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Probability Theory and Related Fields

Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields

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Summary. Let $Y(t)$ $(t \in \mathbb{R}^N)$ be a real-valued, strongly locally nondeterministic Gaussian random field with stationary increments and $Y(0) = 0$. Consider the (N, d) Gaussian random field defined by

$$X(t) = (X_1(t), \ldots, X_d(t)) \ \left(t \in \mathbb{R}^N\right) \ ,$$

where $X_1, \ldots, X_d$ are independent copies of Y. The local and global Hölder conditions in the set variable for the local time of $X(t)$ are established and the exact Hausdorff measure of the level set $X^{-1}(x)$ is evaluated.

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1 Introduction

Let $Y(t)$ $(t \in \mathbb{R}^N)$ be a real-valued, centered Gaussian random field with $Y(0) = 0$. We assume that $Y(t)$ $(t \in \mathbb{R}^N)$ has stationary increments and continuous covariance function $R(t, s) = \mathrm{E}Y(t)Y(s)$ given by

$$R(t,s) = \int_{\mathbb{R}^N} \left(\mathrm{e}^{i\langle t,\lambda\rangle} - 1\right)\left(\mathrm{e}^{-i\langle s,\lambda\rangle} - 1)\Delta(d\lambda\right) , \tag{1.1}$$

where $\langle x, y\rangle$ is the ordinary scalar product in $\mathbb{R}^N$ and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ satisfying

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$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2} \Delta(d\lambda) < \infty \ . \tag{1.2}$$

Then there exists a centered complex-valued Gaussian random measure $W(d\lambda)$ such that

$$Y(t) = \int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) W(d\lambda) \tag{1.3}$$

and for any Borel sets $A,\ B \subseteq \mathbb{R}^N$

$$E\Big(W(A)\overline{W(B)}\Big) = \Delta(A \cap B) \text{ and } W(-A) = \overline{W(A)} \ .$$

It follows from (1.3) that

$$E\left[\left(Y(t+h) - Y(t)\right)^2\right] = 2\int_{\mathbb{R}^N} \left(1 - \cos\ \langle h, \lambda\rangle\right)\ \Delta(d\lambda) \ . \tag{1.4}$$

We assume that there exist constants $\delta_0 > 0,\ 0 < c_1 \le c_2 < \infty$ and a non-decreasing, continuous function σ: $[0, \delta_0) \to [0, \infty)$ which is regularly varying at the origin with index α $(0 < \alpha < 1)$ such that for any $t \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$ with $|h| \le \delta_0$

$$E\left[\left(Y(t+h) - Y(t)\right)^2\right] \le c_1\sigma^2(|h|) \ . \tag{1.5}$$

and for all $t \in \mathbb{R}^N$ and any $0 < r \le \min\{|t|,\ \delta_0\}$

$$\mathrm{Var}\big(Y(t)|Y(s) : r \le |s-t| \le \delta_0\big) \ge c_2\sigma^2(r) \ . \tag{1.6}$$

If (1.5) and (1.6) hold, we shall say that $Y(t)$ $(t \in \mathbb{R}^N)$ is strongly locally σ-nondeterministic. A typical example of strongly locally nondeterministic Gaussian random fields is the so-called fractional Brownian motion in $\mathbb{R}$ of index α $(0 < \alpha < 1)$, i.e. the centered, real-valued Gaussian random field $Y(t)$ $(t \in \mathbb{R}^N)$ with covariance function

$$E(Y(t)Y(s)) = \tfrac{1}{2}\Big(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\Big) \ .$$

See Lemma 7.1 of Pitt (1978) for a proof of the strong local nondeterminism of $Y(t)$. General conditions for strong local nondeterminism of Gaussian processes $Y(t)$ $(t \in \mathbb{R})$ are given by Marcus (1968) and Berman (1972, 1978). They proved that if (i) $\sigma(h) \to 0$ as $h \to 0$ and $\sigma^2(h)$ is concave on $(0, \delta)$; or (ii) $Y(t)$ has stationary increments and the absolutely continuous component of the spectral measure Δ has a density $f(\lambda)$ which satisfies

$$f(\lambda) \ge K|\lambda|^{-\alpha-1}$$

for large $|\lambda|$. Then $Y(t)$ is strongly locally nondeterministic. It is clear that the Gaussian processes considered by Csörgő, Lin and Shao (1995) are strongly locally nondeterministic. We also refer to Monrad and Pitt (1986), Cuzick and Du Peez (1982) and Xiao (1996) for more information on strong local nondeterminism and its use in studying sample path properties of Gaussian random fields.

We associate with $Y(t)$ $(t \in \mathbb{R}^N)$ a Gaussian random field $X(t)$ $(t \in \mathbb{R}^N)$ in $\mathbb{R}^d$ by

$$X(t) = (X_1(t), \ldots, X_d(t)) \ , \tag{1.7}$$

where $X_1, \ldots, X_d$ are independent copies of Y. If $Y(t)$ is the fractional Brownian motion in $\mathbb{R}$ of index α, then $X(t)$ is called d-dimensional fractional Brownian motion of index α (see Kahane (1985)). When $N = 1, \alpha = \frac{1}{2}$, $X(t)$ is the ordinary d-dimensional Brownian motion.

It is known (cf. Pitt (1978), Kahane (1985)) that for any rectangle $I \subseteq \mathbb{R}^N$, if

$$\int_I \int_I \frac{dt\, ds}{\sigma(|t-s|)^d} < \infty \ ,$$

then almost surely the local time $L(x, I)$ of $X(t)$ $(t \in I)$ exists and is square integrable. For many Gaussian random fields including fractional Brownian motion, this condition is also necessary. The joint continuity as well as Hölder conditions in both space variable and (time) set variable of the local times of locally nondeterministic Gaussian processes and fields have been studied by Berman (1969, 1972, 1973), Pitt (1978), Davis (1976), Kôno (1977), Cuzick (1982a), Geman and Horowitz (1980), Geman, Horowitz and Rosen (1984), and recently by Csörgő, Lin and Shao (1995).

This paper is partially motivated by the following beautiful results about the local time of Brownian motion. Let $l(x, t)$ be the local time of a standard Brownian motion $B(t)(t \geq 0)$ in $\mathbb{R}$. Kesten (1965) proved the law of iterated logarithm

$$\limsup_{h \to 0} \sup_x \frac{l(x, h)}{(2h \log\log 1/h)^{1/2}} = 1 \quad a.s. \tag{1.8}$$

Perkins (1981) proved the following global result

$$\limsup_{h \to 0} \sup_{0 \leq t \leq 1-h} \sup_x \frac{l(x, t+h) - l(x, h)}{(2h \log 1/h)^{1/2}} = 1 \quad a.s. \tag{1.9}$$

The first objective of this paper is to prove local and global Hölder conditions analogous to (1.8) and (1.9) for the local times of strongly locally nondeterministic Gaussian random field $X(t)$. The proofs of Kesten (1965) and Perkins (1981) depend on the Markov property of Brownian motion, therefore can not be carried over to the present case. The methods in this paper are based on the work of Berman (1969, 1972), Pitt (1978), Ehm (1981) and Geman, Horowitz and Rosen (1984). The new idea needed to prove our results is the use of strong local nondeterminism.

We will prove the following theorems.

Theorem 1.1 *Let $X(t)(t \in \mathbb{R}^N)$ be the Gaussian random field defined by* (1.7) *and $N > \alpha d$. For any $B \in \mathscr{B}(\mathbb{R}^N)$ define $L^*(B) = \sup_x L(x, B)$. Then there exists a positive finite constant K such that for any $\tau \in \mathbb{R}^N$ almost surely*

$$\limsup_{r\to 0} \frac{L^*(B(\tau,r))}{\phi_1(r)} \le K \ , \tag{1.10}$$

where $B(\tau, r)$ *is the* (*open*) *ball centered at* τ *with radius* r *and*

$$\phi_1(r) = \frac{r^N}{\sigma\Big(r(\log\log 1/r)^{-1/N}\Big)^d} \ .$$

Theorem 1.2 *Assume the same conditions as above. Then for any rectangle* $T \subseteq \mathbb{R}^N$, *there exists a positive finite constant* K *such that almost surely*

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B(t,r))}{\phi_2(r)} \le K \ , \tag{1.11}$$

where

$$\phi_2(r) = \frac{r^N}{\sigma(r(\log 1/r)^{-1/N})^d} \ .$$

If $X(t)$ $(t \in \mathbb{R}^N)$ is the d-dimensional fractional Brownian motion of index α $(0 < \alpha < 1)$, then by a result of Pitt (1978), conditions (1.5) and (1.6) are satisfied with $\sigma(t) = t^\alpha$. The following corollary is an immediate consequence of Theorems 1.1 and 1.2.

