining when R_t is nowhere dense has not been completely solved. In particular, the following problem from Barlow et al. (1986a) remains open.

QUESTION 6.2. Suppose $X = \{X(t), t \in \mathbb{R}_+\}$ is a Lévy process in $\mathbb{R}$ with Lévy measure L such that $\mathsf{L}(-\infty, 0) = \mathsf{L}(0, \infty) = \infty$,

$$\int_{\mathbb{R}} \big(|x| \wedge 1\big) \mathsf{L}(dx) = \infty \quad \text{and} \quad \int_{\mathbb{R}} \operatorname{Re} \frac{1}{1+\psi(\xi)} d\xi < \infty.$$

If a.s. no continuous version of $\ell(x,t)$ exists, does it follow that the range R_t is a.s. nowhere dense?

(b). *Laws of the iterated logarithm and moduli of continuity.* Denoting by $\ell(x,t)$ the local time of Brownian motion W in $\mathbb{R}$, the following laws of the iterated logarithm (LIL) for $\ell(0,t)$ and the maximum local time $\ell^*(t) = \sup_{x\in\mathbb{R}} \ell(x,t)$ of W were established by Kesten (1965):

$$\limsup_{t\to 0+} \frac{\ell(0,t)}{\sqrt{t \log\log t^{-1}}} = \limsup_{t\to 0+} \frac{\ell^*(t)}{\sqrt{t \log\log t^{-1}}} = \sqrt{2} \quad \text{a.s.} \tag{6.6}$$

and

$$\liminf_{t\to 0+} \Big(\frac{\log\log t^{-1}}{t}\Big)^{1/2} \ell^*(t) = K_4 \quad \text{a.s.,} \tag{6.7}$$

where $K_4 > 0$ is a constant. As applications of their large deviation methods, Donsker and Varadhan (1977, p.752) showed the following LIL similar to (6.6) for the local time $\ell(x,t)$ of a symmetric stable Lévy process X of index $\alpha \in (1,2]$:

$$\limsup_{t\to 0+} \frac{\ell(0,t)}{t^{1-1/\alpha}(\log\log t^{-1})^{1/\alpha}} = \limsup_{t\to 0+} \frac{\ell^*(t)}{t^{1-1/\alpha}(\log\log t^{-1})^{1/\alpha}} = c(\alpha) \quad \text{a.s.,} \tag{6.8}$$

where $c(\alpha) > 0$ is an explicit constant. Marcus and Rosen (1994) extended the above results to all symmetric Lévy process with Lévy exponent ψ that is regularly varying of index $\alpha \in (1,2]$. See also Bertoin (1995) for a different approach based on subordinators.

From the inequality

$$t = \int_{\mathbb{R}} \ell(x,t) dx \le 2\ell^*(t) \sup_{0\le s\le t} |X(s)|,$$

one can see that results of the form (6.8) on local times are closely related to the oscillation properties of the sample paths of the process X. This can be made precise by proving the so-called Chung type law of the iterated logarithm [also called the other LIL]. For strictly stable Lévy process X of type A with index α, Taylor (1967, Theorem 4) showed that

$$\liminf_{t\to 0+} \Big(\frac{\log\log 1/t}{t}\Big)^{1/\alpha} \sup_{0\le s\le t} |X(s)| = K_5 \quad \text{a.s.} \tag{6.9}$$

His proof is based on estimates for the small ball probability $\mathbb{P}\big\{\sup_{0\le s\le t} |X(s)| \le \varepsilon\big\}$ [see Li and Shao (2001) for more information on small ball probabilities and their applications]. By using large deviations methods, Donsker and Varadhan (1977, p.752) give another proof of (6.9) for symmetric stable Lévy processes. More generally, Wee (1988) studies the lower functions for a Lévy process. We mention that there is also a uniform version of (6.9) for Brownian motion [cf. Csörgő and Révész (1978)] and other Lévy processes [cf. Hawkes (1971c)].

For the local time $\ell(0,t)$ of a general Lévy process X in $\mathbb{R}$, the uniform modulus of continuity in the time variable t is obtained by Fristedt and Pruitt (1972); see also Bertoin (1995, 642–643). The fast points and slow points of the local time $\ell(0,t)$ have been studied by Marsalle (2000); see Section 12 for related results of Shieh and Taylor (1998). On the other hand, the uniform modulus of continuity of the maximum local time $\ell^*(t)$ has been established by Perkins (1981, 1986) for Brownian motion and strictly stable processes.

Lacey (1990) considered large deviation estimates for the maximum local time $\ell^*(1)$ of a strictly stable Lévy process X of index $\alpha \in (1,2]$ and proved that

$$\log \mathbb{P}\big\{\ell^*(1) > u\big\} \sim -K_6 u^\alpha \quad \text{as } \ u \to \infty, \tag{6.10}$$

where $K_6 > 0$ is an explicit constant, which equals $c(\alpha)$ in (6.8) when X is symmetric. (6.10) matches with the result on $\mathbb{P}\big\{\ell(0,1) > u\big\}$, obtained by Hawkes (1971c). Wee (1997) and Blackburn (2000) have extended (6.10) to a Lévy process with exponent ψ that is regularly varying at 0 with index $\alpha \in (1,2]$.

For a Lévy process X in $\mathbb{R}$, the modulus of continuity of the local time $\ell(x,t)$ in the space variable x has been established by Barlow (1985, 1988) and Marcus and Rosen (1992), using different methods. [For the local times of Brownian motion, the results are due to McKean (1962) and Ray (1963); see e.g., Barlow (1988) for these references]. The following result for stable Lévy processes is from Barlow (1988): If X is a stable Lévy process in $\mathbb{R}$ of index $\alpha > 1$, then almost surely for all intervals $I \subset \mathbb{R}$ and all $t > 0$,

$$\lim_{\delta \downarrow 0} \sup_{\substack{a,\, b \,\in\, I \,:\, |b-a| < \delta \\ 0 \le s \le t}} \frac{|\ell(b,s) - \ell(a,s)|}{|b-a|^{(\alpha-1)/2}\big(\log(1/|b-a|)\big)^{1/2}} = c_\alpha \left(\sup_{x \in I} \ell(x,t)\right)^{1/2},$$

where $c_\alpha > 0$ is an explicit constant depending on the index α and the skewness parameter of $X(1)$ [see (2.10)] only. Applying an isomorphism theorem of Dynkin, Marcus and Rosen (1992, Theorem XIII) prove a similar result for the local times of general symmetric Markov processes.

The liminf law of the iterated logarithm (6.7) for the maximum local times of Brownian motion has been extended to a symmetric stable Lévy process X in $\mathbb{R}$ by Griffiin (1985) and to more general Lévy processes by Wee (1992). Unlike (6.8), no liminf law of the iterated logarithm can hold for $\ell(0,t)$, see Taylor (1986b).

For diffusion processes on fractals, the regularity properties of their local times have been studied in Barlow and Perkins (1988), Barlow and Bass (1992) and Barlow (1998). Large deviation type results and Chung-type LILs analogous to (6.8) for the maximum local times have recently been obtained by Fukushima et al. (1999) of Brownian motion on the nested fractals, and more generally by Bass and Kumagai (2000).

It seems to me that not much work has been done for local times of Feller processes determined by pseudo-differential operators or stable jump diffusions. It would be interesting to see to what extent the above results for Lévy processes are still true for these Feller processes.

6.2. Fractal dimension and measure results. (a). *Dimension results.* The Hausdorff dimension of the zero set $X^{-1}(0)$ was obtained by Taylor (1955) for Brownian motion in $\mathbb{R}$ and by Blumenthal and Getoor (1962) for symmetric stable process in $\mathbb{R}$ with index $\alpha > 1$. They proved that

$$\dim_{\mathrm{H}} X^{-1}(0) = 1 - \frac{1}{\alpha} \quad \text{a.s.}$$

For a general Lévy process X with values in $\mathbb{R}$, Blumenthal and Getoor (1964, pp.63–64) obtained upper and lower bounds for $\dim_{\mathrm{H}} X^{-1}(x)$ in terms of the indices β and β''.

Because of the property (LT), one can always use the result of Horowitz (1968) on the Hausdorff dimension of the range of a subordinator to find the Hausdorff dimension of the level set of a Markov process. It is also possible to obtain the packing dimension of $X^{-1}(0)$ by using the index γ' of the corresponding subordinator. The only possible disadvantage of this approach is that sometimes the dimension is not expressed in terms of the original process X explicitly.

In the case of a Lévy process in $\mathbb{R}$ with exponent ψ, Hawkes (1974) studied the Hausdorff dimension of its zero set directly and proved the following formula in terms of ψ:

$$\dim_{\mathrm{H}} X^{-1}(0) = 1 - \frac{1}{b} \quad \text{a.s.,} \tag{6.11}$$

where

$$\frac{1}{b} = \inf \Big\{ \gamma \le 1 : \big(1 + \mathrm{Re}\,[\psi^{\gamma}]\big)^{-1} \in L^1(\mathbb{R}) \Big\}$$

and the infimum of the empty set is taken as 1 here. Hawkes' proof is based on the results of Kesten (1969) and a subordination [or co-dimension] argument. It is worthwhile to note that Hawkes (1974) has also shown that the parameter b is independent of the other indices of Lévy processes in Section 2.1 and has obtained some results on the relationship between b and $\beta,\ \beta', \gamma$.

Much as in (6.11), it would be interesting to express the packing dimension $\dim_{\mathrm{P}} X^{-1}(0)$ in terms of the exponent ψ. This question is related to Problem 4.16.

The Hausdorff and packing dimensions of the level sets of other Markov processes such as diffusions and Brownian motion on fractals have been considered by Liu and Xiao (1998), Bertoin (1999, Section 9.3).

(b). *Hausdorff and packing measure of the level sets.* Taylor (1973, pp.406–407) describes a recipe for obtaining an exact Hausdorff measure function for the level set of a Markov process. The basic idea is to use (LT) and the result of Fristedt and Pruitt (1971) on the exact Hausdorff measure of the range of a subordinator.

Sometimes, it is more convenient to study the exact Hausdorff measure of the level sets of a Markov process directly. Moreover, this is the only approach for non-Markov processes because no relationship analogous to (LT) between the level set of a non-Markov process and the range of a tractable process has been established in that case. The direct approach uses the local times as a natural measure on the level set $X^{-1}(x)$. Then LIL for $\ell(x, \cdot)$ of the form

$$\limsup_{r \to 0} \frac{\ell(x, t+r) - \ell(x, t-r)}{\varphi(r)} \le K \quad \text{a.s.}$$

and Lemma 3.3 give a positive lower bound for $\varphi\text{-}m\big(X^{-1}(x)\big)$. In order to obtain an upper bound, one can use a covering argument similar to those discussed in Section 5.1; see e.g., Xiao (1997d).

Since the packing measure of the range of an arbitrary subordinator has been studied by Fristedt and Taylor (1992), one can evaluate the packing measure of the level sets of a Markov process by using property (LT) and the results in Fristedt and Taylor (1992). It would be interesting to find conditions on a Lévy process X that ensure that φ-$p\big(X^{-1}(0)\big)$ is 0, positive and finite or ∞, respectively.

Question 6.3. Find an exact Hausdorff measure function for the level sets of Feller processes determined by pseudo-differential operators or stable jump diffusions. Study the packing measure of their level sets.

