Hitting Probabilities and Polar Sets for Fractional Brownian Motion *

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Abstract

Let $X(t)$ $(t \in \mathbf{R}^N)$ be the d-dimensional fractional Brownian motion with index α $(0 < \alpha < 1)$. The upper and lower bounds on the hitting probabilities of $X(t)$ are obtained. Sufficient conditions for a compact set $F \subset \mathbf{R}^d \backslash \{0\}$ to be a polar set for $X(t)$ are proved. It is also proved that if $N \leq \alpha d$, then for any compact set $E \subset \mathbf{R}^N \backslash \{0\}$,

$$\inf\Big\{\dim F : F \in \mathcal{B}(\mathbf{R}^d),\ P\{X(E) \cap F \neq \emptyset\} > 0\Big\} = d - \frac{\mathrm{Dim} E}{\alpha}\ ,$$

and if $N > \alpha d$, then for any compact set $F \subset \mathbf{R}^d \backslash \{0\}$,

$$\inf\Big\{\dim E : E \in \mathcal{B}(\mathbf{R}^N),\ P\{X(E) \cap F \neq \emptyset\} > 0\Big\} = \alpha(d - \mathrm{Dim} F)\ ,$$

where $\mathcal{B}(\mathbf{R}^d)$ denotes the Borel σ-algebra in $\mathbf{R}^d$, and where dim and Dim are Hausdorff dimension and packing dimension respectively.

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1 Introduction

The well known theorem of Kakutani [9] on the hitting probability of Brownian motion in $\mathbf{R}^d$ ($d \geq 3$) is very useful in characterizing the lower functions [4] and the polar sets ([10], see also Kahane [8], Port and Stone [21] and Taylor [27]) for Brownian motion. There have been a lot of efforts to extend these beautiful results to other Gaussian processes and to processes with stationary and independent increments. We refer to Khoshnevisan and Shi [15], and Xiao [35] for recent developments on hitting probabilities of stationary Gaussian random fields and fractional Brownian motion, and to Pruitt and Taylor [22] and Khoshnevisan [14] for hitting probability results for general stable processes and Lévy processes; to Kôno [16], Weber [34] and Khoshnevisan and Shi [15] for results on the lower functions of Gaussian processes including fractional Brownian motion and the Brownian sheet; to Testard [30] [32] and Khoshnevisan [13] for characterizations of polar sets for fractional Brownian motion and the Brownian sheet respectively.

Let $Y(t)$ ($t \in \mathbf{R}^N$) be the centered, real-valued Gaussian random field with covariance function

$$E\Big(Y(t)Y(s)\Big) = \frac{1}{2}\Big(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\Big) ,$$

where $0 < \alpha < 1$ and $|\cdot|$ is the usual Euclidean norm. We will make use of the following fact proved by Pitt [20]: for any $0 \leq r \leq |t|$

$$Var(Y(t)|Y(s) : |s-t| \geq r) \geq r^{2\alpha} . \tag{1.1}$$

This implies that the Gaussian random field $Y(t)$ is strongly locally nondeterministic. See Cuzick and Du Peez [3], Monrad and Pitt [18], Xiao [37] and the references therein for more information on strong local nondeterminism and its applications in studying sample path properties of Gaussian processes.

Associated with $Y(t)$ ($t \in \mathbf{R}^N$), one can define a Gaussian random field $X(t)$ ($t \in \mathbf{R}^N$) in $\mathbf{R}^d$ by

$$X(t) = (X_1(t), \cdots, X_d(t)) ,$$

where $X_1, \cdots, X_d$ are independent copies of Y. The Gaussian random field $X(t)$ is called the d-dimensional fractional Brownian motion of index α or the (N, d, α) Gaussian process (see Kahane [8]). When $N = 1, \alpha = \frac{1}{2}$, $X(t)$ is the ordinary d-dimensional

Brownian motion. If $\alpha = 1/2,\ d = 1$, it is the multiparameter Lévy Brownian motion. It is easy to see that X is a self-similar process of exponent α, i.e. for any $a > 0$,

$$X(a\ \cdot) \stackrel{\mathrm{d}}{=} a^{\alpha} X(\cdot)\ ,$$

where $X \stackrel{\mathrm{d}}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions.

The objective of the present paper is twofold. First we prove lower bounds on the hitting probability and "delayed" hitting probability of fractional Brownian motion, which were left unsolved in Xiao [35]. Together with Theorems 3.1 and 3.2 in [35], these results generalize the classical hitting probability estimates of Kakutani [9] and Dvoretzky and Erdös [4] to fractional Brownian motion. Secondly we study sufficient conditions for a Borel set $F \subset \mathbf{R}^d \backslash \{0\}$ to be polar for fractional Brownian motion in $\mathbf{R}^d$. We also calculate the Hausdorff dimension of the smallest set $F \subset \mathbf{R}^d$ that can be hit by fractional Brownian motion $X(t)$ when t is restricted to some Borel set $E \subset \mathbf{R}^N \backslash \{0\}$. This last problem was raised by Dr. Y. Peres for planar Brownian motion. The author has been informed by Dr. Peres that Bishop and Peres, in their forthcoming paper [2], have solved it for planar Brownian motion and symmetric stable processes and their method depends heavily on the independence of increments of the processes, and hence does not apply to fractional Brownian motion. Our proof is based on a result of Testard [32] and the construction in Xiao [36] and can be extended to more general Lévy stable processes.

The rest of the paper is organized as follows. In Section 2 we prove lower bounds for the hitting probabilities of $X(t)$. In Section 3, we study polar sets for $X(t)$. Our theorem 3.1 and its corollary improve the characterizations of polar sets for $X(t)$ obtained by Testard [30] [32]. As a consequence, we give a simple proof of the fact that if $N/\alpha_1 + N/\alpha_2 \leq d$, then with probability one, two independent fractional Brownian motions $X^1(t)$ and $X^2(t)(t \in \mathbf{R}^N)$ of index $\alpha_i (i = 1, 2)$ in $\mathbf{R}^d$ do not meet each other. Similar result can also be proved for k independent fractional Brownian motions, but we will not pursue further in this direction, instead we refer to Talagrand [25] for a proof of the non-existence of multiple points of multiplicity k for fractional Brownian motion in the critical case of $Nk = (k-1)\alpha d$. Finally in Section 3, we prove the following result, which shows that packing dimension is naturally related to the polarity of $E \times F$ for random field $(t, X(t))$: if $N \le \alpha d$, then for any compact set $E \subset \mathbf{R}^N \backslash \{0\}$,

$$\inf\Big\{\dim F : F \in \mathcal{B}(\mathbf{R}^d),\ P\{X(E) \cap F \neq \emptyset\} > 0\Big\} = d - \frac{\text{Dim} E}{\alpha} ,$$

and if $N > \alpha d$, then for any compact set $F \subset \mathbf{R}^d \backslash \{0\}$,

$$\inf\Big\{\dim E : E \in \mathcal{B}(\mathbf{R}^N),\ P\{X(E) \cap F \neq \emptyset\} > 0\Big\} = \alpha(d - \text{Dim} F) ,$$

where $\mathcal{B}(\mathbf{R}^d)$ denotes the Borel σ-algebra in $\mathbf{R}^d$, and where dim and Dim are Hausdorff dimension and packing dimension respectively. We refer to Falconer [5] and Mattila [17] for more properties of Hausdorff measure, packing measure and related dimensions.

Many of the results in this paper also hold for the Brownian sheet. We refer to Orey and Pruitt [19] for definition and more properties of the Brownian sheet. However, since the Brownian sheet is not locally nondeterministic, the proofs for fractional Brownian motion can not be readily carried over. We will give the detailed proofs elsewhere.

We will use K, K_1, and K_2 to denote positive and finite constants whose precise values are not important and may be different in each appearance.

2 Hitting Probabilities

In Xiao [35], the probability that the d-dimensional fractional Brownian motion $X(t)(t \in \mathbf{R})$ of index α hits a ball $B(y, r)$ in $\mathbf{R}^d$ was studied. In the case of $1 < \alpha d$, upper bounds for hitting probability and delayed hitting probability similar to those for Brownian motion in $\mathbf{R}^d$ with $d \ge 3$ were obtained. These estimates played important roles in calculating the exact packing measure of the image set $X([0, 1])$ in [35]. We should remark that Lemma 3.1 and Theorem 3.1 in [35] are also true for multiparameter fractional Brownian motion $X(t)(t \in \mathbf{R}^N)$. In this section, we will show that the upper bounds in Theorems 3.1 and 3.2 of Xiao [35] also serve as the lower bounds (with different constant factors) for the hitting probabilities. These estimates generalize the classical results of Kakutani [9] and Dvoretzky and Erdös [4] to fractional Brownian motion.

Let $Z_0(t)$ $(t \in \mathbf{R}^N)$ be a centered, real-valued Gaussian random field. We write

$$\sigma^2(t, s) = E(Z_0(t) - Z_0(s))^2 , \quad \sigma^2(t) = E(Z_0(t))^2 .$$

Our arguments extend to Gaussian random fields with a covariance structure somewhat more general than fractional Brownian motion. We will suppose that the following conditions are satisfied: for any $s, t \in \mathbf{R}^N$

$$\sigma^2(t,s) \le \theta^2 |t-s|^{2\alpha} \ , \quad (0<\alpha<1) \tag{2.1}$$

and there exist positive constants $c_1,\ c_2$ and c_3 such that

$$c_1\theta^2 t^{2\alpha} \le \sigma^2(t) \le c_2\theta^2 t^{2\alpha} \ . \tag{2.2}$$

and for any $t, s_1, s_2 \in \mathbf{R}^N$

$$Var\Big(Z_0(t)|Z_0(s_1), Z_0(s_2)\Big) \ge c_3\theta^2 \min\{|t|^{2\alpha},\ |t-s_1|^{2\alpha},\ |t-s_2|^{2\alpha}\} \ . \tag{2.3}$$

It is clear that fractional Brownian motion $Y(t)$ satisfies these conditions with $\theta = 1$. Let $Z_1, \cdots, Z_d$ be independent copies of Z_0 and let

$$Z(t) = (Z_1(t), \cdots, Z_d(t)) \ .$$

The proof of the following lemma is inspired by the argument of Khoshnevisan and Shi [15].