Corollary 1.1 *Let* $X(t)$ $(t \in \mathbb{R}^N)$ *be the* d*-dimensional fractional Brownian motion of index* α $(0 < \alpha < 1)$ *with* $N > \alpha d$. *Then for any* $\tau \in \mathbb{R}^N$ *almost surely*

$$\limsup_{r\to 0} \frac{L^*(B(\tau,r))}{r^{N-\alpha d}(\log\log 1/r)^{\alpha d/N}} \le K \ , \tag{1.12}$$

and for any rectangle $T \subseteq \mathbb{R}^N$, *almost surely*

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B(t,r))}{r^{N-\alpha d}(\log 1/r)^{\alpha d/N}} \le K \ . \tag{1.13}$$

If $\alpha = 1/2$ and $N = d = 1$, then $X(t)$ is Brownian motion in $\mathbb{R}$. By (1.8) and (1.9), we see that (1.12) and (1.13) are the best possible. In the case of real-valued Gaussian processes (i.e. $N = d = 1$), results similar to (1.10) for $L(x, B(\tau, r))$ with $x \in \mathbb{R}$ fixed instead of $L^*(B(t, r))$ have been obtained by Kôno (1977), Cuzick (1982a) and recently by Csörgő, Lin and Shao (1995). Theorem 1.2 confirms a conjecture made by Csörgő, Lin and Shao (1995) for Gaussian processes.

The most important example of Gaussian random fields which are not locally nondeterministic is the Brownian sheet or N-parameter Wiener process $W(t)$ $(t \in \mathbb{R}^N_+)$, see Orey and Pruitt (1973). Results similar to (1.10) and (1.11) for the local time of Brownian sheet were obtained by Ehm (1981). We will adapt some of his arguments to prove our results.

For Gaussian random fields considered in this paper, the problems of proving lower bounds (with different constants) for the limits considered above remain open.

Another problem we consider in this paper is the exact Hausdorff measure of the level sets of Gaussian random field $X(t)$ $(t \in \mathbb{R}^N)$. The exact Hausdorff measure functions for the level sets of Brownian motion and Lévy stable process were obtained by Taylor and Wendel (1966). In the case of Brownian motion in $\mathbb{R}$, Perkins (1981) proved that almost surely for any $x \in \mathbb{R},\ t \in \mathbb{R}_+$,

$$(r \log\log 1/r)^{1/2}\text{-}m\big(X^{-1}(x) \cap [0,t]\big) = \frac{1}{\sqrt{2}}\, L(x,[0,t]) \ , \tag{1.14}$$

where ϕ-m is ϕ-Hausdorff measure. We refer to Falconer (1990) for definition and properties of Hausdorff measure and Hausdorff dimension. The result (1.14) has been extended to some Lévy processes by Barlow, Perkins and Taylor (1986).

The Hausdorff dimension of the level sets of Gaussian random fields were considered by many authors (see Adler (1981), Kahane (1985) and the references therein). The uniform Hausdorff dimension of the level sets and inverse image of strongly locally nondeterministic Gaussian random fields were obtained by Monrad and Pitt (1986). The exact Hausdorff measure of the level sets of certain stationary Gaussian processes was considered by Davies (1976, 1977), in which she adapted partially the method of Taylor and Wendel (1966) and it is essential to assume the stationarity and $N = 1$. In light of our Theorem 1.1, it is natural to conjecture that $\phi_1(r)$ is the exact Hausdorff measure function for $X^{-1}(x)$. The second objective of this paper is to prove the following theorem.

Theorem 1.3 *Let $X(t)(t \in \mathbb{R}^N)$ be the Gaussian random field defined by* (1.7) *with $Y(t)(t \in \mathbb{R}^N)$ further satisfying*

$$\mathrm{E}(Y(t+h) - Y(t))^2 = \sigma^2(|h|)$$

and $N > \alpha d$. Let T be a closed cube in $\mathbb{R}^N\backslash\{0\}$. Then there exists a finite constant $K_1 > 0$ such that for every fixed $x \in \mathbb{R}^d$, with probability 1

$$K_1 L(x,T) \le \phi_1\text{-}m\big(X^{-1}(x) \cap T\big) < \infty \ . \tag{1.15}$$

The proof of this theorem is very much different from those of the previous work on this subject. In fact the proof of the lower bound relies much less on the specific properties of the process than those of Taylor and Wendel (1966) and Davies (1976, 1977), hence can be applied to other random fields such as the Brownian sheet. The proof of the upper bound is based upon the approach of Talagrand (1996). We believe that there exist some finite constant $K_2 > 0$ such that $K_2 L(x,T)$ is an upper bound for ϕ_1-$m(X^{-1}(x) \cap T)$. It seems that this can not be proved by the method of the present paper.

The rest of the paper is organized as follows. In Section 2 we prove some basic facts about regularly varying functions and estimates about the moments of the local time of strongly locally nondeterministic Gaussian random fields. In Section 3 we prove Theorems 1.1 and 1.2. We also apply Theorems 1.1 and 1.2 to study the degree of oscillation of the sample paths. In Section

4, we study the Hausdorff measure of the level sets of $X(t)$ and prove Theorem 1.3.

We will use $K, K_1, \dots, K_4$ to denote unspecified positive finite constants which may not necessarily be the same in each occurrence.

2 Basic estimates

We start with some facts about regularly varying functions. Since $\sigma(s)$ is regularly varying at the origin with index α, it can be written as

$$\sigma(s) = s^{\alpha} L(s) \ ,$$

where $L(s)$: $[0, \delta_0) \to [0, \infty)$ is slowing varying at the origin in the sense of Karamata and hence can be represented by

$$L(s) = \exp\left(\eta(s) + \int_s^a \frac{\epsilon(t)}{t} \, dt \right) \ ,$$

where $\eta(s)$: $[0, \delta_0] \to \mathbb{R}$, $\epsilon(s)$: $(0, a] \to \mathbb{R}$ are bounded measurable functions and

$$\lim_{s \to 0} \eta(s) = c, \ |c| < \infty; \quad \lim_{s \to 0} \epsilon(s) = 0 \ .$$

We lose nothing by restricting attention to those $\sigma(s)$ with

$$L(s) = \exp\left(\int_s^a \frac{\epsilon(t)}{t} \, dt \right) \ . \tag{2.1}$$

It follows from Theorem 1.8.2 in Bingham, Goldie and Teugels (1987) that we may and will further assume $L(s)$ varies smoothly at the origin with index 0. Then

$$\frac{s^n L^{(n)}(s)}{L(s)} \to 0 \text{ as } \ s \to 0 \text{ for } n \geq 1 \ , \tag{2.2}$$

where $L^{(n)}(s)$ is the n-th derivative of $L(s)$. By (2.2) and elementary calculations, we have

Lemma 2.1 *For any $\beta > 0$, let*

$$\tau_\beta(s) = \frac{s}{\sigma(s^{1/N})^\beta} \ .$$

If $\alpha\beta < N$, then there exist $\delta = \delta(\beta, N, \sigma) > 0$ such that $\tau_\beta(s)$ is concave on $(0, \delta)$.

Lemma 2.2 can be deduced from (2.1) by using the dominated convergence theorem (see e.g. Theorem 2.6 in Seneta (1976) or Proposition 1.5.8 in Bingham, Goldie and Teugels (1987)).

Lemma 2.2 *Let σ be a regularly varying function at the origin with index $\alpha > 0$. If $N > \alpha\beta$, then there is a constant $K > 0$ such that for $r > 0$ small enough, we have*

$$\int_0^1 \frac{s^{N-1}}{(\sigma(rs))^{\beta}} \, ds \le K(\sigma(r))^{-\beta} \; .$$

The following lemma generalizes Lemma 2.2 in Xiao (1997).

Lemma 2.3 *Let $\beta > 0$ with $\alpha\beta < N$, $0 < r < \delta$ and $s \in \mathbb{R}^N$. Then for any integer $n \ge 1$ and any distinct $t_1, \dots, t_n \in B(s, r)$, we have*

$$I = \int_{B(s,r)} \frac{dt}{\Big(\sigma(\min\{|t - t_j|, \; j = 1, \dots, n\})\Big)^{\beta}} \le K \, \frac{r^N}{(\sigma(rn^{-1/N}))^{\beta}} \; , \tag{2.3}$$

where $K > 0$ is a finite constant depending on N, d and σ only.