Finally, we mention that the zero set of a Markov process is also related to the collision problem of Markov processes. Let $X_1, X_2, \ldots, X_k$ be k independent Markov processes with values in S. The collision problem concerns the following questions:

(i) Under what conditions does there exist $t > 0$ such that

$$X_1(t) = X_2(t) = \cdots = X_k(t)?$$

(ii) If the $X_1, \ldots, X_k$ do "collide", what are the Hausdorff and packing dimensions of the "set of collision points"

$$C_k = \big\{x \in S : X_1(t) = X_2(t) = \cdots = X_k(t) = x \quad \text{for some} \quad t > 0\big\}$$

and the set of "collision times"

$$D_k = \big\{t > 0 : X_1(t) = X_2(t) = \cdots = X_k(t)\big\}?$$

The above problems were first considered by Jain and Pruitt (1969) for two independent stable processes in $\mathbb{R}$ with indices α_1 and α_2. Assume that $\alpha_2 \le \alpha_1$, for convenience. Jain and Pruitt (1969) showed that collision exists almost surely if both $1 < \alpha_2 \le \alpha_1 \le 2$. This condition was weakened by Hawkes (1971b, c) to $\alpha_1 > 1$. They also obtained the Hausdorff dimensions of C_2 and D_2, as follows:

$$\dim_{\mathrm{H}} C_2 = 1 - \frac{1}{\alpha_1}, \qquad \dim_{\mathrm{H}} D_2 = \alpha_2\big(1 - \frac{1}{\alpha_1}\big) \quad \text{a.s.}$$

See also Hawkes and Pruitt (1974, Theorem 5.3) and the survey of Pruitt (1975) for more information. The above results on the existence of collisions have been extended to Lévy processes on the line by Shieh (1989) and to more general Markov processes by Shieh (1995). Shieh uses local time arguments for proving the existence, and potential theory [see Blumenthal and Getoor (1968, Chapter VI)] for proving the non-existence of collisions. See also Bertoin (1999, Section 9.3) for the study of the collision problem for diffusions.

The following problem has not been solved, even for stable Lévy processes.

Question 6.4. Find, if they exist, exact Hausdorff and packing measure functions for C_k and D_k.

7. Inverse images and hitting probabilities

Let $X = \{X(t), t \ge 0\}$ be a Markov process with values in a metric space S. Again, we just consider the case when $S = \mathbb{R}^d$. This section is concerned with the question of determining when $X^{-1}(F) \cap E \ne \emptyset$ with positive probability, where $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$ are Borel sets, and with the computation of the Hausdorff and packing dimensions of $X^{-1}(F) \cap E$, when this intersection is not empty.

7.1. Conditions for $X^{-1}(F) \cap E \neq \emptyset$. It is well-known that if X is an isotropic stable Lévy process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$ and $E = (0,\infty)$, then a compact set $F \subset \mathbb{R}^d$ satisfies $\mathbb{P}\{X^{-1}(F) \cap E \neq \emptyset\} = 0$ (i.e., F is polar for X) if and only if $\mathrm{Cap}_{d-\alpha}(F) = 0$, where $\mathrm{Cap}_{d-\alpha}(\cdot)$ denotes the Bessel–Riesz capacity of order $d-\alpha$. Kanda (1976) proved that this is true for all non-degenerate stable Lévy processes in $\mathbb{R}^d$ with index $\alpha \neq 1$. For the asymmetric Cauchy process X on the line, Port and Stone (1969) showed earlier that X hits points, and hence that there are no non-empty polar sets. For more information on the potential theory of Lévy processes, we refer to Hawkes (1975, 1979), Bertoin (1996, Chapter 2) and Sato (1999, Chapter 8).

The results on the probability of a general Markov process hitting a Borel set F in the state space can be found in Blumenthal and Getoor (1968), Dellacherie, Maisonneuve and Meyer (1992).

Let $E \subset \mathbb{R}_+$ and $F \subset \mathbb{R}^d$ be compact sets. The question of determining when $\mathbb{P}\big\{X^{-1}(F) \cap E \neq \emptyset\big\} = 0$ is related to the potential theory for the highly singular Markov process $\big\{(t, X(t)), t \in \mathbb{R}_+\big\}$ with values in $\mathbb{R}_+ \times \mathbb{R}^d$. Some sufficient conditions and necessary conditions for $\mathbb{P}\{X^{-1}(F) \cap E \neq \emptyset\} = 0$ have been obtained by Kaufman (1972) for Brownian motion, by Hawkes (1978a) for stable subordinators and Kahane (1983, 1985b) for symmetric stable Lévy processes in $\mathbb{R}^d$. See also Testard (1987) and Xiao (1999) for results on (fractional) Brownian motion. The conditions are best stated in terms of Hausdorff measure and capacity on the product space $\mathbb{R}_+ \times \mathbb{R}^d$ equipped with an appropriate metric.

For any $0 < \eta \leq 1$, we define a metric on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$\rho_\eta\big((s,x),\ (t,y)\big) = \max\big\{\, |s-t|^\eta,\ |x-y| \,\big\}.$$

For any measure function $\varphi \in \Phi$, the φ-Hausdorff measure on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_\eta)$ is denoted by φ-m_η. The corresponding Hausdorff dimension is denoted by $\dim_\eta$.

The following theorem can be proven by methods similar to those in Testard (1987) and Xiao (1999); details will be given elsewhere.

THEOREM 7.1. *Let $X = \{X(t), t \geq 0\}$ be a strictly stable Lévy process of index α in $\mathbb{R}^d$ and let $E \subset (0,\infty)$ and $F \subset \mathbb{R}^d$ be compact sets. Let $\eta = \alpha$ if $0 < \alpha \leq 1$, and $\eta = \alpha^{-1}$ if $1 < \alpha \leq 2$.*

(i). *If s^d-$m_\eta(E \times F) = 0$, then $\mathbb{P}\big\{X^{-1}(F) \cap E \neq \emptyset\big\} = 0$.*

(ii). *If $\mathrm{Cap}_h(E \times F) > 0$, then $\mathbb{P}\{X^{-1}(F) \cap E \neq \emptyset\} > 0$, where*

$$h(s,t;x,y) = \frac{1}{\big[\rho_\eta\big((s,x),(t,y)\big)\big]^d}.$$

There is an obvious gap between the two conditions above. I believe that the condition in (ii) is actually a necessary and sufficient condition for $\mathbb{P}\{X^{-1}(F) \cap E \neq \emptyset\} > 0$. This is supported by a result of Kaufman and Wu (1982), who have shown that if X is a Brownian motion in $\mathbb{R}$, $E \subset (0,\infty)$ is compact and $F = \{x_0\}$, then $\mathbb{P}\big\{X^{-1}(F) \cap E \neq \emptyset\big\} > 0$ if and only if $\mathrm{Cap}_h(E \times F) > 0$.

7.2. Dimension results on $X^{-1}(F) \cap E$. Applying results from the potential theory for isotropic stable Lévy processes and a subordination argument, Hawkes (1971a) proved the following theorem for isotropic stable Lévy processes. It follows from Theorem 5 in Hawkes (1971a) and Theorem 1 of Kanda (1976) [see also

Bertoin (1996, p.61) and Sato (1999, Theorem 42.30)] that the same results hold for all strictly stable Lévy processes.

THEOREM 7.2. *Let X be a strictly stable Lévy process of index α in $\mathbb{R}^d$. Let Γ be the support of the distribution of $X(1)$. If $\alpha \geq d$, then for every Borel set $F \subset \Gamma$,*

$$\dim_{\mathrm{H}} X^{-1}(F) = \frac{\alpha + \dim_{\mathrm{H}} F - d}{\alpha} \qquad a.s.; \tag{7.1}$$

and if $\alpha < d$, then

$$\|\dim_{\mathrm{H}} X^{-1}(F)\|_\infty = \frac{\alpha + \dim_{\mathrm{H}} F - d}{\alpha}, \tag{7.2}$$

where $\|\cdot\|_\infty$ is the L^∞-norm in the underlying probability space.

REMARK 7.3. It follows from Taylor (1967) and Port and Vital (1988) that Γ is a convex cone with the origin as its vertex; and if X is of *type A*, then $\Gamma = \mathbb{R}^d$.

The problem of finding the packing dimension of $X^{-1}(F)$, when X is a strictly stable Lévy process in $\mathbb{R}^d$, has not been solved completely. When $\alpha > d = 1$, it is possible to prove a result analogous to (7.1), in the uniform sense; see Section 8 for more information. However, when $\alpha < d$, we suspect that a result analogous to (7.2) may not hold in general and that $\dim_{\mathrm{P}} F$ alone may not be enough for determining $\dim_{\mathrm{P}} X^{-1}(F)$. It would be interesting to investigate this question.

Not much work on $\dim_{\mathrm{H}} X^{-1}(F)$ has been done for a general Lévy process X or other Markov processes. The following question seems to be of interest.

QUESTION 7.4. Let X be a Lévy process in $\mathbb{R}^d$ with exponent ψ. Is it possible to give a formula for $\dim_{\mathrm{H}} X^{-1}(F)$ in terms of ψ and $\dim_{\mathrm{H}} F$?

Now we turn to the Hausdorff dimension of the intersection $X^{-1}(F) \cap E$. When $X = \{X(t), t \in \mathbb{R}_+\}$ is a Brownian motion in $\mathbb{R}$, the Hausdorff dimensions of $X^{-1}(F) \cap E$ and $X(E) \cap F$ were considered by Kaufman (1972). Hawkes (1978a) generalizes Kaufman's results to stable subordinators. Their results can be stated as

$$\|\dim_{\mathrm{H}} \big(X^{-1}(F) \cap E\big)\|_\infty = \dim_\alpha(E \times F) - \frac{1}{\alpha},$$

where $\alpha = 2$ if X is a Brownian motion in $\mathbb{R}$ and by convention, the fact that the dimension is negative means that the set $X^{-1}(F) \cap E$ is empty. This result can be proved to hold for all strictly stable Lévy processes. However, the packing dimension of $X^{-1}(F) \cap E$ is unknown even for Brownian motion.

A related question is to find the Hausdorff dimension of the smallest set $F \subset \mathbb{R}^d \backslash \{0\}$ that can be hit by a Brownian motion or a stable Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}^d$, when t is restricted to some Borel set $E \subset \mathbb{R}_+$. To be more precise, given $E \subset \mathbb{R}_+$, determine the following infimum:

$$\inf\Big\{\dim_{\mathrm{H}} F : F \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P}\big\{X^{-1}(F) \cap E \neq \emptyset\big\} > 0\Big\}.$$

This question was raised by Y. Peres in 1996 and deals only with the Hausdorff dimension. However, packing dimension is needed in order to answer it.

Xiao (1999) solved this and related problems for Brownian motion X in $\mathbb{R}^d$ and proved that for any compact set $E \subset (0, \infty)$,

$$\inf\Big\{\dim_{\mathrm{H}} F : F \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P}\big\{X^{-1}(F) \cap E \neq \emptyset\big\} > 0\Big\} = d - 2\dim_{\mathrm{P}} E.$$

The exact Hausdorff measure of $X^{-1}(F)$ seems difficult to study in general. It is reasonable to first consider the case when X is a Brownian motion and $F \subset \mathbb{R}^d$ a self-similar set. On the other hand, it is possible to estimate the capacity of $X^{-1}(F)$ in terms of X and the capacity of F. This problem has been considered by Hawkes (1998) for symmetric stable process in $\mathbb{R}$ of index $\alpha \in (0, 2]$ and by Khoshnevisan and Xiao (2003b) for general Lévy processes. See Section 10 for related results.