Lemma 2.1 *For any $a > 0$, let $S = \{t \in \mathbf{R}^N : a \le |t| \le 2a\}$. Consider a centered, real-valued Gaussian random field $Z_0(t)$ $(t \in \mathbf{R}^N)$ that satisfies (2.1), (2.2) and (2.3) on S, where θ may depend on a. Let $Z(t)$ $(t \in \mathbf{R}^N)$ be the associated Gaussian random field with values in $\mathbf{R}^d$ and $N < \alpha d$. Then there exist positive constants K_1 and K_2, depending only on α and d, such that for any $0 < r \le 1$ and any $y \in \mathbf{R}^d$ with $|y| \ge r$, we have*

$$P\Big\{\inf_{t\in S} |Z(t)-y| < r\Big\} \ge K_1 \exp\Big(-\frac{|y|^2}{K_2\theta^2 a^{2\alpha}}\Big) \cdot a^{N-\alpha d} \cdot \Big(\frac{r}{\theta}\Big)^{d-\frac{N}{\alpha}}. \tag{2.4}$$

Proof. For any $r > 0$ and any $y \in \mathbf{R}^d$, let

$$T(r) = \int_S \mathbf{1}_{\{|Z(t)-y|<r\}} dt$$

be the sojourn time of $X(t)$ $(t \in S)$ in $B(y, r)$, where $\mathbf{1}_A$ is the indicator of A. Then

$$\begin{aligned} P\Big\{\inf_{t\in S} |Z(t)-y| < r\Big\} &\ge P\{T(r) > 0\} \\ &\ge \frac{(E(T(r)))^2}{E(T(r)^2)} \end{aligned} \tag{2.5}$$

by the Cauchy-Schwartz inequality. It follows from the Fubini's theorem that

$$
\begin{aligned}
E(T(r)) &= \int_S P\{|Z(t)-y|<r\}dt \\
&= \int_S \int_{\{|u-y|<r\}} \frac{1}{(2\pi)^{d/2}\sigma(t)^d} \exp\Big(-\frac{|u|^2}{2\sigma(t)^2}\Big) du dt \\
&\geq K \int_S \frac{r^d}{\sigma(t)^d} \exp\Big(-2\frac{|y|^2}{\sigma(t)^2}\Big) dt \\
&\geq K \exp\Big(-\frac{|y|^2}{c_1\theta^2 a^{2\alpha}}\Big) \cdot \Big(\frac{r}{\theta}\Big)^d \cdot a^{N-\alpha d} \qquad (2.6)
\end{aligned}
$$

by (2.2) and a simple change of variables. Now we consider

$$
E(T(r)^2) = \int_S \int_S P\{|Z(s)-y|<r, |Z(t)-y|<r\} ds dt \;. \qquad (2.7)
$$

By conditioning, we have

$$
P\{|Z(s)-y|<r, |Z(t)-y|<r\} = \int_{\{|u-y|<r\}} P\{|Z(t)-y|<r | Z(s)=u\} \cdot p_s(u) du \;, \qquad (2.8)
$$

where $p_s(u)$ is the density function of $Z(s)$. Since the conditional distributions in Gaussian processes are still Gaussian and by (2.3), we have

$$
P\{|Z(t)-y|<r | Z(s)=u\} \leq \min\Big\{1, \frac{Kr^d}{\theta^d |t-s|^{\alpha d}}\Big\} \;.
$$

Hence (2.8) is at most

$$
P\{|Z(s)-y|<r\} \cdot \min\Big\{1, \frac{Kr^d}{\theta^d |t-s|^{\alpha d}}\Big\} \leq \min\Big\{1, \frac{Kr^d}{\theta^d |s|^{\alpha d}}\Big\} \cdot \min\Big\{1, \frac{Kr^d}{\theta^d |t-s|^{\alpha d}}\Big\} \;. \qquad (2.9)
$$

It follows from (2.7), (2.8) and (2.9) that

$$
\begin{aligned}
E(T(r)^2) &\leq \int_S \int_S \min\Big\{1, \frac{Kr^d}{\theta^d |s|^{\alpha d}}\Big\} \cdot \min\Big\{1, \frac{Kr^d}{\theta^d |t-s|^{\alpha d}}\Big\} ds dt \\
&\leq K\Big(\frac{r}{\theta}\Big)^{d+\frac{N}{\alpha}} \cdot a^{N-\alpha d} \;. \qquad (2.10)
\end{aligned}
$$

Combining (2.5), (2.6) and (2.10), we obtain (2.4).

The following proposition may be of independent interests, e.g. a similar result will be applied to characterize multiple polar sets for fractional Brownian motion in Xiao [38]. The proof of Proposition 2.1 is a refinement of those of Kôno [16] and Xiao [35].

Proposition 2.1 *Let $S = \{t \in \mathbf{R}^N : a \le |t| \le 2a\}$, $T = \{t \in \mathbf{R}^N : b \le |t| \le 2b\}$ with $2a + 1 \le b$ and let $Z(t)(t \in \mathbf{R}^N)$ be the Gaussian random field in Lemma 2.1, where θ may depend on a or b. Then there exist positive constants K_1 and K_2, depending only on N, d and α, such that for any $0 < r \le 1$ and any $y \in \mathbf{R}^d$ with $|y| \ge r$, we have*

$$\begin{aligned} &P\Big\{\inf_{s\in S}|Z(s)-y|<r,\ \inf_{t\in T}|Z(t)-y|<r\Big\} \\ &\le K_1 \exp\Big(-\frac{|y|^2}{K_2\theta^2 a^{2\alpha}}\Big)\cdot \frac{b^N}{a^{\alpha d-N}(b-2a)^{\alpha d}}\Big(\frac{r}{\theta}\Big)^{2(d-\frac{N}{\alpha})} . \end{aligned} \tag{2.11}$$

Proof. For any bounded set $E \subset \mathbf{R}^N$, we use $N(E, r)$ to denote the smallest number of open balls of radius r that are needed to cover E. Then we have

$$N\Big(S,\ (\frac{r}{\theta})^{1/\alpha}\Big) \le K a^N \Big(\frac{r}{\theta}\Big)^{-\frac{N}{\alpha}}, \quad N\Big(T,\ (\frac{r}{\theta})^{1/\alpha}\Big) \le K b^N \Big(\frac{r}{\theta}\Big)^{-\frac{N}{\alpha}} . \tag{2.12}$$

Let $\{S_p\}$ $(1 \le p \le N(S, (\frac{r}{\theta})^{1/\alpha}))$ be a family of balls of radius $(\frac{r}{\theta})^{1/\alpha}$ that cover S and let $\{T_q\}$ $(1 \le q \le N(T, (\frac{r}{\theta})^{1/\alpha}))$ be a family of balls of radius $(\frac{r}{\theta})^{1/\alpha}$ that cover T. Define the events

$$A = \Big\{\inf_{s\in S}|Z(s)-y|<r,\ \inf_{t\in T}|Z(t)-y|<r\Big\}$$

and

$$A_{p,q} = \Big\{\inf_{s\in S_p}|Z(t)-y|<r,\ \inf_{t\in T_q}|Z(t)-y|<r\Big\} .$$

Then

$$A \subseteq \bigcup_{p=1}^{N(S,(\frac{r}{\theta})^{1/\alpha})} \bigcup_{q=1}^{N(T,(\frac{r}{\theta})^{1/\alpha})} A_{p,q} . \tag{2.13}$$

Now fix p and q with $1 \le p \le N(S, (\frac{r}{\theta})^{1/\alpha})$ and $1 \le q \le N(T, (\frac{r}{\theta})^{1/\alpha})$. Let

$$\epsilon_n = \Big(\frac{r}{\theta}\Big)^{\frac{1}{\alpha}} \exp(-2^{n+1})$$

and let $\{s_i^{(n)},\ 1 \le i \le N(S_p, \epsilon_n)\}$ and $\{t_i^{(n)},\ 1 \le i \le N(T_q, \epsilon_n)\}$ be a set of the centers of open balls with radius ϵ_n that cover S_p and T_q respectively. Denote

$$r_n = \beta\theta d\epsilon_n^\alpha 2^{\frac{n+1}{2}} ,$$

where $\beta > 0$ is a constant whose value will be determined later, and let

$$A_{i,j}^{(n)} = \Big\{|Z(s_i^{(n)})-y| \le r + \sum_{k=n}^{\infty} r_k,\ |Z(t_j^{(n)})-y| \le r + \sum_{k=n}^{\infty} r_k\Big\} ,$$

$$
\begin{aligned}
A^{(n)} &= \bigcup_{k=1}^{n} \bigcup_{i=1}^{N(S_p,\epsilon_k)} \bigcup_{j=1}^{N(T_q,\epsilon_k)} A_{i,j}^{(k)} \\
&= A^{(n-1)} \cup \bigcup_{i=1}^{N(S_p,\epsilon_n)} \bigcup_{j=1}^{N(T_q,\epsilon_n)} A_{i,j}^{(n)} \ . \qquad (2.14)
\end{aligned}
$$