Proof. Let

$$\Gamma_i = \Big\{ t \in B(s,r) : \; |t - t_i| = \min\{|t - t_j|, \; j = 1, \dots, n\} \Big\} \; .$$

Then

$$B(s,r) = \bigcup_{i=1}^{n} \Gamma_i \quad \text{and} \quad \lambda_N(B(s,r)) = \sum_{i=1}^{n} \lambda_N(\Gamma_i) \; , \tag{2.4}$$

where λ_N is the Lebesgue measure in $\mathbb{R}^N$. For any $t \in \Gamma_i$, we write $t = t_i + \rho\theta$, where $\theta \in S_{N-1}$ the unit sphere in $\mathbb{R}^N$ and $0 \le \rho \le \rho_i(\theta)$. Then

$$\begin{aligned} \lambda_N(\Gamma_i) &= C_N \int_{S_{N-1}} v(d\theta) \int_0^{\rho_i(\theta)} \rho^{N-1} \, d\rho \\ &= \frac{C_N}{N} \int_{S_{N-1}} \rho_i(\theta)^N v(d\theta) \; , \end{aligned} \tag{2.5}$$

where v is the normalized surface area in S_{N-1} and C_N is a positive finite constant depending on N only. Hence by Lemmas 2.1, 2.2, (2.4), Jensen's inequality and (2.5), we have

$$\begin{aligned} I &= \sum_{i=1}^{n} \int_{\Gamma_i} \frac{dt}{\sigma(|t - t_i|)^{\beta}} \\ &= \sum_{i=1}^{n} C_N \int_{S_{N-1}} v(d\theta) \int_0^{\rho_i(\theta)} \frac{\rho^{N-1}}{\sigma(\rho)^{\beta}} \, d\rho \\ &\le \sum_{i=1}^{n} K \int_{S_{N-1}} \frac{\rho_i(\theta)^N}{\sigma(\rho_i(\theta))^{\beta}} \, v(d\theta) \\ &\le K \sum_{i=1}^{n} \tau_{\beta}\Big(\int_{S_{N-1}} \rho_i(\theta)^N v(d\theta) \Big) \\ &\le K \sum_{i=1}^{n} \tau_{\beta}(\lambda_N(\Gamma_i)) \end{aligned}$$

$$\leq Kn\ \tau_\beta\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_N(\Gamma_i)\right)$$
$$= K\frac{r^N}{\sigma(rn^{-1/N})^\beta}\ .$$

This proves (2.3).

Lemma 2.4 is due to Cuzick and Du Peez (1982).

Lemma 2.4 *Let $Z_1, \ldots, Z_n$ be the mean zero Gaussian variables which are linearly independent and assume that*

$$\int_{-\infty}^{\infty} g(v)\mathrm{e}^{-\epsilon v^2} dv < \infty$$

for all $\epsilon > 0$. Then

$$\int_{\mathbb{R}^n} g(v_1)\exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^{n} v_j Z_j\right)\right) dv_1 \ldots dv_n$$
$$= \frac{(2\pi)^{n-1}}{(\det \mathrm{Cov}(Z_1, \ldots, Z_n))^{1/2}}\int_{-\infty}^{\infty} g\left(\frac{v}{\sigma_1}\right)\mathrm{e}^{-v^2}\, dv\ ,$$

where $\sigma_1^2 = \mathrm{Var}(Z_1|Z_2, \ldots, Z_n)$ is the conditional variance of Z_1 given $Z_2, \ldots, Z_n$ and $\det \mathrm{Cov}(Z_1, \ldots, Z_n)$ is the determinant of the covariance matrix of $(Z_1, \ldots, Z_n)$.

Now we recall briefly the definition of local time. For an excellent survey on local times of both random and nonrandom vector fields, we refer to Geman and Horowitz (1980) (see also Geman, Horowitz and Rosen (1984)). Let $X(t)$ be any Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $B \subseteq \mathbb{R}^N$, the occupation measure of X is defined by

$$\mu_B(A) = \lambda_N\{t \in B : X(t) \in A\}\ ,$$

for all Borel set $A \subseteq \mathbb{R}^d$. If μ_B is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$, we say that $X(t)$ has a local time on B and define its local time $L(x, B)$ to be the Radon-Nikodym derivative of μ_B.

For a fixed rectangle $T = \prod_{i=1}^{N}[a_i, a_i + h_i]$, if we can choose $L\left(x, \prod_{i=1}^{N}[a_i, a_i + t_i]\right)$ to be a continuous function of $(x, t_1, \ldots, t_N)$, $x \in \mathbb{R}^d$, $0 \leq t_i \leq h_i$ $(i = 1, \ldots, N)$, then X is said to have a jointly continuous local time on T. Throughout this paper, we will always consider the jointly continuous version of the local time. Under this condition, $L(x, \cdot)$ can be extended to be a finite measure supported on the level set

$$X_T^{-1}(x) = \{t \in T : X(t) = x\}\ ,$$

see Adler (1981, Theorem 8.6.1). This fact has been used by Berman (1972), Adler (1978), Ehm (1981), Monrad and Pitt (1986), Rosen (1984) and the author (1995) to study the Hausdorff dimension of the level sets, inverse image and multiple points of stochastic processes.

We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^d$ respectively. It follows from (25.5) and (25.7) in Geman and Horowitz (1980) (see also Geman, Horowitz and Rosen (1984), Pitt (1978)) that for any $x, y \in \mathbb{R}^d$, $B \in \mathscr{B}(\mathbb{R}^N)$ and any integer $n \geq 1$, we have

$$\mathrm{E}[L(x,B)]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \exp\left(-i \sum_{j=1}^{n} \langle u_j, x \rangle\right) \times \mathrm{E} \exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j) \rangle\right) d\,\overline{u}\, d\,\overline{t} \tag{2.6}$$

and for any even integer $n \geq 2$

$$\begin{aligned} &\mathrm{E}[L(x+y,B) - L(x,B)]^n \\ &= (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \left(\exp(-i\langle u_j, x+y\rangle) - \exp(-i\langle u_j, x\rangle)\right) \\ &\qquad \cdot \mathrm{E}\exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j)\rangle\right) d\overline{u}\, d\overline{t}\ , \end{aligned} \tag{2.7}$$

where $\overline{u} = (u_1, \ldots, u_n)$, $\overline{t} = (t_1, \ldots, t_n)$, and each $u_j \in \mathbb{R}^d$, $t_j \in \mathbb{R}^N$. In the coordinate notation we then write $u_j = \left(u_j^1, \ldots, u_j^d\right)$.

Let $Y(t)$ $\left(t \in \mathbb{R}^N\right)$ be a real-valued, centered Gaussian random field with stationary increments. We assume $Y(0) = 0$ and (1.5), (1.6) hold. Let $X(t)$ $\left(t \in \mathbb{R}^N\right)$ be the (N,d) Gaussian random field defined by (1.7).

Lemma 2.5 *There exist $\delta > 0$ such that for any $r \in (0, \delta)$, $B = B(0, r)$, $x, y \in \mathbb{R}^d$, any even integer $n \geq 2$ and any $0 < \gamma < \min\{1, (N/\alpha - d)/2\}$, we have*

$$\mathrm{E}[L(x,B)]^n \leq \frac{K^n\, r^{Nn}}{\prod_{j=1}^{n} (\sigma(rj^{-1/N}))^d}\ , \tag{2.8}$$

$$\mathrm{E}[L(x+y,B) - L(x,B)]^n \leq \frac{K^n |y|^{n\gamma}\, r^{Nn}}{\prod_{j=1}^{n} (\sigma(rj^{-1/N}))^{d+\gamma}}\ (n!)^{2\gamma} \prod_{j=1}^{n} \left(\frac{L(r)}{L(rj^{-1/N})}\right)^{\gamma}, \tag{2.9}$$

where $K > 0$ is a finite constant depending on N, d and σ only.