8. Uniform dimension and measure results

We note that the exceptional null probability events in (4.13) and (7.1) depend on E and $F \subset \mathbb{R}^d$, respectively. In many applications, we have a random time set $E(\omega)$ or $F(\omega) \subset \mathbb{R}^d$ and wish to know the fractal dimensions and fractal measures of $X(E(\omega), \omega)$ and $X^{-1}(F(\omega), \omega)$. For example, for any Borel set $F \subset \mathbb{R}^d$, we can write the intersection $X(\mathbb{R}_+) \cap F$ as $X(X^{-1}(F))$, the set C_k of collision points as $X(D_k)$ and the set M_k of k-multiple points of X as $X(L'_k)$, where L'_k is the projection of L_k into $\mathbb{R}_+$; see Section 9.1. For such problems, the results of the form (4.13) and (7.1) give no information.

8.1. Uniform dimension results for the image. Kaufman (1968) was the first to show that if W is the planar Brownian motion, then

$$\mathbb{P}\Big\{\dim_{\rm H} W(E) = 2\dim_{\rm H} E \ \text{ for all Borel sets } E \subset \mathbb{R}_+\Big\} = 1. \tag{8.1}$$

Since the exceptional null probability event in (8.1) does not depend on E, it is referred to as a *uniform dimension result*. For Brownian motion in $\mathbb{R}$, (8.1) does not hold. This can be seen by taking $E = W^{-1}(0)$. A little surprisingly, Kaufman (1989) showed that with probability one,

$$\dim_{\rm H} W(E+t) = \min\big\{1, 2\dim_{\rm H} E\big\}$$

for all Borel sets $E \subset \mathbb{R}_+$ and almost all $t > 0$. Here the exceptional null probability event does not depend on t or E.

Several authors have worked on the problem of establishing uniform dimension results for the range of stable Lévy processes and other Markov processes. See the survey papers of Pruitt (1975) and Taylor (1986a) for more information. We just mention that Hawkes and Pruitt (1974) proved that for any strictly stable Lévy process X of index α in $\mathbb{R}^d$ with $\alpha \le d$,

$$\mathbb{P}\Big\{\dim_{\rm H} X(E) = \alpha\dim_{\rm H} E \ \text{ for all Borel sets } E \subset \mathbb{R}_+\Big\} = 1. \tag{8.2}$$

They also showed that for any Lévy process X in $\mathbb{R}^d$ with upper index β,

$$\mathbb{P}\Big\{\dim_{\rm H} X(E) \le \beta\dim_{\rm H} E \ \text{ for all Borel sets } E \subset \mathbb{R}_+\Big\} = 1$$

and if, in addition, X is a subordinator, then

$$\mathbb{P}\Big\{\sigma\dim_{\rm H} E \le \dim_{\rm H} X(E) \le \beta\dim_{\rm H} E \ \text{ for all Borel sets } E \subset \mathbb{R}_+\Big\} = 1. \tag{8.3}$$

Hawkes and Pruitt (1974) further showed that the upper and lower bounds in (8.3) are best possible. For a symmetric and transient Lévy process X in $\mathbb{R}^d$, a uniform lower bound for $\dim_{\rm H} X(E)$ in terms of the indices β'', γ and γ' was given by Hendricks (1983):

$$\mathbb{P}\Big\{\dim_{\rm H} X(E) \ge \beta''(d-\gamma')(d-\gamma)^{-1}\dim_{\rm H} E \ \text{ for all Borel sets } E \subset \mathbb{R}_+\Big\} = 1.$$

It follows that for symmetric and transient Lévy processes with $\gamma = \gamma'$, the uniform lower bound for $\dim_{\mathrm{H}} X(E)$ is $\beta'' \dim_{\mathrm{H}} E$. Using Lévy processes with stable components, one can easily show that both upper and lower bounds for $\dim_{\mathrm{H}} X(E)$ can not be improved; see Hendricks (1983).

It is known that for a general transient Lévy process, it is not always possible to find a function $f : [0,1] \to [0,d]$ such that almost surely

$$\dim_{\mathrm{H}} X(E) = f(\dim_{\mathrm{H}} E) \quad \text{ for all Borel sets } \ E \subset \mathbb{R}_+.$$

See Hendricks (1972), Hawkes and Pruitt (1974) for counterexamples. However, given a Lévy process X, it is still an interesting problem to determine whether it is possible to find a function f and a large class $\mathcal{C}$ of Borel sets $E \subset \mathbb{R}_+$ such that almost surely

$$\dim_{\mathrm{H}} X(E) = f(\dim_{\mathrm{H}} E) \quad \text{ for all Borel sets } \ E \in \mathcal{C}. \tag{8.4}$$

Hawkes and Pruitt (1974) studied this question for subordinators and they have shown that for any subordinator X with lower index σ,

$$\mathbb{P}\Big\{\dim_{\mathrm{H}} X(E) = \sigma \dim_{\mathrm{H}} E \ \text{ for all Borel sets } E \in \mathcal{C}\Big\} = 1, \tag{8.5}$$

where $\mathcal{C} = \{E \subset \mathbb{R}_+ : \dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E\}$ [their definition of $\mathcal{C}$ is different. By using an argument in Talagrand and Xiao (1996), one can see that the two definitions are equivalent]. It would be interesting to find the largest class $\mathcal{C}$ on which (8.5) holds. Such a result may be helpful for solving a problem in Hu and Taylor (2000) about "thin points" of the occupation measure of a general subordinator.

Uniform packing dimension results analogous to (8.1) and (8.2) can also be proved. It is worthwhile to note that Perkins and Taylor (1987) have established more precise information by proving uniform results on Hausdorff and packing measures of the images.

Clearly, it is useful to extend the above uniform Hausdorff and packing dimension results to more general Markov processes. If X satisfies a uniform Hölder condition, then upper bounds for both $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$ can be obtained by using Lemma 3.2. For a general Markov process, we follow the approach of Pruitt (1975) and state the following two uniform covering principles which can be applied to prove uniform upper and lower bounds for $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$.

Lemma 8.1 was proved by Hawkes and Pruitt (1974) for Lévy processes. The extension to more general Markov processes is not difficult.

We need some notation. Let $\{t_n,\ n \ge 1\}$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty t_n^p < \infty$ for some $p > 0$, and let $\mathcal{C}_n$ be a class of N_n intervals in $\mathbb{R}_+$ of length t_n with $\log N_n = O(1) |\log t_n|$. For example, we can take $t_n = 2^{-n}$ and $\mathcal{C}_n$ the class of dyadic intervals of order n in, say, $[0,1]$.

LEMMA 8.1. [For proving the upper bounds] *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a strong Markov process in $\mathbb{R}^d$ (or S). If there is a sequence $\{\theta_n\}$ of positive numbers such that for some $\delta > 0$,*

$$\mathbb{P}^x\Big\{\max_{0 \le s \le t_n} |X(s) - x| \ge \theta_n\Big\} \le K_7\, t_n^\delta, \quad \forall x \in \mathbb{R}^d, \tag{8.6}$$

then there exists a positive integer K_8, depending on p and δ only, such that, with probability one, for n large enough, $X(I)$ can be covered by K_8 balls of radius θ_n whenever $I \in \mathcal{C}_n$.

Lemma 8.1 can be applied to a large class of Markov processes including fractional diffusions and stable-like processes. For example, if X is a stable jump diffusion of index α as considered in Kolokoltsov (2000), we can choose $t_n = 2^{-n}$ and $\theta_n = 2^{-n/\beta}$ for some $\beta > \alpha$. It follows from Theorem 6.1 in Kolokoltsov (2000) that (8.6) is satisfied with $\delta = 1 - \alpha/\beta$. Consequently, an easy covering argument using Lemma 8.1 yields $\dim_{\rm H} X(E) \le \alpha \dim_{\rm H} E$ for all Borel sets $E \subset \mathbb{R}_+$. Similar result also holds for the stable-like processes on d-sets considered by Chen and Kumagai (2002).

In order to obtain uniform lower bounds for $\dim_{\rm H} X(E)$ and $\dim_{\rm P} X(E)$, we can use the second covering principle, which requires a condition on the delayed hitting probability of the process. Usually only a transient process X can satisfy (8.7).

LEMMA 8.2. [For proving the lower bounds] *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a strong Markov process in $\mathbb{R}^d$ (or S). Let $\{r_n, n \ge 1\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} r_n^p < \infty$ for some $p > 0$, and let $\mathcal{D}_n$ be a class of N_n balls of diameter r_n in $\mathbb{R}^d$ with $\log N_n = O(1)|\log r_n|$. If there exist a sequence $\{t_n\}$ of positive numbers and constants K_9 and $\delta > 0$ such that*

$$\mathbb{P}^x\Big\{ \inf_{t_n \le s < \infty} |X(s) - x| \le r_n \Big\} \le K_9 r_n^{\delta}, \quad \forall x \in \mathbb{R}^d, \tag{8.7}$$

then there exists a constant K_{10}, depending on p and δ only, such that a.s. for all n large enough, $X^{-1}(B)$ can be covered by at most K_{10} intervals of length r_n, whenever $B \in \mathcal{D}_n$.

8.2. Level sets and inverse image. Uniform Hausdorff and packing dimension results for the level sets of stable Lévy processes in $\mathbb{R}$ with index $\alpha \in (1, 2]$ follow directly from the uniform dimension results for the images of stable subordinators. In fact, for the level sets $X^{-1}(x)$ of a class of Lévy processes, uniform results [i.e., the exceptional null probability event does not depend on x] on the exact Hausdorff measure of $X^{-1}(x)$ have been determined.

When X is a Brownian motion in $\mathbb{R}$, Perkins (1981) has proved that with probability 1,

$$\varphi_5\text{-}m\big(B^{-1}(x)\big) = \ell(x,t) \quad \text{for all } \ (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $\varphi_5(r) = (2r \log\log 1/r)^{1/2}$. This result has been extended by Barlow, Perkins and Taylor (1986b) to a class of Lévy processes with exponents that are regularly varying at ∞. This includes strictly stable Lévy processes with index $\alpha > 1$, Lévy processes with Brownian components and Lévy processes that are close to Cauchy processes.

As for the inverse image $X^{-1}(F)$ of a Markov process X, a uniform dimension result has only been established in the Brownian motion case. Kaufman (1985) has shown that with probability 1,

$$\dim_{\rm H} W^{-1}(F) = \frac{1}{2} + \frac{1}{2}\dim_{\rm H} F \quad \text{for all Borel sets } \ F \subset \mathbb{R}.$$

His proof makes use of the Hölder continuity of B as well as the Hölder continuity of the Brownian local time in the time variable.