Then by the modulus of continuity for $Z(t)$ (see e.g. Kahane [8]), we have

$$
P(A_{p,q}) \le \lim_{n\to\infty} P(A^{(n)}) \ . \qquad (2.15)
$$

By (2.14), we have

$$
P(A^{(n)}) \le P(A^{(n-1)}) + P(A^{(n)} \backslash A^{(n-1)}) \qquad (2.16)
$$

and

$$
P(A^{(n)} \backslash A^{(n-1)}) \le \sum_{i=1}^{N(S_p,\epsilon_n)} \sum_{j=1}^{N(T_q,\epsilon_n)} P\Big(A_{i,j}^{(n)} \backslash A_{i',j'}^{(n-1)}\Big) \ , \qquad (2.17)
$$

where i' and j' are chosen so that $|s_i^{(n)} - s_{i'}^{(n-1)}| < \epsilon_{n-1}$ and $|t_j^{(n)} - t_{j'}^{(n-1)}| < \epsilon_{n-1}$. Now we consider

$$
\begin{aligned}
& P\Big(A_{i,j}^{(n)} \backslash A_{i',j'}^{(n-1)}\Big) \\
&= P\Big\{|Z(s_i^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k \ , \ |Z(s_{i'}^{(n-1)}) - y| > r + \sum_{k=n-1}^{\infty} r_k ; \\
&\qquad\qquad |Z(t_j^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k, \ |Z(t_{j'}^{(n-1)}) - y| > r + \sum_{k=n-1}^{\infty} r_k \Big\} \\
&\le P\Big\{|Z(s_i^{(n)}) - y| < c_4 r \ , \ |Z(s_i^{(n)}) - Z(s_{i'}^{(n-1)})| \ge r_{n-1} ; \\
&\qquad\qquad |Z(t_j^{(n)}) - y| < c_4 r, \ |Z(t_j^{(n)}) - Z(t_{j'}^{(n-1)})| \ge r_{n-1} \Big\} \ . \qquad (2.18)
\end{aligned}
$$

where the last inequality follows from the triangle inequality and the following elementary fact

$$
r + \sum_{k=n}^{\infty} r_k \le r + \sum_{k=0}^{\infty} r_k \ \hat{=}\ c_4 r \ ,
$$

where $A \hat{=} B$ means that A is denoted by B. In order to create independence, for each $t \in T_q$ we denote

$$
E\Big(Z(t) \,\Big|\, Z(s_i^{(n)}), Z(s_{i'}^{(n-1)})\Big) = a_1(t) Z(s_i^{(n)}) + a_2(t) Z(s_{i'}^{(n-1)})
$$

and

$$
\xi(t) = Z(t) - E\Big(Z(t) \,\Big|\, Z(s_i^{(n)}), Z(s_{i'}^{(n-1)})\Big) \ .
$$

Then the $\mathbf{R}^d$-valued Gaussian random field $\xi(t)$ is independent of $Z(s_i^{(n)})$ and $Z(s_{i'}^{(n-1)})$ and for each $t \in T_q$ we can write

$$Z(t) = a_1(t) Z(s_i^{(n)}) + a_2(t) Z(s_{i'}^{(n-1)}) + \xi(t) \ . \tag{2.19}$$

It follows from (2.19) and the independence of $\xi(\cdot)$ and $Z(s_i^{(n)}),\ Z(s_{i'}^{(n-1)})$ that

$$\begin{aligned} &P\Big\{|Z(t_j^{(n)}) - y| < c_4 r,\ |Z(t_j^{(n)}) - Z(t_{j'}^{(n-1)})| \ge r_{n-1} \Big| Z(s_i^{(n)}) = u,\ Z(s_{i'}^{(n-1)}) = v\Big\} \\ &\le P\Big\{\Big|\xi(t_j^{(n)}) + a_1(t_j^{(n)})u + a_2(t_j^{(n)})v - y\Big| < c_4 r,\ \ \Big|\xi(t_j^{(n)}) - \xi(t_{j'}^{(n-1)})\Big| \ge r_{n-1} \\ &\qquad - \Big|a_1(t_j^{(n)}) - a_1(t_{j'}^{(n)})\Big|\,|u| - \Big|a_2(t_j^{(n)}) - a_2(t_{j'}^{(n)})\Big|\,|v|\Big\} \ . \end{aligned} \tag{2.20}$$

By conditioning and (2.20), we obtain that the probability in (2.18) is at most

$$\begin{aligned} &\int\!\!\int_{\{|u-y|<r, |u-v|\ge r_{n-1}\}} P\Big\{\Big|\xi(t_j^{(n)}) + a_1(t_j^{(n)})u + a_2(t_j^{(n)})v - y\Big| < c_4 r, \\ &\qquad\qquad \Big|\xi(t_j^{(n)}) - \xi(t_{j'}^{(n-1)})\Big| \ge \frac{r_{n-1}}{2}\Big\} \cdot p(u,v)\,du\,dv \\ &+ \int\!\!\int_{\{|u-y|<r, |u-v|\ge r_{n-1}\}} P\Big\{\Big|\xi(t_j^{(n)}) + a_1(t_j^{(n)})u + a_2(t_j^{(n)})v - y\Big| < c_4 r\Big\} \\ &\qquad\qquad \cdot \mathbf{1}_{\{|a_1(t_j^{(n)}) - a_1(t_{j'}^{(n)})||u| \ge r_{n-1}/4\}} \cdot p(u,v)\,du\,dv \\ &+ \int\!\!\int_{\{|u-y|<r, |u-v|\ge r_{n-1}\}} P\Big\{\Big|\xi(t_j^{(n)}) + a_1(t_j^{(n)})u + a_2(t_j^{(n)})v - y\Big| < c_4 r\Big\} \\ &\qquad\qquad \cdot \mathbf{1}_{\{|a_2(t_j^{(n)}) - a_2(t_{j'}^{(n)})||v| \ge r_{n-1}/4\}} \cdot p(u,v)\,du\,dv \\ &\hat{=} I_1 + I_2 + I_3 \ , \end{aligned} \tag{2.21}$$

where $p(u,v)$ denotes the density function of $(Z(s_i^{(n)}),\ Z(s_{i'}^{(n-1)}))$. In order to obtain an upper bound for I_1, we fix $(u,v) \in \mathbf{R}^d \times \mathbf{R}^d$ and consider first

$$P\Big\{\Big|\xi(t_j^{(n)}) + a_1(t_j^{(n)})u + a_2(t_j^{(n)})v - y\Big| < c_4 r,\ \Big|\xi(t_j^{(n)}) - \xi(t_{j'}^{(n-1)})\Big| \ge \frac{r_{n-1}}{2}\Big\} \ . \tag{2.22}$$

Since by (2.1) and (2.3), we have

$$E|\xi(t_j^{(n)}) - \xi(t_{j'}^{(n-1)})|^2 \le d\theta^2 |t_j^{(n)} - t_{j'}^{(n-1)}|^{2\alpha}$$

and

$$E|\xi(t_j^{(n)})|^2 \ge c_3 \theta^2 (b - 2a)^{2\alpha} \ .$$

The same argument as that of Xiao ([35], pp.3198 - 3199) gives that (2.22) is at most

$$K \exp\Big(-\frac{(\beta d)^2}{16} 2^n\Big)\Big(\frac{r}{\theta(b-2a)^\alpha}\Big)^d \ . \tag{2.23}$$

Combining (2.22), (2.23) and the above mentioned argument of [35], we have

$$I_1 \le K \exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{8}\ 2^n\Big)\Big(\frac{r}{\theta(b-2a)^\alpha}\Big)^d . \tag{2.24}$$

By the independence of $\xi(\cdot)$ and $Z(s_i^{(n)}),\ Z(s_{i'}^{(n-1)})$, and the fact that $P\{|\xi - x| < r\}$ takes its maximum at $x = 0$, we have

$$I_2 \le P\Big\{\Big|Z(s_i^{(n)}) - y\Big| < c_4 r\ ,\ \Big|Z(s_i^{(n)}) - Z(s_{i'}^{(n-1)})\Big| \ge r_{n-1}\Big\} \cdot P\Big\{|\xi(t_j^{(n)})| < c_4 r\Big\} .$$

Hence using again the proof of [35], we obtain

$$I_2 \le K \exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big)\Big(\frac{r}{\theta(b-2a)^\alpha}\Big)^d . \tag{2.25}$$

Similarly

$$I_3 \le K \exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big)\Big(\frac{r}{\theta(b-2a)^\alpha}\Big)^d . \tag{2.26}$$

Now we take β such that

$$\frac{(\beta d)^2}{16} > 4 .$$

Then by (2.16), (2.17), (2.18), (2.21), (2.24), (2.25) and (2.26), we have

$$\begin{aligned} P(A^{(n)}) \ &\le\ P(A^{(n-1)}) + KN(S_p,\epsilon_n)N(T_q,\epsilon_n)\exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big) \\ &\qquad \cdot \exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r^2}{\theta^2 a^\alpha (b-2a)^\alpha}\Big)^d \\ &\le\ K\Big[N(S_p,\epsilon_0)N(T_q,\epsilon_0) + \sum_{k=1}^{\infty} N(S_p,\epsilon_k)N(T_q,\epsilon_k)\cdot\exp\Big(-\frac{(\beta d)^2}{16}\ 2^k\Big)\Big] \\ &\qquad \cdot \exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r^2}{\theta^2 a^\alpha (b-2a)^\alpha}\Big)^d \\ &=\ K\exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r^2}{\theta^2 a^\alpha (b-2a)^\alpha}\Big)^d . \end{aligned} \tag{2.27}$$

Therefore, by (2.12), (2.13), (2.15) and (2.27), we have

$$\begin{aligned} &P\Big\{\inf_{s\in S}|Z(s) - y| < r,\ \inf_{t\in T}|Z(t) - y| < r\Big\} \\ &\le K\exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\cdot\frac{b^N}{a^{\alpha d - N}(b-2a)^{\alpha d}}\Big(\frac{r}{\theta}\Big)^{2(d-\frac{N}{\alpha})} . \end{aligned}$$

This finishes the proof of (2.11).

Now we prove the main results of this section.