Proof. The proof of (2.8) is rather easy. Since $X_1, \ldots, X_d$ are independent copies of Y, it follows (2.6) that

$$\mathrm{E}[L(x,B)]^n \leq (2\pi)^{-nd} \int_{B^n} \prod_{k=1}^{d} \left[\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} Var\left(\sum_{j=1}^{n} u_j^k Y(t_j)\right)\right) dU^k\right] d\overline{t}\ , \tag{2.10}$$

where $U^k = \left(u_1^k, \ldots, u_n^k\right) \in \mathbb{R}^n$. Denote the covariance matrix of $Y(t_1), \ldots, Y(t_n)$ by $R(t_1, \ldots, t_n)$. For distinct $t_1, \ldots, t_n \in B \backslash \{0\}$, let $(Z_1, \ldots, Z_n)$ be the

Gaussian vector with mean zero and the covariance matrix $R^{-1}(t_1, \ldots, t_n)$. Then the density function of $(Z_1, \ldots, Z_n)$ is

$$(2\pi)^{-n/2}(\det(R(t_1, \ldots, t_n)))^{1/2} \ \exp\left(-\tfrac{1}{2}UR(t_1, \ldots, t_n)U'\right) ,$$

where $U = (u_1, \ldots, u_n) \in \mathbb{R}^n$, U' is the transpose of U and $\det(R)$ denotes the determinant of R. Hence for each $1 \le k \le d$,

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^n u_j^k Y(t_j)\right)\right) dU^k = \frac{(2\pi)^{n/2}}{(\det(R(t_1, \ldots, t_n)))^{1/2}} \ . \tag{2.11}$$

Put (2.11) into (2.10) and notice that the set of $\bar{t} \in \mathbb{R}^{Nn}$ having $t_i = t_j$ for some $i \ne j$ is a set of Nn-dimensional Lebesgue measure 0, we have

$$\mathrm{E}[L(x, B)]^n \le (2\pi)^{-nd/2} \int_{B^n} \frac{1}{(\det(R(t_1, \ldots, t_n)))^{d/2}} \, d\bar{t} \ . \tag{2.12}$$

It is well known that

$$\det(R(t_1, \ldots, t_n)) = \mathrm{Var}(Y(t_1)) \prod_{j=2}^n \mathrm{Var}(Y(t_j)|Y(t_1), \ldots, Y(t_{j-1})) \ , \tag{2.13}$$

where $\mathrm{Var}Y$ and $\mathrm{Var}(Y|Z)$ denote the variance of Y and the conditional variance of Y given Z respectively. It follows from (1.6) and (2.13) that (2.12) is at most

$$K^n \int_{B^n} \prod_{j=1}^n \frac{1}{(\sigma(\min\{|t_j - t_i|, 0 \le i \le j-1\}))^d} \, d\bar{t} \ , \tag{2.14}$$

where $t_0 \hat{=} 0$. Since $N > \alpha d$, we see that (2.8) follows from (2.14) and Lemma 2.3.

Now we turn to the proof of (2.9). By (2.7) and the elementary inequality

$$|\mathrm{e}^{iu} - 1| \le 2^{1-\gamma}|u|^\gamma \qquad \text{for any} \ \ u \in \mathbb{R}, \ 0 < \gamma < 1 \ ,$$

we see that for any even integer $n \ge 2$ and any $0 < \gamma < 1$,

$$\begin{aligned} \mathrm{E}[L(x+y, B) - L(x, B)]^n &\le (2\pi)^{-nd} 2^{(1-\gamma)n} |y|^{n\gamma} \\ &\cdot \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma \ \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^n \langle u_j, \ X(t_j)\rangle\right)\right) d\bar{u} \, d\bar{t} \ . \end{aligned} \tag{2.15}$$

By making the change of variables $t_j = rs_j$, $j = 1, \ldots, n$ and $u_j = \sigma(r)^{-1} v_j$, $j = 1, \ldots, n$ and changing the letters s, v back to t, u, we see that (2.15) equals

$$\begin{aligned} &(2\pi)^{-nd} 2^{(1-\gamma)n} \ |y|^{n\gamma} \ r^{Nn} \ \sigma(r)^{-n(d+\gamma)} \\ &\cdot \int_{B(0,1)^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^n \langle u_j, \ X(rt_j)/\sigma(r)\rangle\right)\right) d\bar{u} \, d\bar{t} \ , \end{aligned} \tag{2.16}$$

To simplify the notations, let

$$Z(t) = \frac{Y(rt)}{\sigma(r)} \quad \text{and} \quad \xi(s) = \frac{\sigma(rs)}{\sigma(r)} \mathrel{\hat{=}} s^{\alpha} l(s) \ .$$

Then $Z(t)$ $\left(t \in \mathbb{R}^N\right)$ satisfies (1.5) and (1.6) with $\sigma(s)$ replaced by $\xi(s)$. It is also easy to verify that $l(s)$ satisfies (2.2), and Lemma 2.3 holds with $\sigma(s)$ and $B(s,r)$ replaced by $\xi(s)$ and $B(s,1)$ respectively. Let $\tilde{Z}(t) = (Z_1(t), \dots, Z_d(t))$, where $Z_1, \dots, Z_d$ are independent copies of Z. Since for any $0 < \gamma < 1$, $|a+b|^{\gamma} \le |a|^{\gamma} + |b|^{\gamma}$, we have

$$\prod_{j=1}^{n} |u_j|^{\gamma} \le \sum{}' \prod_{j=1}^{n} \left| u_j^{k_j} \right|^{\gamma} , \tag{2.17}$$

where the summation $\sum'$ is taken over all $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$. Fix such a sequence $(k_1, \dots, k_n)$, we consider the integral

$$J = \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \left| u_j^{k_j} \right|^{\gamma} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_{j=1}^{n} \langle u_j, \tilde{Z}(t_j) \rangle \right) \right) d\overline{u} \ .$$

For any fixed distinct $t_1, \dots, t_n \in B(0,1) \backslash \{0\}$, $Z_l(t_j)$ $(l = 1, \dots, d$, $j = 1, \dots, n)$ are linearly independent. Then by a generalized Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned} J &\le \prod_{j=1}^{n} \left[\int_{\mathbb{R}^{nd}} \left| u_j^{k_j} \right|^{n\gamma} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_{j=1}^{n} \sum_{l=1}^{d} u_j^l Z_l(t_j) \right) \right) d\overline{u} \right]^{1/n} \\ &= \frac{(2\pi)^{nd-1}}{\left(\det \mathrm{Cov}(Z_l(t_j),\ 1 \le l \le d,\ 1 \le j \le n)\right)^{1/2}} \\ &\quad \cdot \int_{\mathbb{R}} |v|^{n\gamma} \exp\left(-\frac{v^2}{2} \right) dv \ \prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} \\ &\le \frac{K^n \ (n!)^{\gamma}}{\left(\det \mathrm{Cov}(Z(t_1), \dots, Z(t_n))\right)^{d/2}} \prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} \ , \end{aligned} \tag{2.18}$$

where σ_j^2 is the conditional variance of $Z_{k_j}(t_j)$ given $Z_l(t_i)$ $(l \ne k_j$ or $l = k_j,\ i \ne j)$ and the last inequality follows from Stirling's formula. By (1.6) and the independence of $Z_1, \dots, Z_n$, we deduce that

$$\sigma_j^2 \ge c_2 \min\left\{ \xi^2(|t_j - t_i|) :\ i = 0 \text{ or } i \ne j \right\} , \tag{2.19}$$

where $t_0 \mathrel{\hat{=}} 0$. Now we define a permutation π of $\{1, \dots, n\}$ such that

$$|t_{\pi(1)}| = \min\{|t_i|,\ i = 1, \dots, n\}$$

$$|t_{\pi(j)} - t_{\pi(j-1)}| = \min\left\{ |t_i - t_{\pi(j-1)}|,\ i \in \{1, \dots, n\} \backslash \{\pi(1), \dots, \pi(j-1)\} \right\}$$

Then by (2.19) and (2.13) we have

$$\prod_{j=1}^{n}\frac{1}{\sigma_j^{\gamma}} \le K^n \prod_{j=1}^{n} \frac{1}{\min\{(\xi(|t_{\pi(j)}-t_i|))^{\gamma}:\ i=0 \text{ or } i\neq\pi(j)\}}$$
$$\le K^n \prod_{j=1}^{n} \frac{1}{\min\{(\xi(|t_{\pi(j)}-t_{\pi(j-1)}|))^{\gamma},\ (\xi(|t_{\pi(j+1)}-t_{\pi(j)}|))^{\gamma}\}}$$
$$\le K^n \prod_{j=1}^{n} \frac{1}{(\xi(|t_{\pi(j)}-t_{\pi(j-1)}|))^{2\gamma}}$$
$$\le K^n \prod_{j=1}^{n} \frac{1}{(\mathrm{Var}(Z(t_{\pi(j)})|Z(t_{\pi(i)}),\ i=1,\dots,j-1))^{\gamma}}$$
$$\le \frac{K^n}{(\det \mathrm{Cov}(Z(t_1),\dots,Z(t_n)))^{\gamma}}\ . \tag{2.20}$$