For a strictly stable Lévy process X in $\mathbb{R}$ of index $\alpha \in (1, 2)$, the analogous result for $X^{-1}(F)$ should also be true. In fact Kaufman's argument, together with the Hölder conditions for the local times of X established by Donsker and Varadhan (1977), gives a.s.

$$\dim_{\mathrm{H}} X^{-1}(F) \geq 1 - \frac{1}{\alpha} + \frac{1}{\alpha}\dim_{\mathrm{H}} F \quad \text{for all Borel sets } \ F \subset \mathbb{R}.$$

The reverse inequality requires a little more effort, and will be dealt with in a subsequent paper.

9. Multiple points and self-intersection local times

Taylor (1986a, Section 7) contains a historical account of the classical results of Dvoretzky, Erdös, Kakutani and Taylor in the 50's about the multiple points of Brownian motion in $\mathbb{R}^d$. Their original proofs are based on the potential theory of Brownian motion and combinatorial analysis. A nice proof of the existence theorem using an elementary argument based on the self-similarity and Markov property of Brownian motion is given by Khoshnevisan (2003).

Since the late 80's, a lot of progress has been made in the studies of multiple points. Many of the problems and conjectures in Taylor (1986a, Section 7) regarding Lévy processes have been solved by Le Gall (1987a, b), LeGall et al. (1989), Evans (1987a), Fitzsimmons and Salisbury (1989). In this section, we discuss some of their results.

9.1. Existence of the multiple points. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a stochastic process with values in a metric space (S, ρ). A point $x \in S$ is called a *k-multiple point* of X if there exist k distinct times $t_1, t_2, \cdots, t_k \in \mathbb{R}_+$ such that

$$X(t_1) = \cdots = X(t_k) = x.$$

If $k = 2$ (or 3), then x is also called a *double* (or *triple*) point.

The set of k-multiple points is denoted by M_k (or $M_k^{(d)}$ if $S = \mathbb{R}^d$) and the set of *k-multiple times* is denoted by

$$L_k = \big\{(t_1, \cdots, t_k) \in \mathbb{R}_+^k,\ t_1, \ldots, t_k \text{ are distinct and } \ X(t_1) = \cdots = X(t_k)\big\}.$$

When $S = \mathbb{R}^d$, we may also write $L_k^{(d)}$ for L_k.

Given a Markov process X, there are several ways to study the existence of k-multiple points of X:

(a) potential theory for X [see Taylor (1986) and the references therein].

(b) self-intersection local times [Geman et al. (1984), Dynkin (1985), Rosen (1983, 1987), Le Gall, Rosen and Shieh (1989), Rogers (1989), Shieh (1992), etc.]. The idea is that the intersections of Markov processes can be formulated as the zero set problem of a random field, say, Y, which can be effectively studied by the method of local times. The local times of Y are called the self-intersection local times of X [note that in some papers, e.g., Le Gall (1986b), an intersection local time is also referred to as a measure on M_k]. Intersection local times have been under extensive study for their own right. The particular interest here is to use them to define a random measure on the set L_k of multiple times. Using this approach, not only one can prove the existence of k-multiple points, but also the results on the Hausdorff and packing dimensions and measures of L_k and M_k.

(c) Wiener or Lévy sausages [Le Gall (1986b, 1987a, b)]. They are used to define a random measure on the set M_k of multiple points. Exact Hausdorff and packing measure of M_k can be studied.
(d) potential theory for multiparameter processes [Evans (1987a, b), Fitzsimmons and Salisbury (1989), Khoshnevisan and Xiao (2002)].
(e) intersection equivalence to independent percolation [Peres (1996a, 1999)].

Le Gall, Rosen and Shieh (1989) give a sufficient condition for the existence of k-multiple points of a Lévy process by constructing a random measure on L_k. They require Lévy processes having transition density functions.

By using a potential-theoretic and Fourier analytic approach, Evans (1987a) has weakened the conditions of Le Gall, Rosen and Shieh (1989) by only assuming that X has a resolvent density. Applying potential theory for multiparameter Markov processes, Fitzsimmons and Salisbury (1989) prove that Evans' condition is also necessary. Thus, the combined results of the above authors have verified the Hendricks–Taylor conjecture concerning the existence of k-multiple points of a Lévy process.

To state their results, we need some notation. Let X be a Lévy process in $\mathbb{R}^d$ with transition function $P(t,x,A) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to [0,1]$. For all $q > 0, z \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$, set

$$U^q(z,B) = \int_0^\infty e^{-qs} P(s,z,B)ds.$$

Under the assumption that X has a strong Feller resolvent operator, there exists for each $q > 0$ a unique measurable function u^q such that

(i) $U^q(z,B) = \int_B u^q(y-z)dy$ for all $B \in \mathcal{B}(\mathbb{R}^d)$,
(ii) for every y, the function $z \mapsto u^q(y-z)$ is q-excessive,
(iii) $u^q - u^r = (r-q)u^r \star u^q$.

See Hawkes (1979) or Bertoin (1996, Section I.3). $\{u^q, q > 0\}$ is called the family of canonical resolvent densities.

Theorem 9.1. *Let X be a Lévy process in $\mathbb{R}^d$ with canonical resolvent densities $\{u^q, q > 0\}$ and $u^1(0) > 0$. Then, for any integer $k \geq 2$, the sample paths of X have k-multiple points almost surely if and only if*

$$\int_{|x|\leq 1} \left[u^1(x)\right]^k dx < \infty.$$

We mention that the existence of k-multiple points of a Lévy process can also be related to the zero set of an additive Lévy processes, cf. Khoshnevisan and Xiao (2002).

Rogers (1989) has extended the sufficiency part of Theorem 9.1 to certain Markov processes on a complete metric space. His results can be applied to fractional diffusions [cf. Barlow (1998, p.40)] and stable-like processes. It is not clear whether his condition is also necessary for the existence of k-multiple points in these more general settings.

There are two ways to further investigate the existence of multiple points of a Markov process. The first is to restrict the time t to some fractal-type sets. Kahane (1983) has considered the intersection of $X(E)$ and $X(F)$, where $E, F \subset \mathbb{R}_+ \backslash \{0\}$ are disjoint compact sets and X is a symmetric stable Lévy processes. He gives necessary conditions and sufficient conditions for $\mathbb{P}\{X(E) \cap X(F) \neq \emptyset\} > 0$. A necessary and sufficient condition in terms of a suitable capacity of $E \times F$ has recently been obtained by Khoshnevisan and Xiao (2003b); see Section 10.

Using the approach of self-intersection local times, Shieh (1992) proves some sufficient conditions for $X(E)$ to contain k-multiple points, where X is a certain Markov process such as an elliptic diffusion or a Lévy process in $\mathbb{R}^d$. It is a natural question to look for a necessary and sufficient condition similar to (10.8) for such processes.

The second refinement is to ask what set $\Lambda \subset \mathbb{R}^d$ can contain k-multiple points of X. To put it another way, when can $\mathbb{P}\{\Lambda \cap M_k \neq \emptyset\}$ be positive?

When X is a Brownian motion in $\mathbb{R}^d$ $(d = 2, 3)$, this question was considered by Evans (1987b) and Tongring (1988), who proved some sufficient conditions and different necessary conditions for $\mathbb{P}\{\Lambda \cap M_k \neq \emptyset\} > 0$. Fitzsimmons and Salisbury (1989) proved that the sufficient condition of Evans (1987b) and Tongring (1988) for planar Brownian motion is also necessary. By using the approach of intersection equivalence, Peres (1999, Corollary 15.4) proves the following much more general result. In particular, it can be applied to a large class of Lévy processes.

THEOREM 9.2. *Suppose* $\{A_i\}_{i=1}^k$ *are independent random closed sets of* $[0, 1]^d$ *and there exists a constant* $1 \le K < \infty$ *such that*

$$K^{-1}\mathrm{Cap}_{g_i}(\Lambda) \le \mathbb{P}\{A_i \cap \Lambda \neq \emptyset\} \le K\,\mathrm{Cap}_{g_i}(\Lambda)$$

for all closed sets $\Lambda \subset [0, 1]^d$ *and some non-negative and non-increasing functions* g_i $(i = 1, \ldots, k)$*. Then*

$$\mathbb{P}\big\{A_1 \cap \ldots \cap A_k \cap \Lambda \neq \emptyset\big\} > 0 \iff \mathrm{Cap}_{g_1 \cdots g_k}(\Lambda) > 0.$$

Similar results for the intersections of zero sets of independent Lévy processes have been obtained in Khoshnevisan and Xiao (2002, Theorem 6.1), by using potential theory for additive Lévy processes.

9.2. Hausdorff dimension and measure of M_k and L_k. The Hausdorff dimensions of the sets M_k of k-multiple points for Brownian motion in $\mathbb{R}^d$ were obtained by Taylor (1966) for $d = 2$ [and $k \ge 2$] and by Fristedt (1967) for $d = 3$ [and $k = 2$]. These results can also be proved by finding $\dim_{\mathrm{H}} L'_k$ first [recall that L'_k is the projection of L_k into $\mathbb{R}_+$] and then using the uniform Hausdorff dimension result (8.2). If X is either a Brownian motion on $\mathbb{R}^2$ or a symmetric Cauchy process on $\mathbb{R}$, there are points with multiplicity $\mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum. Le Gall (1986a, 1987b) has proved that, given any totally disconnected compact set $E \subset \mathbb{R}_+$, there exists a.s. a $z \in \mathbb{R}^2$ such that $W^{-1}(z)$ has the same order structure as E. In particular, there are points of multiplicity $\aleph_0$ for X. About the size of $W^{-1}(z)$, Taylor (1986a, p.395) raises the question of determining measure functions $\varphi \in \Phi$ such that a.s. φ-$m\big(W^{-1}(z)\big) = 0$ for all $z \in \mathbb{R}^2$. This problem has not been resolved. Taylor (1986a) points out that the results in Perkins and Taylor (1987) imply that if $b > 2$, then a.s. $(\log 1/r)^{-b}$-$m\big(W^{-1}(z)\big) = 0$ for all $z \in \mathbb{R}^2$; and he conjectures that the critical value for b is 1. That is, the function $\varphi_b(s) = (\log 1/r)^{-b}$ satisfies the above condition for $b > 1$, but not for $0 < b < 1$. On the other hand, Bass, Burdzy and Khoshnevisan (1994) have investigated an intersection local time for planar Brownian motion W at points of infinite multiplicity. Their results indicate that the set of points of infinite multiplicity may have a multifractal structure.

The Hausdorff dimension of M_k has been studied by Hawkes (1978c) for an isotropic Lévy process with transition density function, and by Hendricks (1974) and Shieh (1998) for a special class of operator stable Lévy processes.

Hawkes (1978c) proves the following result: if X is an isotropic transient Lévy process with transition density $p_t(x,y) = p_t(|x-y|)$ such that $\int_0^\infty p_t(x)dt$ is monotone in $|x|$, then

$$\dim_{\rm H} M_k = \max\big\{k\gamma - (k-1)d,\ 0\big\} \quad \text{a.s.,} \tag{9.1}$$

where γ is the index of X defined by (2.19). If X is not isotropic, e.g., if X is a Lévy process in $\mathbb{R}^d$ with stable components considered by Pruitt and Taylor (1969), then the index γ alone is not enough to determine $\dim_{\rm H} M_k$. This can be seen from the results of Hendricks (1974) and Shieh (1998) for a class of operator stable Lévy processes. We note that even for this special class of Lévy processes, the problem of finding $\dim_{\rm H} M_k$ has not been settled completely. Hence the following question is interesting.