Theorem 2.1 *Let $X(t)$ $(t \in \mathbf{R}^N)$ be the d-dimensional fractional Brownian motion of index α. If $N < \alpha d$, then there exist positive finite constants K_1 and K_2, depending only on N, d and α, such that for any $r > 0$ small enough and any $y \in \mathbf{R}^d$ with $|y| \geq r$, we have*

$$K_1\Big(\frac{r}{|y|}\Big)^{d-\frac{N}{\alpha}} \leq P\Big\{\exists\ t \in \mathbf{R}^N \ \ \textit{such that } |X(t)-y| < r\Big\} \leq K_2\Big(\frac{r}{|y|}\Big)^{d-\frac{N}{\alpha}}. \tag{2.28}$$

Proof. The upper bound in (2.28) is proved in [35]. To prove the lower bound, we notice that $X(t)$ $(t \in \mathbf{R}^N)$ satisfies (2.1), (2.2) and (2.3) with $\theta = 1$. For any $n \geq 1$, let $S_n = \{t \in \mathbf{R}^N :\ 2^{2n-1} \leq |t| \leq 2^{2n}\}$. Then we have

$$\begin{aligned}
&P\Big\{\exists\ t \in \mathbf{R}^N \text{ such that } |X(t)-y| < r\Big\}\\
&\geq P\Big\{\exists\ n \geq 1 \text{ and } \ t \in S_n \text{ such that } |X(t)-y| < r\Big\}\\
&\geq \sum_{n=1}^{\infty} P\Big\{\inf_{t \in S_n} |X(t)-y| < r\Big\}\\
&\qquad\qquad - \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} P\Big\{\inf_{s \in S_m} |X(s)-y| < r,\ \inf_{t \in S_n} |X(t)-y| < r\Big\}.
\end{aligned} \tag{2.29}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big\{\inf_{t \in S_n} |X(t)-y| < r\Big\}\\
&\geq K \sum_{n=1}^{\infty} \exp\Big(-\frac{|y|^2}{K 2^{4\alpha n}}\Big) \cdot 2^{2n(N-\alpha d)} \cdot r^{d-\frac{N}{\alpha}}\\
&\geq K \int_1^{\infty} x^{N-\alpha d-1} \exp\Big(-\frac{|y|^2}{K x^{2\alpha}}\Big) dx \cdot r^{d-\frac{N}{\alpha}}\\
&= K\Big(\frac{r}{|y|}\Big)^{d-\frac{N}{\alpha}}.
\end{aligned} \tag{2.30}$$

By Proposition 2.1, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} P\Big\{\inf_{s \in S_m} |X(s)-y| < r,\ \inf_{t \in S_n} |X(t)-y| < r\Big\}\\
&\leq K_1 \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} \exp\Big(-\frac{|y|^2}{K_2 2^{4m\alpha}}\Big) \cdot 2^{2m(N-\alpha d)} (2^{2n-1} - 2^{2m})^{N-\alpha d} r^{2(d-\frac{N}{\alpha})}\\
&= K_1 r^{2(d-\frac{N}{\alpha})} \sum_{m=1}^{\infty} \exp\Big(-\frac{|y|^2}{K_2 2^{4m\alpha}}\Big) 2^{4m(N-\alpha d)}
\end{aligned}$$

$$\leq K_1 r^{2(d-\frac{N}{\alpha})} \int_1^\infty \exp\Big(-\frac{|y|^2}{x^{4\alpha}}\Big) \frac{1}{x^{4(\alpha d-N)+1}} dx$$
$$= K\Big(\frac{r}{|y|}\Big)^{2(d-\frac{N}{\alpha})} . \tag{2.31}$$

Putting (2.29), (2.30) and (2.31) together, we obtain (2.28).

Remark. If $X(t)$ ($t \in \mathbf{R}_+$) is a Brownian motion in $\mathbf{R}^d$ with $d \geq 3$, then it is well known that (2.28) holds with equality and $K_1 = K_2 = 1$. This result was stated by Kakutani [9] for $d = 3$. For a proof of the general result, see Port and Stone [21].

The following theorem is a generalization of the well known result of Dvoretzky and Erdös [4] about the delayed hitting probability of Brownian motion in $\mathbf{R}^d$ ($d \geq 3$). We have to assume $N = 1$ to give an ordered parameter set.

Theorem 2.2 *Let $X(t)$ ($t \in \mathbf{R}$) be the d-dimensional fractional Brownian motion of index α ($0 < \alpha < 1$) with $1 < \alpha d$. Then there exist positive constants K_1 and K_2, depending only on α and d, such that for any $T > 0$ and any $0 < r < T^\alpha$ small enough, we have*

$$K_1\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}} \leq P\Big\{\exists\, t \in \mathbf{R} \text{ such that } |t| > T \text{ and } |X(t)| < r\Big\} \leq K_2\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}} . \tag{2.32}$$

Proof. The upper bound in (2.32) is proved in [35]. The proof of the lower bound is a combination of the proof of Theorem 3.2 in [35] and the proof of Theorem 2.1 above. We observe that

$$P\Big\{\exists\, t \in \mathbf{R} \text{ such that } |t| > T \text{ and } |X(t)| < r\Big\}$$
$$\geq \int_{\mathbf{R}^d} P\Big\{\exists\, t > T \text{ such that } |X(t)| < r | X(T) = y\Big\}\, p_T(y) dy , \tag{2.33}$$

where $p_T(y)$ is the density function of $X(T)$,

$$p_T(x) = \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \frac{1}{T^{\alpha d}} \cdot \exp\Big(-\frac{|x|^2}{2T^{2\alpha}}\Big) .$$

In order to estimate the conditional probability in (2.33), we write

$$X(tT + T) = X^1(t) + c(t)X(T) , \tag{2.34}$$

where X^1 is independent of $X(T)$ and

$$c(t) = \frac{(1+t)^{2\alpha} + 1 - t^{2\alpha}}{2} .$$

It is clear that $c(t) \geq \frac{1}{2}$ and $c(t) \equiv 1$ if $\alpha = \frac{1}{2}$; $c(t)$ is decreasing if $\alpha < \frac{1}{2}$ and $c(t)$ is increasing if $\alpha > \frac{1}{2}$. Let

$$Z(t) = \frac{X^1(t)}{c(t)} \ .$$

Now we verify that $Z(t)$ satisfies the conditions in Lemma 2.1. To this end, we denote by $Z_0(t)$ the first component of $X^1(t)$. It is proved in [35] that for any $s, t \in \mathbf{R}$

$$\sigma^2(t,s) \leq \frac{KT^{2\alpha}}{c(t)c(s)} \ |t-s|^{2\alpha} \ ,$$

and $\sigma^2(t)$ satisfies (2.2) on $S = [a,\ 2a]$ with

$$\theta = \frac{T^\alpha}{m} \ , \qquad \text{where } m = \min\{c(a),\ c(2a)\} \ .$$

It follows from (1.1) that for any $s_1, s_2, t \in \mathbf{R}_+$,

$$\begin{aligned}
&Var\Big(Z_0(t)|Z_0(s_1), Z_0(s_2)\Big) \\
&\geq Var\Big(\frac{Y(tT+T)}{c(t)} - Y(T)\,|Y(s_1T+T),\ Y(s_1T+T),\ Y(T)\Big) \\
&\geq \frac{1}{c^2(t)} Var\Big(Y(tT+T)\,|Y(s_1T+T),\ Y(s_1T+T),\ Y(T)\Big) \\
&\geq \frac{T^{2\alpha}}{c^2(t)} \min\Big\{|t|^{2\alpha},\ |t-s_1|^{2\alpha},\ |t-s_2|^{2\alpha}\Big\} \ .
\end{aligned}$$

Therefore, (2.3) is satisfied. By Lemma 2.1, for $S = [a, 2a]$ we have

$$\begin{aligned}
&P\Big\{\inf_{t\in S} |X_1(t) + c(t)y| < r\Big\} \\
&\geq P\Big\{\inf_{t\in S} |Y(t) + y| < \frac{r}{\tilde{m}}\Big\} \qquad (\tilde{m} = \max\{c(a),\ c(2a)\}) \\
&\geq K \exp\Big(-\frac{|y|^2\tilde{m}^2}{KT^{2\alpha}a^{2\alpha}}\Big) a^{1-\alpha d} \cdot \Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}} \ . \qquad (2.35)
\end{aligned}$$

By Proposition 2.1 and using an argument similar to the proof of Theorem 2.1, we obtain that for $r > 0$ small enough and $y \in \mathbf{R}^d$ with $r \leq |y| \leq 2T^\alpha$

$$P\Big\{\exists\ t > 0 \ \text{ such that } |X_1(t) + c(t)y| < r\Big\} \geq K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}} \ . \qquad (2.36)$$

Putting (2.36) into (2.33), we see that (2.33) is at least

$$K \int_{\{r\leq|y|\leq 2T^\alpha\}} \Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}} \frac{1}{T^{\alpha d}} \exp\Big(-\frac{|y|^2}{2T^{2\alpha}}\Big) dy \geq K\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}} \ .$$

This completes the proof of (2.32).

3 Polar Sets for Fractional Brownian Motion

Let $B(t)$ $(t \in \mathbf{R}_+)$ be the d-dimensional Brownian motion and let $F \subset \mathbf{R}^d \backslash \{0\}$ be a compact set. It is well known that F is a polar set for $B(t)$, that is

$$P\{\exists t \in \mathbf{R}_+ \ \text{ such that } \ B(t) \in F\} = 0 \ ,$$

if and only if

$$\mathrm{Cap}_{d-2}(F) = 0 \qquad \text{if } \ d \geq 3, \tag{3.1}$$

$$\mathrm{Cap}_{\log}(F) = 0 \qquad \text{if } \ d = 2, \tag{3.2}$$

where $\mathrm{Cap}_{d-2}(F)$ is the capacity of F with respective to the kernel

$$k(x,y) = \frac{1}{|x-y|^{d-2}}$$

and $\mathrm{Cap}_{\log}(F)$ is is the capacity of F with respective to the kernel

$$k(x,y) = \log \frac{1}{|x-y|} \ .$$

See, e.g. Port and Stone [21] or Kahane [8] for a proof.

These results are due to Kakutani [10] and have been extended partially to more general processes with stationary and independent increments by Hawkes [6] and Kahane [7], to fractional Brownian motion by Testard [30] [32], and recently to the Brownian sheet by Khoshnevisan [13]. Taylor and Watson [29] also proved similar results for the polar sets for the heat equation.