Combining (2.18) and (2.20), we obtain

$$J \le \frac{K^n (n!)^{\gamma}}{(\det\ \mathrm{Cov}(Z(t_1),\dots,Z(t_n)))^{d/2+\gamma}}$$
$$\le \frac{K^n (n!)^{\gamma}}{\prod_{j=1}^{n}(\xi(\min\{|t_j-t_i|, 0\le i\le j-1\}))^{d+2\gamma}}\ . \tag{2.21}$$

Take

$$0<\gamma<\min\left\{1,\ \frac{1}{2}\left(\frac{N}{\alpha}-d\right)\right\}\ .$$

Then it follows from (2.15), (2.16), (2.17), (2.21) and Lemma 2.3 with $\beta = d+2\gamma$ that

$$\mathrm{E}[L(x+y,B)-L(x,B)]^n \le K^n |y|^{n\gamma}(n!)^{\gamma} r^{Nn}\sigma(r)^{-n(d+\gamma)}$$
$$\cdot \int_{B(0,1)^n}\prod_{j=1}^{n}\frac{1}{(\xi(\min\{|t_j-t_i|, 0\le i\le j-1\}))^{d+2\gamma}}$$
$$\le K^n |y|^{n\gamma}(n!)^{\gamma} r^{Nn}\sigma(r)^{-n(d+\gamma)}\ \frac{1}{\prod_{j=1}^{n}(\xi(j^{-1/N}))^{d+2\gamma}}$$
$$\le \frac{K^n|y|^{n\gamma}\ r^{Nn}}{\prod_{j=1}^{n}(\sigma(rj^{-1/N}))^{d+\gamma}}\ (n!)^{2\gamma}\ \prod_{j=1}^{n}\left(\frac{L(r)}{L(rj^{-1/N})}\right)^{\gamma}.$$

This proves (2.9).

Remark. Under the extra condition $\sigma(ar)\ge a^{\alpha}\sigma(r)$ for $0\le a\le 1$ and $r>0$ small enough (see Csörgő, Lin and Shao (1995)), (2.9) becomes

$$\mathrm{E}[L(x+y,B)-L(x,B)]^n \le K^n|y|^{n\gamma}r^{Nn}\sigma(r)^{-n(d+\gamma)}(n!)^{\frac{\alpha d}{N}+2\gamma}$$

and the above proof can be simplified.

Since $X(t)$ has stationary increments, the above arguments also prove the following lemma.

Lemma 2.6 *For any* $\tau \in \mathbb{R}^N$, *let* $B = B(\tau, r)$ *with* $r \in (0, \delta)$. *Then for any* $x,\ y \in \mathbb{R}^d$, *even integer* $n \geq 2$ *and any* $0 < \gamma < \min\{1, (N/\alpha - d)/2\}$

$$\mathrm{E}[L(x + X(\tau), B)]^n \leq K^n \frac{r^{Nn}}{\prod_{j=1}^{n} (\sigma(rj^{-1/N}))^d}, \tag{2.22}$$

$$\begin{aligned}\mathrm{E}[L(x + y + X(\tau), B) - L(x + X(\tau), B)]^n &\leq \frac{K^n\ |y|^{n\gamma}\ r^{Nn}}{\prod_{j=1}^{n} (\sigma(rj^{-1/N}))^{d+\gamma}} (n!)^{2\gamma} \\ &\quad \times \prod_{j=1}^{n} \left(\frac{L(r)}{L(rj^{-1/N})} \right)^{\gamma}. \end{aligned} \tag{2.23}$$

Lemma 2.7 *With the notations of Lemma 2.6, for any* $c > 0$ *there exists a finite constant* $A > 0$, *depending on* N, d *and* σ *only, such that for any* $u > 0$ *small enough,*

$$P\left\{ L(x + X(\tau), B) \geq \frac{Ar^N}{(\sigma(ru))^d} \right\} \leq \exp\left(-c/u^N\right), \tag{2.24}$$

$$\begin{aligned} &P\left\{ |L(x + y + X(\tau), B) - L(x + X(\tau), B)| \geq \frac{Ar^N |y|^{\gamma}}{(\sigma(ru))^{d+\gamma} u^{2N\gamma}} \right\} \\ &\leq \exp\left(-c/u^N\right). \end{aligned} \tag{2.25}$$

Proof. We only prove (2.24). The proof is similar to that of Theorem 1 in Kôno (1977a). Let

$$\Lambda = \frac{L(x + X(\tau), B)}{r^N} \quad \text{and} \quad u_n = \frac{1}{n^{1/N}}.$$

Then by Chebyshev's inequality and Lemma 2.5, we have

$$\begin{aligned} P\left\{ \Lambda \geq A/(\sigma(ru_n))^d \right\} &\leq \frac{E(\Lambda^n)(\sigma(ru_n))^{nd}}{A^n} \\ &\leq \left(\frac{K}{A}\right)^n (\sigma(ru_n))^{nd} \prod_{j=1}^{n} \frac{1}{(\sigma(rj^{-1/N}))^d} \\ &= \left(\frac{K}{A}\right)^n \left(\frac{r}{n^{1/N}}\right)^{n\alpha d} L\left(\frac{r}{n^{1/N}}\right)^{nd} \\ &\quad \times \prod_{j=1}^{n} \left(\frac{j^{1/N}}{r}\right)^{\alpha d} L\left(\frac{r}{j^{1/N}}\right)^{-d}. \end{aligned} \tag{2.26}$$

It follows from Stirling's formula that (2.26) is at most

$$\left(\frac{K}{A}\right)^n \exp\left(-\frac{\alpha d}{N}n\right)(2\pi n)^{\alpha d/(2N)} L\left(\frac{r}{n^{1/N}}\right)^{nd} \prod_{j=1}^{n} L\left(\frac{r}{j^{1/N}}\right)^{-d} . \tag{2.27}$$

By using the representation (2.1), we obtain

$$\begin{aligned}
&L\left(\frac{r}{n^{1/N}}\right)^{nd} \prod_{j=1}^{n} L\left(\frac{r}{j^{1/N}}\right)^{-d} \\
&\quad = L\left(\frac{r}{n^{1/N}}\right)^{nd} \exp\left(-nd\int_{\frac{r}{n^{1/N}}}^{a} \frac{\epsilon(t)}{t}\, dt + d\sum_{j=1}^{n-1} \int_{\frac{r}{n^{1/N}}}^{\frac{r}{j^{1/N}}} \frac{\epsilon(t)}{t}\, dt\right) \\
&\quad = \exp\left(d\sum_{j=1}^{n-1} j\int_{\frac{r}{(j+1)^{1/N}}}^{\frac{r}{j^{1/N}}} \frac{\epsilon(t)}{t}\, dt\right) \\
&\quad \le \exp\left(\frac{\epsilon d}{N}\sum_{j=1}^{n-1} j\log\left(1+\frac{1}{j}\right)\right) \qquad \left(0<\epsilon<\frac{\alpha}{2}\right) \\
&\quad \le \exp\left(\frac{\epsilon d}{N}\left(n-\frac{1}{2}\log n-\frac{1}{2}\log(2\pi)\right)\right)
\end{aligned} \tag{2.28}$$

It follows from (2.26), (2.27) and (2.28) that for any $c>0$ we can choose a constant $A>K$ and an integer n_0 large enough such that for any $n \ge n_0$

$$\begin{aligned}
&P\left\{\Lambda \ge A/(\sigma(ru_n))^d\right\} \\
&\quad \le \exp\left(n\left(\log\left(\frac{K}{A}\right) - \frac{(\alpha-\epsilon)d}{2N}\right) + \frac{(\alpha-\epsilon)d}{2N}(\log n + \log(2\pi))\right) \\
&\quad \le \exp(-2cn) = \exp\left(-2c/u_n^N\right) .
\end{aligned} \tag{2.29}$$

Finally for any $u>0$ small enough, there is $n>n_0$ such that

$$u_{n+1} \le u < u_n .$$

Hence by (2.29) and the fact that for every $n \ge 1$

$$\left(\frac{n}{n+1}\right)^{1/N} \ge \frac{1}{2} ,$$

we have

$$\begin{aligned}
P\left\{\Lambda \ge A/(\sigma(ru))^d\right\} &\le P\left\{\Lambda \ge A/(\sigma(ru_n))^d\right\} \\
&\le \exp\left(-2c/u_n^N\right) \\
&\le \exp\left(-c/u^N\right) .
\end{aligned}$$

This completes the proof of (2.24).