Question 9.3. Let X be a Lévy processes in $\mathbb{R}^d$. Find general formulas for $\dim_{\rm H} M_k$ and $\dim_{\rm P} M_k$.

Now we turn to the problem of finding the exact Hausdorff measure function for the set M_k of k-multiple points. For Brownian motion $W = \{W(t), t \in \mathbb{R}_+\}$ on $\mathbb{R}^d$, this problem has been completely resolved by Le Gall (1986b, 1987a, 1989). To restate his results in Le Gall (1989), let

$$h_k(r) = r^2\big(\log 1/r \log\log\log 1/r\big)^k, \qquad k \ge 2$$

and

$$\tilde{h}_2(r) = r\big(\log\log 1/r\big)^2.$$

Let $\ell_k^{(d)}(\cdot)$ be the image measure of the k-th order self-intersection local time $\alpha_k^{(d)}$ of W [note that $\alpha_k^{(d)}$ is a random measure on $L_k^{(d)}$] under the mapping $(t_1, \cdots, t_k) \mapsto W(t_1)$. This is a random measure carried by $M_k^{(d)}$ and it is called the *projected self-intersection local time.*

Theorem 9.4. *Let $W = \{W(t), t \ge 0\}$ be a Brownian motion in $\mathbb{R}^d$.*

(i). *If $d = 2$, then for every integer $k \ge 2$, there exists a positive constant c_k such that a.s.*

$$h_k\text{-}m(F \cap M_k^{(2)}) = c_k \ell_k^{(2)}(F) \quad \textit{for all} \ \ F \in \mathcal{B}(\mathbb{R}^2).$$

(ii). *If $d = 3$, then there exists a positive constant K_{11} such that a.s.*

$$\tilde{h}_2\text{-}m(F \cap M_2^{(3)}) = K_{11} \ell_2^{(3)}(F) \quad \textit{for all} \ \ F \in \mathcal{B}(\mathbb{R}^3).$$

Partial results on the Hausdorff measure of M_k for general Lévy processes have also been obtained by Le Gall (1987b). His approach consists of two parts. In the first part, he considers the set N_k of the intersection points of $X_1, \ldots, X_k$, which are k independent copies of X, and constructs directly a random measure μ_k on N_k as the normalized limit of the Lebesgue measure of the sausages:

$$\mu_k(A) = \lim_{\epsilon \to 0} \big[C(\epsilon)\big]^{-k} \lambda_d\Big(S_1(\epsilon) \cap S_2(\epsilon) \cdots \cap S_k(\varepsilon) \cap A\Big),$$

where $C(\varepsilon)$ is the capacity of the ball $B(0,\varepsilon)$ and $S_i(\varepsilon)$ is the ε-sausage of X_i defined by

$$S_i(\varepsilon) = \bigcup_{s\in\mathbb{R}} \big(X_i(s) + B(0,\varepsilon)\big).$$

Then he establishes bounds on the moments of $\mu_k(A)$ and apply them to derive upper and lower bounds for the Hausdorff measure of N_k. More specifically, he has found measure functions φ^* and ψ^* such that

$$\varphi^*\text{-}m\big(N_k \cap I\big) < \infty \qquad \text{for all compact sets } I \subset \mathbb{R}^d$$

and

$$\psi^*\text{-}m\big(N_k \cap A\big) \geq K\,\mu_k(A) \qquad \text{for all Borel sets } \ A \subset \mathbb{R}^d,$$

where $K > 0$ is a constant depending on d, k and the laws of X only. The second part of his argument is easy: since M_k can be identified with the set N_k of intersection points of independent copies $X_1, \ldots, X_k$ of X with different starting points, the result on M_k follows. However, it is not known when we can have $\varphi^* \asymp \psi^*$. Hence no exact Hausdorff measure function for M_k has yet been determined.

The Hausdorff dimension of the set $L_k^{(d)}$ of multiple times for Brownian motion W in $\mathbb{R}^d$ $(d = 2, 3)$ has been obtained by Rosen (1983) as follows:

$$\dim_{\rm H} L_2^{(3)} = \frac{1}{2} \qquad \text{and} \qquad \dim_{\rm H} L_k^{(2)} = 1 \ \text{ for all } \ k \geq 2. \tag{9.2}$$

He also conjectured that an exact Hausdorff measure function for $L_k^{(d)}$ is

$$\varphi_k^{(d)}(r) = r^{2-d/2}\big(\log\log 1/r\big)^{d(k-1)/2}.$$

Zhou (1994) verifies this conjecture for $d = 3$; i.e., an exact Hausdorff measure function for $L_2^{(3)}$ is

$$\varphi_2^{(3)}(r) = r^{1/2}\big(\log\log 1/r\big)^{3/2}.$$

For $d = 2$, the analogous problem remains open.

In general, the Hausdorff dimension of L_k is not known for Lévy processes. However, if X is a symmetric Lévy process in $\mathbb{R}^d$ with exponent ψ such that $\xi \mapsto e^{-t\psi(\xi)}$ is in $L^1(\mathbb{R}^d)$, then $\dim_{\rm H} L_k$ can be derived from Theorem 1.10 in Khoshnevisan and Xiao (2002):

$$\dim_{\rm H} L_k = \sup\Big\{b > 0 : \int_{[0,1]^k} \frac{1}{|\mathbf{s}|^b}\Phi(\mathbf{s})d\mathbf{s} < \infty\Big\},$$

where Φ is the gauge function on $\mathbb{R}^k$ defined by

$$\Phi(\mathbf{s}) = (2\pi)^{-d} \int_{\mathbb{R}^{(k-1)d}} \exp\Big(-\sum_{j=1}^k |s_j|\psi\big(\xi_j - \xi_{j-1}\big)\Big)d\xi$$

for $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbb{R}^k$. In particular, if X is a symmetric stable Lévy processes in $\mathbb{R}^d$ with index $\alpha \in (0, 2]$ and such that $\alpha k > (k-1)d$ [i.e., $L_k \neq \emptyset$], then

$$\dim_{\rm H} L_k = k - \frac{(k-1)d}{\alpha}.$$

This extends Rosen's result (9.2).

Since Problem 5.4 has not been solved, it may be relatively easier to consider the following less general problem.

QUESTION 9.5. Let X be a strictly stable Lévy processes in $\mathbb{R}^d$ or a fractional diffusion. Find Hausdorff measure functions h and φ such that

$$0 < h\text{-}m(M_k) < \infty \quad \text{and} \quad 0 < \varphi\text{-}m(L_k) < \infty.$$

Related to this problem, Le Gall (1987b, p.372) conjectures that if X is a symmetric stable Lévy process of index α in $\mathbb{R}^d$, then an exact Hausdorff measure function for M_k is

$$h(r) = r^a \big(\log\log 1/r\big)^k$$

if $\alpha < d$ and $a = k\alpha - (k-1)d > 0$; and for $\alpha = d = 1$,

$$h(r) = r\big(\log 1/r \, \log\log\log 1/r\big)^k.$$

Finally, we consider the exact packing measure of the set M_k of k-multiple points. Le Gall (1987b) proves that, if X is a Brownian motion in $\mathbb{R}^2$, then, for every integer $k \ge 2$, $M_k^{(2)}$ does not have an exact packing measure function and he gives an integral test for $\varphi\text{-}p(M_k^{(2)}) = 0$ or ∞. More precisely, the following is Theorem 5.1 of Le Gall (1987b).

THEOREM 9.6. *Suppose* $f : (0, \infty) \to \mathbb{R}_+$ *is a decreasing function such that* $r \mapsto r^k f(r)$ *is increasing for* r *large enough. Let*

$$\varphi(r) = r^2 \big(\log 1/r\big)^k f\big(\log 1/r\big).$$

Then

$$\varphi\text{-}p(M_k^{(2)}) = \begin{cases} 0 \\ \infty \end{cases} \quad \textit{according to whether} \quad \sum_{n=1}^{\infty} f(2^n) \begin{cases} < 0 \\ = \infty. \end{cases}$$

For Brownian motion in $\mathbb{R}^3$, Le Gall (1987b) has only obtained the following partial result for the packing measure of $M_2^{(3)}$: For $\beta > 0$, let

$$\varphi_\beta(r) = r\,\big(\log 1/r\big)^{-\beta},$$

then (i) there exists a $\beta > 0$ such that $\varphi_\beta\text{-}p(M_2^{(3)}) = \infty$ a.s. and (ii) $\varphi_\beta\text{-}p(M_2^{(3)}) = 0$ a.s. if $\beta > 1$.

The problems of finding the exact packing measure functions for $M_2^{(3)}$ and $L_k^{(d)}$ have not yet been solved. It is plausible that in the Brownian motion case, $M_2^{(3)}$ has an exact packing measure function, but $L_k^{(d)}$ ($d = 2$ and 3) do not. The latter problems are related to the liminf behavior of the self-intersection local times, which seems to be more difficult to study than the limsup behavior.

10. Exact capacity results

In the following, we discuss some exact capacity results of Kahane (1983), Hawkes (1978b, 1998), Khoshnevisan and Xiao (2003b) for the range and inverse image of Lévy processes, and their applications to intersections of Lévy processes.

Let X be a stable Lévy process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$ (including the asymmetric Cauchy process), Blumenthal and Getoor (1960b) proved that for any Borel set $E \subset \mathbb{R}_+$,

$$\dim_{\text{H}} X(E) = \min\big\{d,\ \alpha \dim_{\text{H}} E\big\} \quad \text{a.s.} \tag{10.1}$$

On the other hand, Hawkes (1971a) considered the Hausdorff dimension of the inverse image

$$X^{-1}(F) = \{t \in \mathbb{R}_+ : X(t) \in F\}$$

of a strictly stable process X in $\mathbb{R}^d$ of index α [cf. Theorem 7.2]. Once $\dim_{\mathrm{H}} X(E)$ or $\dim_{\mathrm{H}} X^{-1}(F)$ is known, it is of interest to further investigate the exact Hausdorff measure functions for $X(E)$ and $X^{-1}(F)$ or their capacities. For the former, even though there have been a lot of work in the case when $E = \mathbb{R}_+$ and $F = \{0\}$, few results exist for general E and F and the problems seem to be quite difficult. For the latter, several authors have worked on characterizing the capacities of $X(E)$ and $X^{-1}(F)$ in terms of the capacities of E and F for any Borel sets $E \subset \mathbb{R}_+$ and $F \subset \mathbb{R}^d$.

When X is a symmetric stable process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$, Kahane (1985b, Theorem 8) proved that for any Borel set $E \subset \mathbb{R}_+$,

$$s^{\gamma}\text{-}m(E) = 0 \Longrightarrow s^{\alpha\gamma}\text{-}m(X(E)) = 0 \quad \text{a.s.} \tag{10.2}$$

and if $\alpha\gamma < d$, then

$$\mathrm{Cap}_{\gamma}(E) > 0 \Longrightarrow \mathrm{Cap}_{\alpha\gamma}\big(X(E)\big) > 0 \ \text{ a.s.} \tag{10.3}$$

On the other hand, Hawkes (1998) has recently proved that if X is a stable subordinator of index $\alpha \in (0, 1)$, then for any Borel set $E \subset \mathbb{R}_+$ and $\gamma \in (0, 1)$,

$$\mathrm{Cap}_{\gamma}(E) > 0 \Longleftrightarrow \mathrm{Cap}_{\alpha\gamma}\big(X(E)\big) > 0 \ \text{ a.s.} \tag{10.4}$$

See also Hawkes (1978b) for a related result. We note that Hawkes' argument uses specific properties of stable subordinators and does not work for other stable processes; further, while Kahane's proof of (10.2) depends crucially on the self-similarity of strictly stable processes, it does not apply to general Lévy processes either.