For any $0 < \alpha < 1$, we define a metric on $\mathbf{R}^N \times \mathbf{R}^d$ by

$$\rho_\alpha((s,x), \ (t,y)) = \max\{|s-t|^\alpha, \ |x-y|\} \ ,$$

where $|\cdot|$ is the usual Euclidean norm. Let Φ be the class of functions $\phi : (0,\delta) \to (0,1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leq K, \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

For any function $\phi \in \Phi$ and any set $A \subseteq \mathbf{R}^N \times \mathbf{R}^d$, we can define the ϕ-Hausdorff measure of A under the metric ρ_α by

$$\phi\text{-}m_\alpha(E) = \lim_{\epsilon \to 0} \ \inf \left\{ \sum_i \phi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B_\alpha(u_i, r_i), \ r_i < \epsilon \right\} \ , \tag{3.3}$$

where $B_\alpha(u, r)$ is the open ball in the metric space $(\mathbf{R}^N \times \mathbf{R}^d, \rho_\alpha)$ centered at u with radius r. We will denote by $B(u, r)$ the open ball in the Euclidean space centered at u with radius r, and by ϕ-m the ordinary ϕ-Hausdorff measure.

It is known that ϕ-m_α is a metric outer measure in the sense of Carathéodory and hence every Borel set in $(\mathbf{R}^N \times \mathbf{R}^d, \rho_\alpha)$ is ϕ-m_α measurable (see Rogers [23]). Results such as density theorems and Frostman's lemma analogous to those for ordinary Hausdorff measure still hold. See Taylor and Watson [29]. The Hausdorff dimension $\dim_\alpha A$ of $A \subset \mathbf{R}^N \times \mathbf{R}^d$ is defined by

$$\dim_\alpha A = \inf\{\gamma > 0 : s^\gamma\text{-}m_\alpha(A) = 0\} \ . \tag{3.4}$$

These type of Hausdorff measure and Hausdorff dimension have been applied by Hawkes [6], Taylor and Watson [29] and Testard [30] [32] to characterize the polar sets for stochastic processes and the heat equation, and by Testard [31] to study double polar sets for Brownian motion.

Similar to the definition of packing measure given by Taylor and Tricot [28], one can also define packing measure ϕ-p_α and packing dimension Dim_α on the metric space $(\mathbf{R}^N \times \mathbf{R}^d, \rho_\alpha)$. For example, the packing dimension $\mathrm{Dim}_\alpha A$ is defined by

$$\mathrm{Dim}_\alpha A = \inf\{\gamma > 0 : s^\gamma\text{-}p_\alpha(A) = 0\} \ . \tag{3.5}$$

The following inequalities, which are analogous to the inequalities for the ordinary Hausdorff dimension and packing dimension in Tricot [33], are proved by Testard [32]. For any Borel sets $E \subseteq \mathbf{R}^N, F \subseteq \mathbf{R}^d$

$$\begin{aligned} \frac{\dim E}{\alpha} + \dim F \ &\leq \ \dim_\alpha(E \times F) \leq \min\Big\{\frac{\mathrm{Dim} E}{\alpha} + \dim F,\ \frac{\dim E}{\alpha} + \mathrm{Dim} F\Big\} \\ &\leq \ \mathrm{Dim}_\alpha(E \times F) \leq \frac{\mathrm{Dim} E}{\alpha} + \mathrm{Dim} F \ . \end{aligned} \tag{3.6}$$

Let $X(t)$ $(t \in \mathbf{R}^N)$ be the d-dimensional fractional Brownian motion in $\mathbf{R}^d$ of index α. For compact sets $E \subseteq \mathbf{R}^N \backslash \{0\}$ and $F \subseteq \mathbf{R}^d \backslash \{0\}$, Testard [30] [32] proved sufficient conditions and necessary conditions on E and F for $E \times F$ to be a polar set for the random field $(t, X(t))$, or equivalently, for

$$P\{X(E) \cap F \neq \emptyset\} = 0. \tag{3.7}$$

In particular, he proved the following result.

Lemma 3.1 *If ϕ-$m_\alpha(E \times F) = 0$, where $\phi(s) = t^d(\log 1/t)^{N/(2\alpha)}$, then (3.7) holds; if*

$$\mathrm{Cap}_h(E \times F) > 0 \ ,$$

where

$$h(s,t;x,y) = \frac{1}{\max\{|t-s|^{\alpha d}, |x-y|^d\}} \ , \tag{3.8}$$

then $P\{X(E) \cap F \neq \emptyset\} > 0$. In particular, if $\mathrm{Cap}_{d-N/\alpha}(F) > 0$ then $P\{X^{-1}(F) \neq \emptyset\} > 0$.

Remark. 1. If $X(t)$ is Brownian motion in $\mathbf{R}$, conditions which imply (3.7) and the Hausdorff dimensions of $X(E) \cap F$ and $E \cap X^{-1}(F)$ were first considered by Kaufman [11]. Hawkes [6] and Kahane [7] generalized these results to stable subordinators and Lévy stable processes respectively.

2. If $X(t)$ is Brownian motion in $\mathbf{R}$, then Kaufman and Wu [12] have shown that for $E \subset \mathbf{R}_+$, $F = \{x_0\}$,

$$P\{X(E) \cap F \neq \emptyset\} > 0 \ \text{ if and only if } \ \mathrm{Cap}_h(E \times F) > 0 \ . \tag{3.9}$$

It is not known, even for Brownian motion, whether (3.9) holds for a general F.

In this section, we will first prove a sufficient condition for $E \times F$ to be a polar set for $(t, X(t))$, which improves the results of Testard mentioned above, and can be applied to study the intersections of independent fractional Brownian motions. Then we will apply Lemma 3.1 to calculate the Hausdorff dimension of the smallest sets $F \subset \mathbf{R}^d$ that can be hit by $X(t)$ when t is restricted to a compact set $E \subset \mathbf{R}^N \backslash \{0\}$ and the Hausdorff dimension of the smallest sets $E \subset \mathbf{R}^N \backslash \{0\}$ whose image under $X(t)$ can intersect $F \subset \mathbf{R}^d$. See Theorems 3.3 and 3.4 below.

The following lemma is similar to Lemma 3.1 in Xiao [35], the proof is easier.

Lemma 3.2 *Let $X(t)$ $(t \in \mathbf{R}^N)$ be the fractional Brownian motion in $\mathbf{R}^d$ of index α. For any $\delta > 0$, there exist positive finite constant K and r_0, which depend on N, α, d and δ only, such that for any $x \in \mathbf{R}^N$ with $|x| \geq \delta$, any $0 < r < r_0$ and any $y \in \mathbf{R}^d$*

$$P\Big\{\inf_{t \in B(x, r^{1/\alpha})} |X(t) - y| < r\Big\} \leq K r^d \ .$$

Theorem 3.1 *Let $X(t)$ $(t \in \mathbf{R}^N)$ be the d-dimensional Brownian motion in $\mathbf{R}^d$ of index α and let $E \subseteq \mathbf{R}^N \backslash \{0\}$, $F \subseteq \mathbf{R}^d$ be compact sets. If $s^d\text{-}m_\alpha(E \times F) = 0$, then (3.7) holds.*

Proof. Since $E \subseteq \mathbf{R}^N \backslash \{0\}$ is compact, there exists $\delta > 0$ such that $d(0, E) \geq 2\delta$. Hence every open ball with radius less than δ intersecting E has distance from 0 greater than δ. Since $s^d\text{-}m_\alpha(E \times F) = 0$, it follows from Theorem 32 in Rogers [23] that there exists a sequences of open balls

$$B_\alpha((t_n, y_n), r_n) = B(t_n, r_n^{1/\alpha}) \times B(y_n, r_n) \quad (n = 1, 2, \cdots)$$

in the metric space $(\mathbf{R}^N \times \mathbf{R}^d, \rho_\alpha)$ such that

$$E \times F \subseteq \limsup_{n \to \infty} B_\alpha((t_n, y_n), r_n) \tag{3.10}$$

and

$$\sum_{n=1}^{\infty} r_n^d < \infty \ . \tag{3.11}$$

Let

$$D_n = \{X(B(t_n, r_n^{1/\alpha})) \cap B(y_n, r_n) \neq \emptyset\} \ ,$$

then by Lemma 3.2 we have $P(D_n) \leq K r_n^d$. It follows from (3.11) that

$$\sum_{n=1}^{\infty} P(D_n) < \infty \ .$$

Therefore the Borel-Cantelli lemma implies

$$P\{\limsup_{n \to \infty} D_n\} = 0 \ .$$

On the other hand, by (3.10) we have

$$\{X(E) \cap F \neq \emptyset\} \subseteq \limsup_{n \to \infty} D_n \ .$$

This finishes the proof of Theorem 3.1.

Corollary 3.1 *For any compact set $F \subseteq \mathbf{R}^d \backslash \{0\}$ with*

$$s^{d - N/\alpha}\text{-}m(F) = 0 \ , \tag{3.12}$$

then $X^{-1}(F) = \emptyset$ a. s..

Proof. By Theorem 3.1, it suffices to prove that for any closed cube $I \subseteq \mathbf{R}^N \backslash \{0\}$, we have

$$s^d\text{-}m_\alpha(I \times F) = 0 \ . \tag{3.13}$$

Since $s^{d-N/\alpha}\text{-}m(F) = 0$, there exists a sequences of open balls $\{B(y_n, r_n)\}$ in $\mathbf{R}^d$ such that

$$F \subseteq \limsup_{n\to\infty} B(y_n, r_n) \quad \text{and} \quad \sum_{n=1}^{\infty} r_n^{d-N/\alpha} < \infty \ . \tag{3.14}$$

For each $n \geq 1$, I can be covered by $N(I,\ r_n^{1/\alpha})$ balls of radius $r_n^{1/\alpha}$ and

$$N(I,\ r_n^{1/\alpha}) \leq \frac{K}{r_n^{N/\alpha}} \ . \tag{3.15}$$

Hence

$$I \times F \subseteq \limsup_{n\to\infty} \Big(\bigcup_{i=1}^{N(I, r_n^{1/\alpha})} B(t_i, r_n^{1/\alpha}) \times B(y_n, r_n) \Big)$$

and by (3.14) and (3.15) we have

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N(I, r_n^{1/\alpha})} r_n^d \leq \sum_{n=1}^{\infty} \frac{K}{r_n^{N/\alpha}} \cdot r_n^d < \infty \ .$$

Using Theorem 32 in Rogers [23] again, we obtain (3.13).