3 Hölder conditions of the local time

Let $Y(t)$ $\left(t \in \mathbb{R}^N\right)$ be a real-valued, centered Gaussian random field with stationary increments. We assume that $Y(0) = 0$ and (1.5), (1.6) hold. Let $X(t)$ $\left(t \in \mathbb{R}^N\right)$ be the (N, d) Gaussian random field defined by (1.7). In this section, we adapt the arguments of Ehm (1981) and Geman, Horowitz and Rosen (1984) to prove Theorems 1.1 and 1.2.

We start with the following lemma, which is a consequence of Lemma 2.1 in Talagrand (1995).

Lemma 3.1 *Let* $Y(t)$ $\left(t \in \mathbb{R}^N\right)$ *be a Gaussian random field satisfying* (1.5) *and* $Y(0) = 0$. *Then for any* $r > 0$ *small enough and* $u \geq K\sigma(r)$, *we have*

$$P\left\{\sup_{|t| \leq r} |Y(t)| \geq u\right\} \leq \exp\left(-\frac{u^2}{K\,\sigma^2(r)}\right) . \tag{3.1}$$

Proof of Theorem 1.1. For any fixed $\tau \in \mathbb{R}^N$, let $B_n = B(\tau, 2^{-n})$ $(n = 1, 2, \ldots)$. It follows from Lemma 3.1 that

$$P\left\{\sup_{t \in B_n} |X(t) - X(\tau)| \geq \sigma(2^{-n})\sqrt{2K \log n}\right\} \leq n^{-2} .$$

Then by the Borel-Cantelli lemma, almost surely there exist $n_1 = n_1(\omega)$ such that

$$\sup_{t \in B_n} |X(t) - X(\tau)| \leq \sigma(2^{-n})\sqrt{2K \log n} \quad \text{for} \quad n \geq n_1 . \tag{3.2}$$

Let $\theta_n = \sigma\left(2^{-n}/(\log\log 2^n)^{1/N}\right)(\log\log 2^n)^{-2}$ and

$$G_n = \left\{x \in \mathbb{R}^d : |x| \leq \sigma(2^{-n})\sqrt{2K \log n}, \quad x = \theta_n p \ \text{ for some } \ p \in \mathbb{Z}^d\right\} ,$$

where $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$. The cardinality of G_n satisfies

$$\# G_n \leq K(\log n)^{3d+1} \tag{3.3}$$

at least when n is large enough. Recall that

$$\phi_1(r) = \frac{r^N}{\sigma\left(r(\log\log 1/r)^{-1/N}\right)^d} .$$

It follows from Lemma 2.7 that

$$\begin{aligned} &P\{L(x + X(\tau), B_n) \geq A\phi_1(2^{-n}) \ \text{ for some } \ x \in G_n\} \\ &\quad \leq K(\log n)^{3d+1} \exp(-2\log n) \\ &\quad = K(\log n)^{3d+1} n^{-2} . \end{aligned}$$

Hence by the Borel-Cantelli lemma there is $n_2 = n_2(\omega)$ such that almost surely

$$\sup_{x\in G_n} L(x+X(\tau),B_n) \le A\phi_1(2^{-n}) \quad \text{for} \quad n \ge n_2 \; . \tag{3.4}$$

For any fixed integers n with $n^2 > 2^d$, $h \ge 1$ and any $x \in G_n$, define

$$F(n,h,x) = \left\{ y \in \mathbb{R}^d : y = x + \theta_n \sum_{j=1}^{h} \epsilon_j 2^{-j} \text{ for } \epsilon_j \in \{0,1\}^d \right\} . \tag{3.5}$$

A pair of points $y_1,\ y_2 \in F(n,h,x)$ is said to be linked if $y_2 - y_1 = \theta_n \epsilon 2^{-h}$ for some $\epsilon \in \{0,1\}^d$. Then by (2.25) we have

$$P\Bigg\{ |L(y_1+X(\tau),B_n) - L(y_2+X(\tau),B_n)| \ge \frac{A2^{-nN}|y_1-y_2|^\gamma\ (h\log\log 2^n)^{2\gamma}}{\Big(\sigma\Big(2^{-n}/(h\log\log 2^n)^{1/N}\Big)\Big)^{d+\gamma}}$$

$$\text{for some } x \in G_n, h \ge 1 \text{ and some linked pair } y_1, y_2 \in F(n,h,x) \Bigg\}$$

$$\le \#G_n \sum_{h=1}^{\infty} 2^{hd} \exp(-2h\log n)$$

$$\le K(\log n)^{3d+1} \, \frac{2^d/n^2}{1-2^d/n^2} \; .$$

Since

$$\sum_{n=[2^{d/2}]+1}^{\infty} (\log n)^{3d+1} \, \frac{2^d/n^2}{1-2^d/n^2} < \infty \; ,$$

there exist $n_3 = n_3(\omega)$ such that for almost surely for $n \ge n_3$

$$|L(y_1+X(\tau),B_n) - L(y_2+X(\tau),B_n)| \le \frac{A2^{-Nn}|y_1-y_2|^\gamma\ (h\log\log 2^n)^{2\gamma}}{\Big(\sigma\Big(2^{-n}/(h\log\log 2^n)^{1/N}\Big)\Big)^{d+\gamma}} \tag{3.6}$$

for all $x \in G_n$, $h \ge 1$ and any linked pair $y_1,\ y_2 \in F(n,h,x)$. Let Ω_0 be the event that (3.2), (3.4) and (3.6) hold eventually. Then $P(\Omega_0) = 1$. Fix an $n \ge n_4 = \max\{n_1,n_2,n_3\}$ and any $y \in \mathbb{R}^d$ with $|y| \le \sigma(2^{-n})\sqrt{2K\log n}$. We represent y in the form $y = \lim_{h\to\infty} y_h$, where

$$y_h = x + \theta_n \sum_{j=1}^{h} \epsilon_j 2^{-j} \qquad \Big(y_0 = x,\ \epsilon_j \in \{0,1\}^d\Big)$$

for some $x \in G_n$. Then each pair $y_{h-1},\ y_h$ is linked, so by (3.6) and the continuity of $L(\cdot,B_n)$ we have

$$|L(y+X(\tau),B_n) - L(x+X(\tau),B_n)|$$

$$\le A2^{-Nn} \sum_{h=1}^{\infty} \left|\theta_n 2^{-h}\right|^\gamma \frac{(h\log\log 2^n)^{2\gamma}}{\Big(\sigma\Big(2^{-n}/(h\log\log 2^n)^{1/N}\Big)\Big)^{d+\gamma}}$$

$$\leq K 2^{-Nn} \theta_n^\gamma \sum_{h=1}^{\infty} 2^{-h\gamma} \; h^{2\gamma+\alpha(d+\gamma)/N} \; \frac{(\log\log 2^n)^{2\gamma}}{\left(\sigma\left(2^{-n}/(\log\log 2^n)^{1/N}\right)\right)^{d+\gamma}}$$
$$\leq K 2^{-Nn} \frac{\theta_n^\gamma \; (\log\log 2^n)^{2\gamma}}{\left(\sigma\left(2^{-n}/(\log\log 2^n)^{1/N}\right)\right)^{d+\gamma}}$$
$$= K\phi_1(2^{-n}) \; .$$

It follows from (3.4) and (3.7) that almost surely for $n \geq n_4$

$$L(y + X(\tau), B_n) \leq K\phi_1(2^{-n}) \tag{3.8}$$

for any $y \in \mathbb{R}^d$ with $|y| \leq \sigma(2^{-n})\sqrt{2K \log n}$. Therefore

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} L(x, B_n) &= \sup_{x \in \overline{X(B_n)}} L(x, B_n) \\ &\leq K\phi_1(2^{-n}) \; . \end{aligned} \tag{3.9}$$

Finally for any $r > 0$ small enough, there exists an $n \geq n_4$ such that $2^{-n} \leq r < 2^{-n+1}$. Hence by (3.9) we have

$$\sup_{x \in \mathbb{R}^d} L(x, B(\tau, r)) \leq K\phi_1(r) \; .$$

This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is quite similar to that of Theorem 1.1.