The following theorem of Khoshnevisan and Xiao (2003b) strengthens and extends the results of Kahane (1985b) and Hawkes (1998, Theorem 4) mentioned above.

Theorem 10.1. *Let X be a symmetric Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ. For any $0 < \gamma < d$ and any Borel set $E \subset \mathbb{R}_+$, the event $\{\mathrm{Cap}_{\gamma}\big(X(E)\big) > 0\}$ satisfies a zero-one law; and*

$$\mathrm{Cap}_{\gamma}\big(X(E)\big) > 0 \ \ \textit{a.s.} \quad \Longleftrightarrow \quad \mathrm{Cap}_{\Phi_1}\big(E \times \mathbb{R}_+^M\big) > 0, \tag{10.5}$$

where

$$\Phi_1(s, x) = \int_{\mathbb{R}^d} \exp\Big(-|s|\psi(\xi) - \sum_{j=1}^{M} |x_j| \cdot |\xi|^{\beta}\Big) d\xi, \quad (s, x) \in \mathbb{R} \times \mathbb{R}^M$$

and $M \in \mathbb{N}$ and $\beta \in (0, 2]$ are chosen to satisfy $\gamma = d - M\beta$.

When X is a strictly stable Lévy process of index $\alpha \in (0, 2]$, the kernel Φ_1 in Theorem 10.1 can be replaced by a Bessel–Riesz type kernel with respect to a different (asymmetric) metric, as shown by the following Corollary 10.2.

Corollary 10.2. Let X be a symmetric Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ satisfying

$$\psi(\xi) \asymp |\xi|^{\alpha}, \quad \xi \in \mathbb{R}^d,$$

for some $\alpha \in (0, 2]$. Then for any $0 < \gamma < d$,

$$\mathrm{Cap}_\gamma\big(X(E)\big) > 0 \ \text{ a.s.} \iff \mathrm{Cap}_{\Phi_2}\big(E \times \mathbb{R}_+^M\big) > 0, \tag{10.6}$$

where

$$\Phi_2(s, x) = \frac{1}{\max\{|s|^{d/\alpha},\ |x|^{d/\beta}\}}, \qquad (s, x) \in \mathbb{R} \times \mathbb{R}^M$$

and $M \in \mathbb{N}$ and $\beta \in (0, \alpha)$ are chosen so that $\gamma = d - M\beta$.

REMARK 10.3. Since we have chosen $\beta \in (0, \alpha)$, the following formula

$$\rho\big((s, x), (t, y)\big) = \max\big\{|s - t|^{\beta/\alpha},\ |x - y|\big\}$$

defines a metric on $\mathbb{R} \times \mathbb{R}^M$. The kernel Φ_2 can be written in the form

$$\Phi_2(s, x) = \frac{1}{\rho\big((0, 0), (s, x)\big)^{d/\beta}} \qquad (s, x) \in \mathbb{R} \times \mathbb{R}^M.$$

Thus, we can view Φ_2 as a Bessel–Riesz type kernel with respect to the metric ρ.

Capacities are also useful in studying self-intersections of Lévy processes. Kahane (1983) proved the following result: Let E_1 and E_2 be two compact sets contained in disjoint intervals and let X be a symmetric stable Lévy process in $\mathbb{R}^d$ of index α. Then

$$\begin{aligned} \mathrm{Cap}_{d/\alpha}(E_1 \times E_2) > 0 \ &\Longrightarrow\ \mathbb{P}\big\{X(E_1) \cap X(E_2) \neq \emptyset\big\} > 0 \\ &\Longrightarrow\ s^{d/\alpha}\text{-}m(E_1 \times E_2) > 0. \end{aligned} \tag{10.7}$$

Kahane (1983, p.90) conjectured that $\mathrm{Cap}_{d/\alpha}(E_1 \times E_2) > 0$ is necessary and sufficient for $\mathbb{P}\big\{X(E_1) \cap X(E_2) \neq \emptyset\big\} > 0$. This has been recently proven by Khoshnevisan and Xiao (2003b).

THEOREM 10.4. *Let X_1 and X_2 be two independent symmetric Lévy processes in $\mathbb{R}^d$ with Lévy exponents ψ_1 and ψ_2, respectively. We assume that for all $t > 0$, $X_1(t)$ has a density that is positive a.e. Then for any disjoint Borel sets $E, F \subset \mathbb{R}_+\backslash\{0\}$,*

$$\mathbb{P}\Big\{X_1(E) \cap X_2(F) \neq \emptyset\Big\} > 0 \iff \mathrm{Cap}_{\Phi_3}\big(E \times F\big) > 0, \tag{10.8}$$

where

$$\Phi_3(t) = \int_{\mathbb{R}^d} \exp\Big(- |t_1|\psi_1(\xi) - |t_2|\psi_2(\xi)\Big) d\xi, \qquad t = (t_1, t_2) \in \mathbb{R}^2.$$

The methods for proving Theorems 10.1 and 10.4 are based on the potential-theoretic results for additive Lévy processes established in Khoshnevisan and Xiao (2002, 2003a) and Khoshnevisan, Xiao and Zhong (2003).

Now we consider the exact capacity of the inverse image $X^{-1}(F)$ of a Lévy process X with values in $\mathbb{R}^d$. Hawkes (1998) proves that if X is a symmetric stable Lévy process in $\mathbb{R}$ of index $\alpha \in (0, 2]$ and $0 < \beta < 1$ satisfies $\alpha + \beta > 1$, then for every Borel set $F \subset \mathbb{R}$,

$$\mathrm{Cap}_{(\alpha+\beta-1)/\alpha}\big(X^{-1}(F)\big) = 0 \iff \mathrm{Cap}_\beta(F) = 0 \quad \text{a.s.}$$

Appealing to the potential theory of Lévy processes, Khoshnevisan and Xiao (2003b) extend this result to more general Lévy process.

Along similar lines, several authors have studied the following "capacitary modulus" problem for the range of a Lévy process. According to Rosen (2000), a function $h(x) : \mathbb{R}^d \to \mathbb{R}_+$ is called a *capacitary modulus* for $\Lambda \subset \mathbb{R}^d$ if there exist constants $0 < K_{12} \le K_{13} < \infty$ such that

$$\Big[K_{13} \int_{\mathbb{R}^d} f(|x|)h(x)dx\Big]^{-1} \le \mathrm{Cap}_f(\Lambda) \le \Big[K_{12} \int_{\mathbb{R}^d} f(|x|)h(x)dx\Big]^{-1}$$

for all $f : \mathbb{R}_+ \to [0, \infty]$. The point is that the constants K_{12} and K_{13} are independent of the kernel f. This type of results are closely related to intersections of independent Markov processes. See Section 9 for more information.

Let $W = \{W(t), t \ge 0\}$ be a Brownian motion in $\mathbb{R}^d$. Pemantle, Peres and Shapiro (1996) prove that the function

$$h(x) = \begin{cases} |x|^{-(d-2)} & \text{if } d \ge 3 \\ |\log x| & \text{if } d = 2 \end{cases} \tag{10.9}$$

is a capacity modulus for the range $W([0, 1])$. They have also shown that the function $h(x) = x^{-1/2}$ is a capacitary modulus for the zero set $W^{-1}(0)$. Their results have been extended by Rosen (2000) to a class of Lévy processes including the isotropic stable Lévy processes and subordinators.

Rosen (2000) believes the similar results should still hold for Lévy processes in the domain of attraction of general strictly stable Lévy processes in $\mathbb{R}^d$. It would be interesting to solve this problem, as well as to consider the capacitary modulus problem for other Markov processes such as diffusions on fractals.

11. Average densities and tangent measure distributions

The average density for the zero set of Brownian motion was studied by Bedford and Fisher (1992). Let W be a Brownian motion in $\mathbb{R}$ and let μ be the $(2r \log\log 1/r)^{1/2}$-Hausdorff measure of $W^{-1}(0)$. Bedford and Fisher (1992) proved that the order-two density of μ with respect to the gauge function $r \mapsto \sqrt{r}$ exists and $D_2^{1/2}(\mu, t) = 2/\sqrt{\pi}$ for μ-a.e. $t \in \mathbb{R}_+$. Falconer and Xiao (1995) proved the existence of order-two densities of the range $X([0, 1])$ of any strictly stable processes in $\mathbb{R}^d$ with index $\alpha < d$, thereby extending the result of Bedford and Fisher (1992). Similar problems for the range of planar Brownian motion was studied by Mörters (1998). It is interesting to note that Mörters (1998) proves that the order two densities of the range of planar Brownian motion do not exist, but the order-three densities do. The following is the result for Brownian motion from Falconer and Xiao (1995) and Mörters (1998).

THEOREM 11.1. *Let μ be the occupation measure of Brownian motion in $\mathbb{R}^d$ defined by*

$$\mu(B) = \lambda_1\{t \in [0, 1] : W(t) \in B\}, \qquad \forall B \in \mathcal{B}(\mathbb{R}^d).$$

Then with probability 1

(i). *If $d \ge 3$, then $D_2^2(\mu, x) = 2/(d-2)$ μ-a.e. $x \in \mathbb{R}^d$.*

(ii). *If $d = 2$, let $\varphi(r) = r^2 \log(1/r)$, then for μ-a.e. $x \in \mathbb{R}^2$, $D_2^\varphi(\mu, x)$ does not exist. However, $D_3^\varphi(\mu, x) = 2$ for μ-a.e. $x \in \mathbb{R}^2$.*

Similar results for the average densities of the set of multiple points of Brownian motion have been proven by Mörters and Shieh (1999).

For tangent measure distributions of the occupation measure μ of Brownian motion in $\mathbb{R}^2$, Mörters (2000, Theorem 1.2) proves the following result.

THEOREM 11.2. *Let μ be the occupation measure of Brownian motion in $\mathbb{R}^2$. Let $\varphi(r) = r^2 \log 1/r$. Then a.s. the φ-tangent measure distribution of order three of μ exists for μ-a.e. $x \in \mathbb{R}^2$ and is given by*

$$w\text{-}\mathrm{lim}_{\delta\downarrow 0}\frac{1}{\log|\log\delta|}\int_{\varepsilon}^{1/e} \mathbb{1}_M\Big(\frac{\mu_{x,r}}{\varphi(r)}\Big)\frac{dr}{r|\log r|} = \int_0^{\infty} \mathbb{1}_M(\frac{a}{\pi}\lambda_2)\, a\, e^{-a} da$$

for all Borel sets $M \subset \mathcal{M}(\mathbb{R}^2)$, where w-lim means weak convergence in $\mathcal{M}(\mathbb{R}^2)$, the space of all locally finite Borel measures on $\mathbb{R}^2$, and λ_2 is the Lebesgue measure on $\mathbb{R}^2$.