If $N > \alpha d$, then by a result of Monrad and Pitt [18], with probability 1 for any Borel set $F \subseteq \mathbf{R}^d$, $X^{-1}(F) \neq \emptyset$ and

$$\dim X^{-1}(F) = N - \alpha d + \alpha \dim F \ .$$

In the case of $N < \alpha d$, there is an obvious gap between (3.12) and $\text{Cap}_{d-N/\alpha}(F) = 0$. It is natural to ask whether $\text{Cap}_{d-N/\alpha}(F) = 0$ is a necessary and sufficient condition for $X^{-1}(F) = \emptyset$ *a. s.*.

Now we apply Corollary 3.1 to study the existence of intersections of two independent fractional Brownian motions. Let $X^1(t)$ $(t \in \mathbf{R}^N)$ and $X^2(t)$ $(t \in \mathbf{R}^N)$ be two independent fractional Brownian motions in $\mathbf{R}^d$ with index $0 < \alpha < 1$. It is known that if $2N < \alpha d$, then with probability 1, X^1 and X^2 do not intersect; if $2N > \alpha d$, then with probability 1, X^1 and X^2 intersect. We can decide the critical case $2N = \alpha d$ by using Corollary 3.1 and a theorem of Talagrand [24].

Theorem 3.2 *Let $X^1(t)$ $(t \in \mathbf{R}^N)$ and $X^2(t)$ $(t \in \mathbf{R}^N)$ be two independent fractional Brownian motions in $\mathbf{R}^d$ with index α_1 and α_2 $(0 < \alpha_1, \alpha_2 < 1)$ respectively. If*

$$\frac{N}{\alpha_1} + \frac{N}{\alpha_2} \leq d,$$

then with probability 1,

$$X^1(\mathbf{R}^N) \cap X^2(\mathbf{R}^N \backslash \{0\}) = \emptyset \ . \tag{3.16}$$

In particular, if $\alpha_1 = \alpha_2$ and $2N = \alpha d$, then (3.16) holds.

Proof. It is sufficient to prove that for any closed cube $J \subseteq \mathbf{R}^N \backslash \{0\}$,

$$X^1(\mathbf{R}^N) \cap X^2(J) = \emptyset \ \ a.\ s. \tag{3.17}$$

By a theorem of Talagrand [24], we have

$$0 < s^{N/\alpha_2} \log\log \tfrac{1}{s}\text{-}m\left(X^2(J)\right) < \infty \ .$$

Hence

$$s^{d-N/\alpha_1}\text{-}m\left(X^2(J)\right) = 0 \ .$$

Let $F = X^2(J)$ in Corollary 3.1, by the independence of X^1 and X^2, we obtain (3.17).

Remark. By studying multiple polar sets for $X(t)$, we can prove similar results for the intersection of k independent fractional Brownian motions. Recently, Talagrand [25] studied self-intersection of fractional Brownian motion $X(t) (t \in \mathbf{R}^N)$ of index α in $\mathbf{R}^d$ and proved that if $Nk = (k-1)\alpha d$, then with probability 1, $X(t)$ has no multiple points of multiplicity k.

A compact set $E_\gamma \subset \mathbf{R}^N$ is called a Cantor-type set if $E_\gamma = \cap_{n=1}^{\infty} E_n$, where

$$E_n = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_n=1}^{N_{i_1 \cdots i_{n-1}}} I_{i_1 \cdots i_n} \quad (n = 1,\ 2,\ \cdots)$$

is a decreasing sequence of compact sets and for each $n \geq 1$, $I_{i_1 \cdots i_n}$ $(i_n = 1, \cdots, N_{i_1 \cdots i_{n-1}})$ are disjoint closed subcubes of $I_{i_1 \cdots i_{n-1}}$. The following lemma is proved in Talagrand and Xiao [26] for $N = 1$. The proof for the case $N > 1$ is the same.

Lemma 3.3 *Let $E \subseteq \mathbf{R}^N$ be compact. For every $0 < \gamma < \mathrm{Dim} E$, there exists a Cantor-type set $E_\gamma = \cap_{n=1}^{\infty} E_n$ with $E_\gamma \subseteq E$ and satisfies the following properties* *(i)* $E_1 = \cup_{i_1=1}^{N_0} I_{i_1}$, *where*

$$N_0 = \left[\frac{1}{\eta_0^\gamma}\right] + 1 \quad (\textit{with } \eta_0^{\gamma/d} \log \frac{1}{\eta_0} < \frac{1}{2}) \ , \tag{3.18}$$

$[x]$ is the integer part of x , and $\{I_{i_1}\}$ $(i_1 = 1, \cdots, N_0)$ are closed intervals of length $2\delta_0 \le \frac{1}{2}\eta_0$ with gap between any two of them greater than η_0.

(ii) *For $n \ge 2$, $E_n = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_n=1}^{N_{i_1 \cdots i_{n-1}}} I_{i_1 \cdots i_n}$ with*

$$N_{i_1 \cdots i_{n-1}} = \left[\frac{1}{\eta_{i_1 \cdots i_{n-1}}^\gamma}\right] + 1 \ , \tag{3.19}$$

and

$$|I_{i_1 \cdots i_n}| = 2\delta_{i_1 \cdots i_{n-1}} \le \frac{1}{2}\eta_{i_1 \cdots i_{n-1}} \ ,$$

where $|I|$ denotes the diameter of I. Each interval $I_{i_1 \cdots i_{n-1}}$ of E_{n-1} contains $N_{i_1 \cdots i_{n-1}}$ closed subintervals $I_{i_1 \cdots i_n}$ $(i_n = 1, 2, \cdots, N_{i_1 \cdots i_{n-1}})$ with gaps greater than $\eta_{i_1 \cdots i_{n-1}}$.

(iii) *There exists a Borel probability measure σ on $\mathbf{R}$ with $\sigma(E_\gamma) = 1$ such that, for each interval I_{i_1} $(i_1 = 1, \cdots, N_0)$,*

$$\sigma(I_{i_1}) = N_0^{-1} \ ,$$

and for each interval $I_{i_1 \cdots i_n}$ in E_n,

$$\sigma(I_{i_1 \cdots i_n}) = \sigma(I_{i_1 \cdots i_{n-1}}) N_{i_1 \cdots i_{n-1}}^{-1} \ . \tag{3.20}$$

(iv) *For every open set $V \subseteq \mathbf{R}$ that intersects E_γ, we have $\mathrm{Dim}(E_\gamma \cap V) \ge \gamma$.*

Remark. By the construction of E_γ, once $\eta_{i_1 \cdots i_{n-1}}$ has been chosen, we can choose $\eta_{i_1 \cdots i_{n-1} i_n}$ as small as we please. In particular, we will assume η's satisfying

$$\eta_{i_1 \cdots i_n}^{\gamma/d} \ \frac{\log 1/\eta_{i_1 \cdots i_{n-1}}}{\log 1/\eta_{i_1 \cdots i_n}} < \frac{1}{2}. \tag{3.21}$$

and

$$\sum_{n=1}^{\infty} \left(\log \frac{1}{\eta_{i_1 \cdots i_n}}\right)^{-1} < \infty \ . \tag{3.22}$$

Now we are in a position to prove the following theorem.

Theorem 3.3 *Let $X(t)(t \in \mathbf{R}^N)$ be the fractional Brownian motion in $\mathbf{R}^d$ of index α with $N \le \alpha d$. Let $E \subset \mathbf{R}^N \backslash \{0\}$ be a Borel set, Then with probability 1,*

$$\inf\Big\{\dim F : F \in \mathcal{B}(\mathbf{R}^d),\ P\{X(E) \cap F \ne \emptyset\} > 0\Big\} = d - \frac{1}{\alpha}\mathrm{Dim} E \ . \tag{3.23}$$

Proof. For any Borel set $F \subset \mathbf{R}^d$ with $\dim F < d - \frac{1}{\alpha}\mathrm{Dim}E$, by (3.6) we have $\dim_\alpha(E \times F) < d$ and hence $s^d\text{-}m_\alpha(E \times F) = 0$. It follows from Theorem 3.1 that $P\{X(E) \cap F \ne \emptyset\} = 0$. This implies that

$$\inf\Big\{\dim F : F \in \mathcal{B}(\mathbf{R}^d),\ P\{X(E) \cap F \ne \emptyset\} > 0\Big\} \ge d - \frac{1}{\alpha}\mathrm{Dim} E \ .$$

To prove the reverse inequality, it suffices to show that for every $0 < \gamma < \mathrm{Dim}E$, there exists a compact set $F \subset \mathbf{R}^d$ such that

$$\dim F \le d - \frac{\gamma}{\alpha} \quad \text{and} \quad \mathrm{Cap}_h(E \times F) > 0 \ , \tag{3.24}$$

where h is the function defined by (3.8). The method of construction of a Cantor-type set $F \subset \mathbf{R}^d$ satisfying (3.24) is similar to the method in Xiao [36], but since the second inequality in (3.24) is stronger than the conclusion in Xiao [36], several improvements have to be made.

Fix a $0 < \gamma < \mathrm{Dim}E$, let E_γ be the Cantor-type set in Lemma 3.3. Now we construct a decreasing sequence of closed subsets $F_n \subset [0,1]^d$ inductively and then define $F = \cap_{n=1}^\infty F_n$. Naturally, the construction of $\{F_n\}$ depends on the structure of $\{E_n\}$ and the metric ρ_α. To simplify the notations, from now on, we will not distinguish a positive number from its integer part.