Proof of Theorem 1.2. For simplicity, we only consider the case $T = [0, 1]^N$. Let $\mathscr{D}_n$ be the family of 2^{nN} dyadic cubes of order n in T. Let $\theta_n = \sigma\left(2^{-n}/(\log 2^n)^{1/N}\right)(\log 2^n)^{-2}$ and let

$$G_n = \left\{x \in \mathbb{R}^d : |x| \leq n, \; x = \theta_n p \text{ for some } \; p \in \mathbb{Z}^d\right\} \; .$$

Then

$$\#G_n \leq K \frac{n^{3d+1}}{(\sigma(2^{-n}))^d} \; . \tag{3.10}$$

It follows from (3.10) and the proof of Lemma 2.7 that there is a finite constant $A > 0$ such that

$$\begin{aligned} &P\{L(x, B) \geq A\phi_2(2^{-n}) \;\text{ for some } x \in G_n \;\text{ and }\; B \in \mathscr{D}_n\} \\ &\quad \leq K 2^{Nn} \#G_n \exp(-2N \log 2^n) \\ &\quad = \frac{K n^{3d+1} 2^{-Nn}}{(\sigma(2^{-n}))^d} \; . \end{aligned} \tag{3.11}$$

Since $N > \alpha d$, we have

$$\sum_{n=1}^{\infty} \frac{K n^{3d+1} 2^{-Nn}}{(\sigma(2^{-n}))^d} < \infty \; .$$

Hence by the Borel-Cantelli lemma, there exist $n_5 = n_5(\omega)$ such that almost surely

$$\sup_{x \in G_n} L(x, B) \leq A\phi_2(2^{-n}) \quad \text{for all} \quad n \geq n_5 \quad \text{and all} \quad B \in \mathscr{D}_n \ . \tag{3.12}$$

For any fixed integers $n,\ h \geq 1$ and $x \in G_n$, we still define $F(n,h,x)$ as in (3.5). Similar to (3.11), we have for $n > d/(2N)$

$$\begin{aligned} &P\Bigg\{ |L(y_1, B) - L(y_2, B)| \geq \frac{A2^{-nN}|y_1 - y_2|^{\gamma}\ (h \log 2^n)^{2\gamma}}{\left(\sigma\left(2^{-n}/(h \log 2^n)^{1/N}\right)\right)^{d+\gamma}} \quad \text{for some} \\ &\qquad B \in \mathscr{D}_n,\ x \in G_n,\ h \geq 1 \text{ and some linked pair } y_1, y_2 \in F(n,h,x) \Bigg\} \\ &\qquad \leq K2^{Nn}\ \#G_n \sum_{h=1}^{\infty} 2^{hd} \exp(-2Nh \log 2^n) \\ &\qquad \leq K2^{Nn}\ \frac{n^{3d+1}}{(\sigma(2^{-n}))^d}\ 2^{-2Nn} \\ &\qquad = K\ \frac{n^{3d+1} 2^{-Nn}}{(\sigma(2^{-n}))^d}\ , \end{aligned} \tag{3.13}$$

and these terms are summable over n. Hence there is an integer $n_6 = n_6(\omega)$ such that almost surely the event in (3.13) does not occur for $n \geq n_6$.

Finally, since $X(t)$ is almost surely continuous on T, there exist $n_7 = n_7(\omega)$ such that almost surely

$$\sup_{t \in T} |X(t)| \leq n_7 \ . \tag{3.14}$$

Let $n \geq n_8 \hat{=} \max\{n_5, n_6, n_7\}$. For any $y \in \mathbb{R}^d$, if $|y| > n$, then $L(y, T) = 0$; whereas if $|y| \leq n$, then we can write

$$y = \lim_{h \to \infty} y_h$$

with

$$y_h = x + \theta_n \sum_{j=1}^{h} \epsilon_j 2^{-j}$$

for some $x \in G_n$. Similar to (3.7), we have

$$\begin{aligned} |L(y, B) - L(x, B)| &\leq \sum_{h=1}^{\infty} \frac{A2^{-nN}\left(\theta_n 2^{-h}\right)^{\gamma}\ (h \log 2^n)^{2\gamma}}{\left(\sigma\left(2^{-n}/(h \log 2^n)^{1/N}\right)\right)^{d+\gamma}} \\ &\leq K2^{-Nn}\ \theta_n^{\gamma} \sum_{h=1}^{\infty} \frac{2^{-\gamma h}\ h^{2\gamma}(\log 2^n)^{2\gamma}}{\left(\sigma\left(2^{-n}/(h \log 2^n)^{1/N}\right)\right)^{d+\gamma}} \\ &\leq K\phi_2(2^{-n})\ . \end{aligned} \tag{3.15}$$

Then (1.11) follows easily from (3.12), (3.15) and a monotonicity argument.

Remark. The methods in this paper can be applied to prove sharp Hölder conditions in set variable for the self-intersection local times of Gaussian random fields considered by Geman, Horowitz and Rosen (1984), Rosen (1984).

The Hölder conditions for the local times of Gaussian random field $X(t)$ are closely related to the irregularity of the sample paths of $X(t)$ (cf. Berman (1972)). To end this section, we apply Theorem 1.1 and Theorem 1.2 to derive results about the degree of oscillation of the sample paths of $X(t)$.

Theorem 3.1 *Let $X(t)\left(t \in \mathbb{R}^N\right)$ be the Gaussian random field defined by* (1.7). *For any $\tau \in \mathbb{R}^N$, there is a finite constant $K > 0$ such that*

$$\liminf_{r\to 0} \sup_{s\in B(\tau,r)} \frac{|X(s) - X(\tau)|}{\sigma\left(r/(\log\log 1/r)^{1/N}\right)} \geq K \ \ a.s. \tag{3.16}$$

For any rectangle $T \subseteq \mathbb{R}^N$

$$\liminf_{r\to 0} \inf_{t\in T} \sup_{s\in B(t,r)} \frac{|X(s) - X(t)|}{\sigma\left(r/(\log 1/r)^{1/N}\right)} \geq K \ \ a.s. \tag{3.17}$$

In particular, $X(t)$ is almost surely nowhere differentiable in $\mathbb{R}^N$.

Proof. Clearly it is sufficient to consider the case of $d = 1$. For any rectangle $Q \subseteq \mathbb{R}^N$,

$$\begin{aligned}\lambda_N(Q) &= \int_{\overline{X(Q)}} L(x, Q)\, dx \\ &\leq L^*(Q) \cdot \sup_{s,t\in Q} |X(s) - X(t)| \ . \end{aligned}\tag{3.18}$$

Let $Q = B(\tau, r)$. Then (3.16) follows immediately from (3.18) and (1.10). Similarly, (3.17) follows from (3.18) and (1.11).

Remark. With a little more effort, we can prove that under the conditions of Theorem 3.1, for any $\tau \in \mathbb{R}^N$

$$\liminf_{r\to 0} \sup_{s\in B(\tau,r)} \frac{|X(s) - X(\tau)|}{\sigma\left(r/(\log\log 1/r)^{1/N}\right)} \leq K \ \ a.s.$$

The proof is a modification of that of Theorem 3.2 of Monrad and Rootzén (1995), using some technical lemmas in Xiao (1996). In particular, if $X(t)$ $\left(t \in \mathbb{R}^N\right)$ is a d-dimensional fractional Brownian motion of index α, then for any $\tau \in \mathbb{R}^N$

$$\liminf_{r\to 0} \sup_{s\in B(\tau,r)} \frac{|X(s) - X(\tau)|}{r^\alpha/(\log\log 1/r)^{\alpha/N}} = K \ \ a.s.$$

where the constant K depends on N, d and α only. This generalizes Chung's law of iterated logarithm (Chung (1948)) and Theorem 3.3 of Monrad and Rootzén (1995) to multiparameter cases.