The corresponding problems regarding average densities and tangent measure distributions for the occupation measures of general Lévy processes [e.g., Cauchy processes] as well as diffusions on fractals have not been solved. It would be of interest to study them.

12. Multifractal analysis of Markov processes

In recent years, there has been a lot of interest in verifying the multifractal formalism and in evaluating the multifractal spectrum of various deterministic and random measures; see Olsen (2000) and the references therein. For a self-similar measure μ satisfying certain separation conditions, the multifractal spectra $f_\mu(\alpha)$ and $F_\alpha(\mu)$ of μ are defined through its local dimension and can be represented nicely as the Legendre transform of a convex function τ; see Section 3.4. However, for random measures associated to stochastic processes, as shown first by Perkins and Taylor (1998) for super Brownian motion and by Hu and Taylor (1997) for a stable subordinator, this fails to be of much use because either the function τ needed for the multifractal formalism has no valid definition [this is the case if μ is the occupation measure of a stable subordinator] or the local dimensions are the same everywhere on the support of the random measure μ [this is the case for the occupation measure of Brownian motion]. Thus, in order to capture the delicate fluctuations of the random measures involved, a refined notion of multifractal analysis is required.

To be more specific, we describe the results of Hu and Taylor (1997) on the occupation measure of a stable subordinator $X = \{X(t),\, t \in \mathbb{R}_+\}$ of index $\alpha \in (0,1)$ [see Dolgopyat and Sidorov (1995) for the special case of $\alpha = 1/2$]. Let μ be the occupation measure of X defined by (6.1). It follows from results related to the Hausdorff and packing dimension of $X([0,1])$ that a.s.

$$d(\mu,x) = \lim_{r\downarrow 0}\frac{\log\mu(x-r,x+r)}{\log r} = \alpha \quad \mu\text{-a.e. } x \in \mathbb{R}. \tag{12.1}$$

However, on the exceptional set where (12.1) is false, $d(\mu,x)$ does not exist. For $x \in \mathrm{supp}(\mu)$, consider the lower and upper local dimensions of μ at x:

$$\underline{d}(\mu,x) = \liminf_{r\to 0}\frac{\log\mu(x-r,x+r)}{\log r}$$

and $\overline{d}(\mu,x)$ defined similarly, but with a lim sup. Hu and Taylor (1997) show that a.s. $\underline{d}(\mu,x) = \alpha$ for every $x \in \mathrm{supp}(\mu)$; while for the random sets

$$C_\beta = \big\{x \in \mathrm{supp}(\mu) : \overline{d}(\mu,x) \ge \beta\big\}$$

and

$$D_\beta = \big\{x \in \mathrm{supp}(\mu) : \overline{d}(\mu,x) = \beta\big\},$$

they prove that $C_\beta = \emptyset$ for $\beta < \alpha$ or for $\beta > 2\alpha$, and $D_\beta \neq \emptyset$ for $\alpha \le \beta \le 2\alpha$. Moreover, in the latter case,

$$\dim_{\mathrm{H}} C_\beta = \dim_{\mathrm{H}} D_\beta = \frac{2\alpha^2}{\beta} - \alpha, \qquad \text{a.s.}$$

When $\alpha = 1/2$, $\dim_{\mathrm{H}} C_\beta$ has also been given by Proposition 2 of Dolgopyat and Sidorov (1995). We further remark that Hu and Taylor (2000) have extended the above results to a general subordinator X in $\mathbb{R}$. Since it is not known whether a uniform Hausdorff dimension result holds for the images of X [cf. (8.4)], their multifractal spectrum is given for the time set.

12.1. Limsup random fractals. First we recall some results on "limsup random fractals". This class of random fractals has been introduced by Dembo et al. (2000a, b), Khoshnevisan, Peres and Xiao (2000) to approximate random sets arising from the multifractal analysis of occupation measures and the sample paths of Brownian motion. They are also useful in studying various exceptional sets related to more general stochastic processes. The results in this section are from the above references. Some of the dimension properties of limsup random fractals can be found in Orey and Taylor (1974), Deheuvels and Mason (1998).

Let $N \ge 1$ be a fixed integer. For every integer $n \ge 1$, let $\mathcal{D}_n$ denote the collection of all hyper-cubes in $\mathbb{R}_+^N$ of the form

$$\big[k^1 2^{-n}, (k^1+1)2^{-n}\big] \times \cdots \times \big[k^N 2^{-n}, (k^N+1)2^{-n}\big],$$

where $k \in \mathbb{Z}_+^N$ is any N-dimensional positive integer. Suppose for each integer $n \ge 1$, $\big\{Z_n(I);\ I \in \mathcal{D}_n\big\}$ denotes a collection of random variables, each taking values in $\{0,1\}$. By a discrete *limsup random fractal*, we mean a random set of the form $A \hat{=} \limsup_n A(n)$, where,

$$A(n) \hat{=} \bigcup_{I \in \mathcal{D}_n : Z_n(I)=1} I^o,$$

where I^o denotes the interior of I.

In order to determine the hitting probabilities for a discrete limsup random fractal A, we assume the following two conditions on the random variables $\big\{Z_n(I);\ I \in \mathcal{D}_n\big\}$.

Condition 1: the index assumption. Suppose that for each $n \ge 1$, the mean $p_n \hat{=} \mathbb{E}\big[Z_n(I)\big]$ is the same for all $I \in \mathcal{D}_n$ and that

$$\lim_{n\to\infty} \frac{1}{n} \log_2 p_n = -\gamma,$$

for some $\gamma > 0$, where $\log_2$ is the logarithm in base 2. We refer to γ as the *index* of the limsup random fractal A.

Condition 2: a bound on the correlation length. For each $\varepsilon > 0$, define

$$f(n,\varepsilon) = \max_{I \in \mathcal{D}_n} \#\Big\{J \in \mathcal{D}_n :\ \mathrm{Cov}\big(Z_n(I), Z_n(J)\big) \ge \varepsilon \mathbb{E}\big[Z_n(I)\big]\mathbb{E}\big[Z_n(J)\big]\Big\}.$$

Suppose that $\delta > 0$ satisfies

$$\forall \varepsilon > 0, \qquad \limsup_{n\to\infty} \frac{1}{n} \log_2 f(n,\varepsilon) \le \delta\,.$$

If Condition 2 holds for every $\delta > 0$, then we say that **Condition** 2_* holds. The following theorem is from Khoshnevisan, Peres, and Xiao (2000) and Dembo et al. (2000a). Similar results under weaker conditions can be found in Dembo et al. (2000b).

THEOREM 12.1. *Suppose that* $A = \limsup_n A(n)$ *is a discrete limsup random fractal which satisfies Condition 1 with index* γ, *and Condition 2 for some* $\delta > 0$. *Then for any analytic set* $E \subset \mathbb{R}_+^N$,

$$\mathbb{P}\big(A \cap E \neq \emptyset\big) = \begin{cases} 1 & \text{if } \dim_{\mathrm{P}}(E) > \gamma + \delta, \\ 0 & \text{if } \dim_{\mathrm{P}}(E) < \gamma. \end{cases}$$

Moreover, if Condition 2_* *is satisfied, then for any analytic set* $E \subset \mathbb{R}_+^N$,

$$\dim_{\mathrm{H}}(E) - \gamma \le \dim_{\mathrm{H}}\big(A \cap E\big) \le \dim_{\mathrm{P}}(E) - \gamma \quad a.s. \tag{12.2}$$

In particular, $\dim_{\mathrm{H}}(A) = N - \gamma$, *a.s.*

Let W be a Brownian motion in $\mathbb{R}^d$ and let $\mathcal{U}$ be the class of sequences $\{(u_n, v_n)\}$ such that $u_n, v_n \ge 0$ and $u_n + v_n \downarrow 0$. Let h be a positive continuous function such that $h(x) \uparrow \infty$ as $x \downarrow 0$. Kôno (1977) studied the exact Hausdorff measure of the set F_h of "two-sided fast points" of W defined by

$$F_h = \Big\{t \in [0,1] :\ \exists\{(u_n, v_n)\} \in \mathcal{U},\ \ |W(t+u_n) - W(t-v_n)| \ge \sqrt{u_n + v_n}\, h(u_n+v_n)\Big\}$$

and showed that φ-$m(F_h) = 0$ or ∞ according to an integral test involving φ and h. Since F_h can be regarded approximately as a limsup random fractal, Kôno's result suggests that it would be interesting to study the exact Hausdorff measure of more general limsup random fractals. A solution of the following problem will have several interesting applications.

QUESTION 12.2. Let A be a limsup random fractal satisfying Conditions 1 and 2_*. Study the exact Hausdorff measure of A.

We note that Dembo et al. (2000a) have obtained some partial results about φ-$m(A)$.

12.2. Fast points of Brownian motion. Let $W = \{W(t), t \in \mathbb{R}_+\}$ be a linear Brownian motion. For $\lambda \in (0,1]$, Orey and Taylor (1974) have considered the set of λ-*fast points* for W, defined by

$$\mathrm{F}(\lambda) = \Big\{t \in [0,1] :\ \limsup_{h \to 0^+} \frac{\big|W(t+h) - W(t)\big|}{\sqrt{2h|\log h|}} \ge \lambda\Big\} \tag{12.3}$$

and have proved that

$$\forall \lambda \in (0,1], \quad \dim_{\mathrm{H}}\big(\mathrm{F}(\lambda)\big) = 1 - \lambda^2 \quad \text{a.s.} \tag{12.4}$$

Kaufman (1975) subsequently showed that any analytic set E with $\dim_{\mathrm{H}}(E) > \lambda^2$, a.s. intersects $\mathrm{F}(\lambda)$. The next theorem from Khoshnevisan, Peres and Xiao (2000) shows that packing dimension is the right index for deciding which sets intersect $\mathrm{F}(\lambda)$.

THEOREM 12.3. *Let* W *denote linear Brownian motion. For any analytic set* $E \subset \mathbb{R}_+$,

$$\sup_{t \in E} \limsup_{h \to 0+} \frac{\big|W(t+h) - W(t)\big|}{\sqrt{2h|\log h|}} = \big(\dim_{\mathrm{P}}(E)\big)^{1/2}, \quad a.s.$$

Equivalently,

$$\forall \lambda > 0, \quad \mathbb{P}\big(\mathrm{F}(\lambda) \cap E \neq \emptyset\big) = \begin{cases} 1 & \text{if } \dim_{\mathrm{P}}(E) > \lambda^2, \\ 0 & \text{if } \dim_{\mathrm{P}}(E) < \lambda^2. \end{cases} \tag{12.5}$$

Moreover, if $\dim_{\mathrm{P}}(E) > \lambda^2$ *then* $\dim_{\mathrm{P}}(F(\lambda) \cap E) = \dim_{\mathrm{P}}(E)$ *a.s.*

Remark 12.4. Condition (12.5) can be sharpened to a necessary and sufficient criterion for a compact set E to contain λ-fast points; see Khoshnevisan, Peres and Xiao (2000) for details.

Hausdorff dimension results for the exceptional times related to the functional laws of the iterated logarithm have been obtained by Deheuvels and Lifshits (1997), Deheuvels and Mason (1998) and Lucas (2002). Their arguments are based on those of Orey and Taylor (1974). The basic idea of Khoshnevisan, Peres and Xiao (2000) is that such exceptional times sets as $\mathrm{F}(\lambda)$ can be approximated by limsup random fractals and Theorem 12.4 follows from the general results on hitting probabilities of limsup random fractals. As another application of their arguments, Khoshnevisan, Peres and Xiao (2000) have strengthened the results of Deheuvels and Mason (1998).