For $n = 1$, let

$$b_1 = \left(\frac{1}{{\eta_0}^{\alpha - \gamma/d}} \log \frac{1}{\eta_0}\right)^d \ , \tag{3.25}$$

where η_0 is the constant in (3.18), and let $F_1 = \cup_{j_1=1}^{b_1} J_{j_1}$, where J_{j_1} $(j_1 = 1, \cdots, b_1)$ are closed subcubes of $[0,1]^d$ of side length η_0^α with gaps at least $\tilde{\eta}_0$ and

$$\tilde{\eta}_0 \approx \frac{\eta_0^{\alpha-\gamma/d}}{2 \log \frac{1}{\eta_0}},$$

where $a \approx b$ means $\frac{1}{2}b \le a \le b$. F_1 is well defined since

$$\frac{1}{{\eta_0}^{\alpha-\gamma/d}} \log \frac{1}{\eta_0}\Big(\tilde{\eta}_0 + \eta_0^\alpha\Big) \le \frac{1}{2} + \eta_0^{\gamma/d}\Big(\log\frac{1}{\eta_0}\Big) < 1 \ ,$$

we can construct $b_1^{1/d}$ closed subintervals of length η_0^α with gaps at least $\tilde{\eta}_0$ in $[0,1]$ in each coordinate direction, and then let J_{j_1} be the Cartesian products of these closed subintervals. By (3.25), we also have

$$\sum_{j_1=1}^{b_1} \psi(|J_{j_1}|) \le K_{\psi,N} \ ,$$

where

$$\psi(s) = \frac{s^{d-\gamma/\alpha}}{(\log 1/s)^d}$$

and $K_{\psi,N}$ is a finite constant depending on ψ and N only.

For $n = 2$, by Lemma 3.3 (ii), $E_2 = \cup_{i_1=1}^{N_0} \cup_{i_2=1}^{N_{i_1}} I_{i_1 i_2}$. In order to construct F_2, we construct F_{2,i_1} for $i_1 = 1, \cdots, N_0$, and then define $F_2 = \cup_{i_1=1}^{N_0} F_{2,i_1}$.

For each fixed i_1, to construct F_{2,i_1}, let

$$b_{2,i_1} = \left(\frac{\eta_0^\alpha}{{\eta_{i_1}}^{\alpha-\gamma/d}} \cdot \frac{\log 1/\eta_{i_1}}{\log 1/\eta_0} \right)^d .$$

In each cube J_{j_1} of F_1, we construct b_{2,i_1} closed subcubes $J_{j_1 j_2}^{(i_1)}$ of side length $\eta_{i_1}^\alpha$ with gaps at least $\tilde{\eta}_{i_1}$ and

$$\tilde{\eta}_{i_1} \approx \frac{1}{2} \eta_{i_1}^{\alpha-\gamma/d} \frac{\log 1/\eta_0}{\log 1/\eta_{i_1}} \ .$$

This is possible since

$$\frac{\eta_0^\alpha}{{\eta_{i_1}}^{\alpha-\gamma/d}} \cdot \frac{\log 1/\eta_{i_1}}{\log 1/\eta_0} \left(\tilde{\eta}_{i_1} + \eta_{i_1}^\alpha \right) \le \left(\frac{1}{2} + \eta_{i_1}^{\gamma/d} \frac{\log 1/\eta_{i_1}}{\log 1/\eta_0} \right) \cdot \eta_0^\alpha < \eta_0^\alpha .$$

We set

$$F_{2,i_1} = \bigcup_{j_1=1}^{b_1} \bigcup_{j_2=1}^{b_{2,i_1}} J_{j_1 j_2}^{(i_1)}$$

for $i_1 = 1, 2, \cdots, N_0$ and let

$$F_2 = \bigcup_{i_1=1}^{N_0} F_{2,i_1} \ .$$

Then by the choices of N_0, b_1 and b_{2,i_1}

$$\sum_{i_1=1}^{N_0} \sum_{j_1=1}^{b_1} \sum_{j_2=1}^{b_{2,i_1}} \psi(|\ J_{j_1 j_2}^{(i_1)}\ |) \le K_{\psi,N} \ .$$

Suppose now that

$$F_{n-1} = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_{n-2}=1}^{N_{i_1 \cdots i_{n-3}}} F_{n-1, i_1 \cdots i_{n-2}}$$

has been constructed with

$$F_{n-1,i_1\cdots i_{n-2}} = \bigcup_{j_1=1}^{b_1} \cdots \bigcup_{j_{n-1}=1}^{b_{n-1,i_1\cdots i_{n-2}}} J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})} ,$$

where $J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})}$ has side length$\eta_{i_1\cdots i_{n-2}}^{\alpha}$, and

$$\sum_{i_1=1}^{N_0} \cdots \sum_{i_{n-2}=1}^{N_{i_1\cdots i_{n-3}}} b_1 b_{2,i_1} \cdots b_{n-1,i_1\cdots i_{n-2}} \cdot \psi(\eta_{i_1\cdots i_{n-2}}^{\alpha}) \leq K_{\psi,N}. \tag{3.26}$$

We will construct F_n in the same way as that for $n = 2$. By Lemma 3.3 (ii),

$$E_n = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n} .$$

For each fixed $i_1 \cdots i_{n-1}$, we define

$$b_{n,i_1\cdots i_{n-1}} = \left(\frac{\eta_{i_1\cdots i_{n-2}}^{\alpha}}{\eta_{i_1\cdots i_{n-1}}^{\alpha-\gamma/d}} \cdot \frac{\log 1/\eta_{i_1\cdots i_{n-1}}}{\log 1/\eta_{i_1\cdots i_{n-2}}} \right)^d . \tag{3.27}$$

In each cube $J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})}$ of $F_{n-1,i_1\cdots i_{n-2}}$, we construct $b_{n,i_1\cdots i_{n-1}}$ closed subcubes $J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})}$ of side length $\eta_{i_1\cdots i_{n-1}}^{\alpha}$ and gaps at least $\tilde{\eta}_{i_1\cdots i_{n-1}}$ with

$$\tilde{\eta}_{i_1\cdots i_{n-1}} \approx \frac{1}{2} \eta_{i_1\cdots i_{n-1}}^{\alpha-\gamma/d} \frac{\log 1/\eta_{i_1\cdots i_{n-2}}}{\log 1/\eta_{i_1\cdots i_{n-1}}}. \tag{3.28}$$

This is possible since by (3.27), (3.28) and (3.21)

$$\begin{aligned} & \frac{\eta_{i_1\cdots i_{n-2}}^{\alpha}}{\eta_{i_1\cdots i_{n-1}}^{\alpha-\gamma/d}} \cdot \frac{\log 1/\eta_{i_1\cdots i_{n-1}}}{\log 1/\eta_{i_1\cdots i_{n-2}}} \left(\tilde{\eta}_{i_1\cdots i_{n-1}} + \eta_{i_1\cdots i_{n-1}}^{\alpha} \right) \\ & \leq \left(\frac{1}{2} + \eta_{i_1\cdots i_{n-1}}^{\gamma/d} \frac{\log 1/\eta_{i_1\cdots i_{n-2}}}{\log 1/\eta_{i_1\cdots i_{n-1}}} \right) \cdot \eta_{i_1\cdots i_{n-2}}^{\alpha} \\ & < \eta_{i_1\cdots i_{n-2}}^{\alpha} . \end{aligned}$$

Let $F_{n,i_1\cdots i_{n-1}}$ be the union of all the subcubes $J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})}$,

$$F_{n,i_1\cdots i_{n-1}} = \bigcup_{j_1=1}^{b_1} \bigcup_{j_2=1}^{b_{2,i_1}} \cdots \bigcup_{j_n=1}^{b_{n,i_1\cdots i_{n-1}}} J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})} .$$

As $i_1 \cdots i_{n-1}$ varies, we obtain a sequence $\{F_{n,i_1\cdots i_{n-1}}\}$ of compact sets. Let

$$F_n = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_{n-1}=1}^{N_{i_1\cdots i_{n-2}}} F_{n,i_1\cdots i_{n-1}} .$$

Then by (3.26) and (3.27), we have

$$\sum_{i_1=1}^{N_0} \cdots \sum_{i_{n-1}=1}^{N_{i_1\cdots i_{n-2}}} b_1 b_{2,i_1} \cdots b_{n,i_1\cdots i_{n-1}} \cdot \psi(\eta^{\alpha}_{i_1\cdots i_{n-1}}) \le K_{\psi,N} \ . \tag{3.29}$$

By induction, we have constructed a decreasing sequence of closed sets $\{F_n\}$ satisfying (3.28) and (3.29). Set $F = \cap_{n=1}^{\infty} F_n$; then $F \subseteq [0,1]^d$ is a compact set.

Now we verify that F satisfies the two inequalities in (3.24). Since for each $n \ge 1$, $F \subseteq F_n$, equation (3.29) implies that

$$\dim F \le d - \gamma/\alpha.$$

To prove the second inequality in (3.24), we observe that for each $n \ge 1, E_n \times F_n \supseteq G_n$, where $G_1 = E_1 \times F_1$ and for $n \ge 2$,

$$G_n = \bigcup_{i_1=1}^{N_0} \cdots \bigcup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} \bigcup_{j_1=1}^{b_1} \bigcup_{j_2=1}^{b_{2,i_1}} \cdots \bigcup_{j_n=1}^{b_{n,i_1\cdots i_{n-1}}} I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}.$$

Let $G = \cap_{n=1}^{\infty} G_n$; then G is compact and $G \subseteq E_\gamma \times F$. So it is sufficient to prove that

$$\mathrm{Cap}_h(G) > 0. \tag{3.30}$$

To this end, we first define a Borel measure μ on $\mathbf{R}^{N+d}$ with $\mu(G) = 1$ as follows. Let σ be the Borel measure carried by E_γ in Lemma 3.3 (iii). For each rectangle $I_{i_1} \times J_{j_1}$ in G_1, we define

$$\mu(I_{i_1} \times J_{j_1}) = \frac{\sigma(I_{i_1})}{b_1} = \frac{\eta_0^{\alpha d}}{(\log 1/\eta_0)^d}.$$

For each rectangle $I_{i_1 i_2} \times J^{(i_1)}_{j_1 j_2}$ in G_2, we define

$$\begin{aligned} \mu(I_{i_1 i_2} \times J^{(i_1)}_{j_1 j_2}) &= \frac{\sigma(I_{i_1 i_2})}{b_1 b_{2,i_1}} \\ &= (\eta_0 \eta_{i_1})^{\gamma} \cdot \frac{\eta_{i_1}^{\alpha d - \gamma}}{\eta_0^{\gamma} (\log 1/\eta_{i_1})^d} \\ &= \frac{\eta_{i_1}^{\alpha d}}{(\log 1/\eta_{i_1})^d} \ . \end{aligned}$$

Similarly, for each rectangle $I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}$ in G_n, we define

$$\begin{aligned} \mu(I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}) &= \frac{\sigma(I_{i_1\cdots i_n})}{b_1 b_{2,i_1} \cdots b_{n,i_1\cdots i_{n-1}}} \\ &= \frac{\eta^{\alpha d}_{i_1\cdots i_{n-1}}}{(\log 1/\eta_{i_1\cdots i_{n-1}})^d} \ . \end{aligned} \tag{3.31}$$

Finally, for each $n \geq 1$, we define $\mu(\mathbf{R}^{N+d} \setminus G_n) = 0$. Then by the mass distribution principle (see Falconer [5]) μ can be extended to a Borel measure on $\mathbf{R}^{N+d}$ with $\mu(G) = 1$.

For every $(s, x) \in G$, there exist two sequences $\mathbf{i} = i_1 i_2 \cdots i_n \cdots$ and $\mathbf{j} = j_1 j_2 \cdots j_n \cdots$, such that

$$\{(s, x)\} = \bigcap_{n=1}^{\infty} I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}.$$

For any $0 < r < \eta_0^\alpha$, we consider $\mu(B_\alpha((s, x), r))$. Let n be the positive integer satisfying

$$\eta_{i_1 \cdots i_{n-1}}^\alpha \leq r < \eta_{i_1 \cdots i_{n-2}}^\alpha \qquad (i_0 \hat{=} 0).$$

Consider first the case $\eta_{i_1 \cdots i_{n-1}}^\alpha \leq r < \tilde{\eta}_{i_1 \cdots i_{n-1}}$. Since $B_\alpha((s, x), r) = B(s, r^{1/\alpha}) \times B(x, r)$ and the gaps between any two subcubes $I_{i_1 \cdots i_n}$ are at least $\eta_{i_1 \cdots i_{n-1}}$, $B(x, r^{1/\alpha})$ can intersect at most

$$K \Big(\frac{r^{1/\alpha}}{\eta_{i_1 \cdots i_{n-1}}} \Big)^N \leq K \frac{r^d}{\eta_{i_1 \cdots i_{n-1}}^{\alpha d}}$$

rectangles $I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}$. So by (3.31), we have

$$\begin{aligned} \mu\Big(B_\alpha((s, x), r)\Big) &\leq K \frac{r^d}{\eta_{i_1 \cdots i_{n-1}}^{\alpha d}} \, \mu\Big(I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}\Big) \\ &\leq K \frac{r^d}{(\log 1/r)^d} \, . \end{aligned} \tag{3.32}$$

In the second case $\tilde{\eta}_{i_1 \cdots i_{n-1}} \leq r < \eta_{i_1 \cdots i_{n-2}}^\alpha$, an inequality similar to (3.32) does not hold in general. Otherwise, (3.30) would follow from the same proof as that of Frostman's theorem (see Kahane [8]). For each $1 \leq k < b_{n, i_1 \cdots i_{n-1}}^{1/d}$, since the gaps between any two cubes $J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}$ are at least $\tilde{\eta}_{i_1 \cdots i_{n-1}}$ and the gaps between any two cubes $I_{i_1 \cdots i_{n-1}}$ are greater than $\eta_{i_1 \cdots i_{n-2}}$, we see that

$$B_\alpha\Big((s, x), (k+1)\tilde{\eta}_{i_1 \cdots i_{n-1}}\Big) \setminus B_\alpha\Big((s, x), k\tilde{\eta}_{i_1 \cdots i_{n-1}}\Big)$$

can intersect at most $\eta_{i_1 \cdots i_{n-1}}^{-\gamma}$ rectangles $I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}$. So by (3.31), we have

$$\begin{aligned} &\mu\Big(B_\alpha\Big((s, x), (k+1)\tilde{\eta}_{i_1 \cdots i_{n-1}}\Big) \setminus B_\alpha\Big((s, x), k\tilde{\eta}_{i_1 \cdots i_{n-1}}\Big)\Big) \\ &\leq \eta_{i_1 \cdots i_{n-1}}^{-\gamma} \cdot \mu\Big(I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}\Big) \\ &= \frac{\eta_{i_1 \cdots i_{n-1}}^{\alpha d - \gamma}}{(\log 1/\eta_{i_1 \cdots i_{n-1}})^d} \, . \end{aligned} \tag{3.33}$$

Consider the integral

$$
\begin{aligned}
&\int_G \frac{d\mu(t,y)}{\max\{|t-s|^{\alpha d}, |y-x|^d\}} \\
&= \sum_{n=1}^{\infty} \int_{\{\eta^{\alpha}_{i_1\cdots i_{n-1}} \le \rho_\alpha((s,x),(t,y)) < \eta^{\alpha}_{i_1\cdots i_{n-2}}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} + \frac{1}{\eta_0^{\alpha d}} \\
&= \sum_{n=1}^{\infty} \Big[\int_{\{\eta^{\alpha}_{i_1\cdots i_{n-1}} \le \rho_\alpha((s,x),(t,y)) < \tilde\eta_{i_1\cdots i_{n-1}}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \\
&\qquad + \int_{\{\tilde\eta_{i_1\cdots i_{n-1}} \le \rho_\alpha((s,x),(t,y)) < \eta^{\alpha}_{i_1\cdots i_{n-2}}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \Big] + \frac{1}{\eta_0^{\alpha d}} \ . \qquad (3.34)
\end{aligned}
$$

Let

$$
k_n = \min\Big\{k : \ \tilde\eta_{i_1\cdots i_{n-1}} 2^{-k} \le \eta^{\alpha}_{i_1\cdots i_{n-1}}\Big\} \ .
$$

Then

$$
k_n \le K \log \frac{1}{\eta_{i_1\cdots i_{n-1}}} \ . \qquad (3.35)
$$

By (3.32) and (3.35) we have

$$
\begin{aligned}
&\int_{\{\eta^{\alpha}_{i_1\cdots i_{n-1}} \le \rho_\alpha((s,x),(t,y)) < \tilde\eta_{i_1\cdots i_{n-1}}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \\
&\le \sum_{k=1}^{k_n} \int_{\{\tilde\eta_{i_1\cdots i_{n-1}} 2^{-k} \le \rho_\alpha((s,x),(t,y)) < \tilde\eta_{i_1\cdots i_{n-1}} 2^{-k+1}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \\
&\le \sum_{k=1}^{k_n} \frac{(\tilde\eta_{i_1\cdots i_{n-1}} 2^{-k+1})^d}{(\tilde\eta_{i_1\cdots i_{n-1}} 2^{-k})^d} \cdot \frac{1}{(\log 1/(\tilde\eta_{i_1\cdots i_{n-1}} 2^{-k+1}))^d} \\
&\le \frac{K}{(\log 1/\eta_{i_1\cdots i_{n-1}})^{d-1}} \ . \qquad (3.36)
\end{aligned}
$$

It follows from (3.33) that

$$
\begin{aligned}
&\int_{\{\tilde\eta_{i_1\cdots i_{n-1}} \le \rho_\alpha((s,x),(t,y)) < \eta^{\alpha}_{i_1\cdots i_{n-2}}\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \\
&\le \sum_{k=1}^{b^{1/d}_{n,i_1\cdots i_{n-1}}-1} \int_{\{\tilde\eta_{i_1\cdots i_{n-1}} k \le \rho_\alpha((s,x),(t,y)) < \tilde\eta_{i_1\cdots i_{n-1}}(k+1)\}} \frac{d\mu(t,y)}{\rho_\alpha((s,x),(t,y))^d} \\
&\le \sum_{k=1}^{b^{1/d}_{n,i_1\cdots i_{n-1}}-1} \frac{\eta^{\alpha d-\gamma}_{i_1\cdots i_{n-1}}}{(k\tilde\eta_{i_1\cdots i_{n-1}} \cdot \log 1/\eta_{i_1\cdots i_{n-1}})^d} \\
&\le \frac{K}{(\log 1/\eta_{i_1\cdots i_{n-2}})^d} \ . \qquad (3.37)
\end{aligned}
$$

Combining (3.34), (3.36), (3.37) and (3.22), we have shown that there exists a finite constant $K > 0$ such that for every $(s, x) \in G$

$$\int_G \frac{d\mu(t,y)}{\max\{|t-s|^{\alpha d}, |y-x|^d\}} \le K \ .$$

This proves (3.30) and hence Theorem 3.3.

Remark The proof of Theorem 3.3 also yields the following result analogous to those in Xiao [36] and Bishop and Peres [1]: For any Borel set $E \subset \mathbf{R}^N$

$$\frac{\text{Dim}E}{\alpha} = \sup\Big\{\dim_\alpha(E \times F) - \dim F : \ F \subseteq \mathbf{R}^d \text{ compact } \Big\} \ .$$

Theorem 3.4 *Let $X(t)(t \in \mathbf{R}^N)$ be the fractional Brownian motion in $\mathbf{R}^d$ of index α with $N > \alpha d$. Let $F \subset \mathbf{R}^d \backslash \{0\}$ be a Borel set. Then with probability 1,*

$$\inf\Big\{\dim E : E \in \mathcal{B}(\mathbf{R}^N \backslash \{0\}), \ P\{X(E) \cap F \neq \emptyset\} > 0\Big\} = \alpha(d - \text{Dim}F) \ . \qquad (3.38)$$

Proof. The upper bound follows easily from (3.6) and Theorem 3.1. The proof of the lower bound is similar to the proof of Theorem 3.3. We leave the details to the interested reader.

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