Several authors have also studied the Hausdorff measure of the exceptional sets for Brownian motion, see Orey and Taylor (1974), Kôno (1977), Lucas (2002); but the problems of determining the exact Hausdorff measure of these exceptional sets have not been solved except for the case considered by Kôno (1977). It would be useful to develop some general techniques for studying the Hausdorff measure of a limsup type random fractals; see Problem 12.2.

When X is a symmetric stable Lévy process in $\mathbb{R}$ of index α, Orey and Taylor (1974) stated that for every $0 < \gamma < \alpha^{-1}$,

$$\dim_{\mathrm{H}}\Big\{t \in [0,1] : \limsup_{h \to 0} \frac{\big|X(t+h) - X(t)\big|}{h^\gamma} = \infty\Big\} = \alpha\gamma. \tag{12.6}$$

We note that the results in Khoshnevisan, Peres and Xiao (2000) are not applicable to processes with discontinuities. It would be interesting to relax some of the conditions there so that the general methods can be applied to Lévy processes and other Markov processes. In particular, such a result will be useful to study the fractal properties of the exceptional sets related to the following result of Hawkes (1971c) for a stable subordinator X of index α:

$$\lim_{\epsilon \to 0} \inf_{\substack{0 \le t \le 1 \\ 0 < h < \epsilon}} \frac{X(t+h) - X(t)}{\varphi(h)} = c_\alpha, \tag{12.7}$$

where $\varphi(h) = h^{1/\alpha} |\log h|^{-(1-\alpha)/\alpha}$ and $c_\alpha > 0$ is an explicit constant depending on α only.

Finally, it is worthwhile to mention that it has been proven [cf. Kahane (1985a)] that on the sample paths of a Brownian motion, there exist *slow points* t at which the oscillation of $W(t+h) - W(t)$ is of the order $\sqrt{h}$. The fractal properties of the set of slow points are different from those of fast points and we will not discuss them here. Instead we refer to Taylor (1986a) for more information and related references.

12.3. Thick and thin points for occupation measures. In this subsection, we describe the results of Shieh and Taylor (1998), Dembo et al. (1999, 2002a, b, 2001) on the thick and thin points for the occupation measures of Brownian motion and stable Lévy processes.

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Brownian motion in $\mathbb{R}^d$ $(d \geq 2)$ or a stable subordinator in $\mathbb{R}$ of index $\alpha \in (0,1)$. Let μ be the occupation measure of X. When X is Brownian motion, we will write μ as μ_W. It follows from the modulus of continuity and the uniform dimension results of Kaufman (1969) and Perkins and Taylor (1987) that the local dimension of μ_W at x is 2 for every $x \in W([0,1])$. For the occupation measure μ of a stable subordinator, the local dimension $d(\mu, x) = \alpha$ for μ-a.e. $x \in \mathrm{supp}(\mu)$, while at those exceptional points, $d(\mu, x)$ does not exist. Thus the ordinary multifractal spectrum defined in terms of the local dimension for μ is not of much use. However, the delicate fluctuation structure of μ can be captured by means of logarithmic corrections.

Shieh and Taylor (1998) study the random sets

$$A_\theta = \Big\{x \in X([0,1]) : \limsup_{r \to 0} \frac{\mu(x-r, x+r)}{c_\alpha r^\alpha (\log r^{-1})^{1-\alpha}} \geq \theta\Big\}$$

and

$$B_\theta = \Big\{x \in X([0,1]) : \limsup_{r \to 0} \frac{\mu(x-r, x+r)}{c_\alpha r^\alpha (\log r^{-1})^{1-\alpha}} = \theta\Big\},$$

where $c_\alpha > 0$ is an explicit constant. They have proved the following theorem:

Theorem 12.5. *Let μ be the occupation measure of a stable subordinator $X = \{X(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}$ with index $\alpha \in (0,1)$. If $\theta > 1$, then $A_\theta = \emptyset$ a.s. If $0 \leq \theta \leq 1$ then $B_\theta \neq \emptyset$ a.s. Moreover,*

$$\dim_{\mathrm{H}} A_\theta = \dim_{\mathrm{H}} B_\theta = \alpha\big(1 - \theta^{1/(1-\alpha)}\big). \tag{12.8}$$

They refer to (12.8) as the *logarithmic multifractal spectrum* of μ. We will follow the terminology of Dembo et al. (2000a, b, 2001) and call A_θ and B_θ the sets of thick points of the occupation measure μ. We mention that a similar result for the thick points of a subordinator with Laplace exponent that is regularly varying at infinity has been proven by Marsalle (1999).

In a series of papers, Dembo, Peres, Rosen and Zeitouni (2000a, b, 2001) have investigated two different types of logarithmic multifractal spectra for μ_W: thick points and thin points. A point $x \in \mathbb{R}^d$ $(d \geq 3)$ is called a *thick point* for μ_W if

$$\limsup_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 |\log \varepsilon|} = a$$

for some $a > 0$. Similarly, $x \in \mathbb{R}^d$ is called a *thin point* for μ_W if

$$\liminf_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 / |\log \varepsilon|} = a \tag{12.9}$$

for some $a > 0$.

Among other beautiful results, Dembo, Peres, Rosen and Zeitouni (2000a, b, 2001) obtain the Hausdorff dimensions of the sets of thick and thin points of the occupation measure μ_W. Theorem 12.6 deals with thick points [note that the scaling functions for $d \geq 3$ and $d = 2$ are different].

Theorem 12.6. *Let $W = \{W(t), t \in \mathbb{R}_+\}$ be a Brownian motion in $\mathbb{R}^d$.*

(i). *If $d \geq 3$, then for all $0 \leq a \leq 4/q_d^2$,*

$$\dim_{\mathrm{H}} \Big\{ x \in \mathbb{R}^d : \limsup_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 |\log \varepsilon|} = a \Big\} = 2 - \frac{a q_d^2}{2} \quad a.s., \tag{12.10}$$

where q_d is the first positive zero of the Bessel function $J_{d/2-2}(x)$.

(ii). *If $d = 2$, then for any $0 < a \leq 2$,*

$$\dim_{\mathrm{H}} \Big\{ x \in \mathbb{R}^2 : \limsup_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 |\log \varepsilon|^2} = a \Big\} = 2 - a \quad a.s.$$

In both cases, the packing dimension of the sets of thick points equals 2 a.s.

Remark 12.7. Dembo, Peres, Rosen and Zeitouni (2001) have also proved the existence of *consistently thick points* for the occupation measure of a planar Brownian motion: $x \in \mathbb{R}^2$ is called consistently thick for μ_W if

$$\liminf_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 |\log \varepsilon|^2} = a$$

for some $a > 0$. They show that, unlike in the case of the set of thick points of μ_W of a planar Brownian motion, the packing dimension of the set of consistently thick points equals $2 - a$.

Theorem 12.8 gives the Hausdorff dimension of the sets of thin points.

Theorem 12.8. *Let $W = \{W(t), t \in \mathbb{R}_+\}$ be a Brownian motion in $\mathbb{R}^d$ and $d \geq 2$. Then for all $a > 1$,*

$$\dim_{\mathrm{H}} \Big\{ x \in \mathbb{R}^d : \liminf_{\varepsilon \to 0} \frac{\mu_W\big(B(x,\varepsilon)\big)}{\varepsilon^2 / |\log \varepsilon|} = a \Big\} = 2 - \frac{2}{a} \quad a.s.$$

The packing dimension of the set of thin points equals 2 a.s.

A result similar to (12.10) for the set of thick points of symmetric stable processes in $\mathbb{R}^d$ with index $\alpha < d$ has been proven by Dembo, Peres, Rosen and Zeitouni (1999). However, it seems that no results on thick points for Cauchy processes or more general Lévy processes have been established. It is also natural to ask for the spectrum of thin points in a sense similar to (12.9) [different logarithmic or other corrections may be allowed] for certain class of Lévy processes, say, a subordinator with an exact packing measure function. Compared to (12.9), the results of Hu and Taylor (1997, 2000) deal with "extremely thin" points of the occupation measure of a subordinator.

We mention that the thick points for the projected intersection local times of independent Brownian motions in $\mathbb{R}^d$ $(d = 2, 3)$ have recently been studied by König and Mörters (2002), Dembo, Peres, Rosen and Zeitouni (2002).

12.4. Local Hölder exponents and spectrum of singularities. Comparing with Section 12.2, a different way of characterizing the multifractal structure of the sample paths of a stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$ is to use the local Hölder exponents. For every $t_0 \in \mathbb{R}_+$, recall from Section 3.4 that the local Hölder exponent of X at t_0 is defined by

$$h_X(t_0) = \sup\big\{\ell > 0 : X \in C^\ell(t_0)\big\},$$

where $X \in C^{\ell}(t_0)$ is defined in Example 3.12. Let $S(h) = \{t : h_X(t) = h\}$. Then $d(h) = \dim_{\mathrm{H}} S(h)$ is called the *spectrum of singularities* of X. Note that $d(h) < 0$ means that $S(h) = \emptyset$.

If W is a Brownian motion in $\mathbb{R}^d$, then the local Hölder exponent of W is $1/2$ everywhere on the sample paths, thus the spectrum of singularities of W is trivial and the set of "fast points" can be studied in order to gain more information. Jaffard (1999) shows that, however, the sample paths of a general Lévy process in $\mathbb{R}^d$ may have an interesting spectrum of singularities. More precisely, for $\beta > 0$, define

$$d_\beta(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta], \\ -\infty & \text{otherwise;} \end{cases}$$

$$\overline{d}_\beta(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/2], \\ 1 & \text{if } h = 1/2, \\ -\infty & \text{otherwise.} \end{cases}$$

THEOREM 12.9. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure L and upper index $\beta > 0$. Let*

$$C_j = \int_{2^{-j-1} \le |x| \le 2^{-j}} \mathsf{L}(dx) \qquad \textit{for } \ j \ge 1,$$

and assume that

$$\sum_{j=1}^{\infty} 2^{-j} \sqrt{C_j \log(1 + C_j)} < \infty.$$

(i). *If X has no Brownian component, then a.s. the spectrum of singularities of X is $d_\beta(h)$.*

(ii). *If X has a Brownian component, then a.s. the spectrum of singularities of X is $\overline{d}_\beta(h)$.*

It is easy to verify that the conditions in Theorem 12.9 are satisfied by all stable Lévy processes of index α. To compare Theorem 12.9 with (12.6), we note that when X is a symmetric stable Lévy process of index $\alpha \in (0, 2)$, then (12.6) implies that the Hausdorff dimension of the set of points where the Hölder exponent of X at t is at most h is αh; however, as Jaffard (1999, p.210) points out, (12.6) does not fully characterize the regularity of X at these points. It is worthwhile to mention that the Lévy processes in Theorem 12.9 serve as examples of multifractal functions with a dense set of discontinuities.

We believe that local Hölder exponents and spectrum of singularities are useful for analyzing sample paths of more general Markov processes such as those determined by pseudo-differential operators or stable-like processes. So far, not much work has been done for these processes.

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