If $X(t)$ is Brownian motion in $\mathbb{R}$ and we take $T = [0, 1]$, then (3.17) becomes

$$\liminf_{r\to 0} \inf_{0\le t\le 1} \sup_{0\le s\le r} \frac{|X(t+s) - X(t)|}{(r/\log 1/r)^{1/2}} \ge K \;\; a.s.$$

Csörgő and Révész (1979) proved the following more precise result:

$$\lim_{r\to 0} \inf_{0\le t\le 1} \sup_{0\le s\le r} \frac{|X(t+s) - X(t)|}{(r/\log 1/r)^{1/2}} = \sqrt{\frac{\pi}{8}} \;\; a.s.$$

It seems natural to believe that with the conditions of Theorem 3.1, the following inequality holds for any rectangle $T \subseteq \mathbb{R}^N$,

$$\limsup_{r\to 0} \inf_{t\in T} \sup_{s\in B(t,r)} \frac{|X(s) - X(t)|}{\sigma\Big(r/(\log 1/r)^{1/N}\Big)} \le K \;\; a.s.$$

4 The Hausdorff measure of the level sets

Let $X(t)$ $\big(t \in \mathbb{R}^N\big)$ be a Gaussian random field with values in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$, let

$$X^{-1}(x) = \big\{t \in \mathbb{R}^N : X(t) = x\big\}$$

be the x-level set of $X(t)$. The Hausdorff dimension of the level sets of fractional Brownian motion and more general Gaussian random fields have been studied by several authors, see Kahane (1985) and Adler (1981). In this section, we investigate the exact Hausdorff measure of the level sets of strongly locally σ-nondeterministic Gaussian random fields.

We start with the following lemma. Let $T \subset \mathbb{R}^N$ be a closed cube.

Lemma 4.1 *Let μ_ω be a random measure on T and let $f_k(t) = f_k(t, \omega)$ be a sequence of positive random functions. If there exist a positive constant K such that for any positive integers n, k,*

$$E\int_T [f_k(t)]^n \mu_\omega(dt) \le K^n \frac{2^{-nNk}}{\prod_{j=1}^{n} \sigma(2^{-k} j^{-1/N})^d} \;. \tag{4.1}$$

Then with probability 1 for μ_ω almost all $t \in T$

$$\limsup_{k\to\infty} \frac{f_k(t)}{\phi_1(2^{-k})} \le K \;. \tag{4.2}$$

Proof. Let $A > 0$ be a constant which will be determined later and let

$$A_k(\omega) = \left\{ t \in T : f_k(t) \ge A\phi_1\left(2^{-k}\right) \right\} .$$

Then by (4.1) we have

$$\begin{aligned} E\mu_\omega(A_k) &\le \frac{E\int_T [f_k(t)]^n \mu_\omega(dt)}{(A\phi_1(2^{-k}))^n} \\ &\le \left(\frac{K}{A}\right)^n \sigma\left(2^{-k}\left(\log\log 2^k\right)^{-1/N}\right)^{nd} \prod_{j=1}^n \frac{1}{\sigma(2^{-k}j^{-1/N})^d} \\ &\le \left(\frac{K}{A}\right)^n (\log k)^{-n\alpha d/N} n^{n\alpha d} L\left(2^{-k}(\log k)^{1/N}\right)^{nd} \prod_{j=1}^n \frac{1}{L(2^{-k}j^{-1/N})^d} . \end{aligned}$$

By taking $n = \log k$ and similar to (2.28), we can choose A large such that

$$E\mu_\omega(A_k) \le k^{-2} .$$

This implies that

$$E\left(\sum_{k=1}^\infty \mu_\omega(A_k)\right) < \infty .$$

Therefore with probability 1 for μ_ω almost all $t \in T$, (4.2) holds.

Proposition 4.1 *Assume that the conditions of Theorem* 1.1 *are satisfied. Let* $x \in \mathbb{R}^d$ *be fixed and let* $L(x,\cdot)$ *be the local time of* $X(t)$ *at* x *which is a random measure supported on* $X^{-1}(x)$. *Then with probability 1 for* $L(x,\cdot)$ *almost all* $t \in T$

$$\limsup_{r\to 0} \frac{L(x, B(t,r))}{\phi_1(r)} \le K , \tag{4.3}$$

where $K > 0$ *is a finite constant which does not depend on* x.

Proof. Let $f_k(t) = L\left(x, B\left(t, 2^{-k}\right)\right)$ and μ_ω be the restriction of $L(x,\cdot)$ on T, that is, for any Borel set $B \subseteq \mathbb{R}^N$

$$\mu_\omega(B) = L(x, B\cap T) .$$

Then, by an argument similar to the proof of Proposition 3.1 of Pitt (1977), we have that for any positive integer $n \ge 1$

$$\begin{aligned} &\mathrm{E}\int_T [f_k(t)]^n L(x,dt) \\ &= \left(\frac{1}{2\pi}\right)^{(n+1)d} \int_T \int_{B(t,2^{-k})^n} \int_{\mathbb{R}^{(n+1)d}} \exp\left(-i\sum_{j=1}^{n+1}\langle x, u_j\rangle\right) \\ &\quad \cdot \mathrm{E}\exp\left(i\sum_{j=1}^{n+1}\langle u_j, X(s_j)\rangle\right) d\bar{u}\, d\bar{s} , \end{aligned} \tag{4.4}$$

where $\overline{u} = (u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1}$ and $\overline{s} = (t, s_1, \ldots s_n) \in T \times B(t, 2^{-k})^n$. Similar to the proof of (2.8) we have that (4.4) is at most

$$\begin{aligned} K^n \int_T \int_{B(t,2^{-k})^n} & \frac{d\overline{s}}{\sqrt{\det \mathrm{Cov}(X(t), X(s_1), \ldots, X(s_n))}} \\ & \leq K^n \frac{2^{nNk}}{\prod_{j=1}^{n} \sigma(2^{-k} j^{-1/N})^d} , \end{aligned} \tag{4.5}$$

where $K > 0$ is a finite constant depending on N, d, σ and T only. It is clear that (4.3) follows immediately from (4.5) and Lemma 4.1.

Theorem 4.1 *Assume that the conditions of Theorem* 1.1 *are satisfied. Let $T \subset \mathbb{R}^N$ be a closed cube and let $L(x, T)$ be the local time of $X(t)$ on T. Then there exists a positive constant K such that for every $x \in \mathbb{R}^d$ with probability* 1

$$\phi_1\text{-}m\big(X^{-1}(x) \cap T\big) \geq KL(x, T) , \tag{4.6}$$

Proof. As we mentioned, $L(x, \cdot)$ is a locally finite Borel measure in $\mathbb{R}^d$ supported on $X^{-1}(x)$. Let

$$D = \left\{ t \in T : \limsup_{r \to 0} \frac{L(x, B(t, r))}{\phi_1(r)} > K \right\} ,$$

where K is the constant in (4.3). Then D is a Borel set and by Proposition 4.1, $L(x, D) = 0$ almost surely. Using the upper density theorem of Rogers and Taylor (1969), we have almost surely

$$\begin{aligned} & \phi_1\text{-}m\big(X^{-1}(x) \cap T\big) \\ & \quad \geq \phi_1\text{-}m\big(X^{-1}(x) \cap (T \backslash D)\big) \\ & \quad \geq KL(x, T \backslash D) \\ & \quad = KL(x, T) . \end{aligned}$$

This completes the proof of (4.6).

Now we will use an approach similar to that of Talagrand (1996) to study the upper bound of $\phi_1\text{-}m\big(X^{-1}(x) \cap T\big)$. In addition to the conditions in Theorem 1.1, we further assume that

$$\mathrm{E}(Y(t+h) - Y(t))^2 = \sigma^2(|h|) . \tag{4.7}$$

Let T be a closed cube in $\mathbb{R}^N \backslash \{0\}$ and denote

$$\epsilon = \min_{t \in T} \sigma(|t|) .$$

Then $\epsilon > 0$ and there exists a positive constant η such that for every $0 < h \leq \eta$, $\sigma(h) < \epsilon/2$.

It is clear that there is a positive integer M, depending on T and η, such that T can be covered by M closed balls, say $\{\overline{B}(t_j, \eta), j = 1, \ldots, M\}$, of radius η. In order to prove that almost surely

$$\phi_1\text{-}m\big(X^{-1}(x) \cap T\big) < \infty$$

it suffices to show for each $j = 1, \ldots, M$

$$\phi_1\text{-}m\big(X^{-1}(x) \cap \overline{B}(t_j, \eta)\big) < \infty \ \ a.s.$$

To simplify the notations, we will assume $T = \overline{B}(t_0, \eta)$, where $t_0 \in \mathbb{R}^N \backslash \{0\}$ is fixed. For any $t \in T$, let

$$X^1(t) = X(t) - X^2(t), \qquad X^2(t) = E(X(t)|X(t_0)) \ .$$

The two random fields X^1 and X^2 are independent.

Lemma 4.2 *For any* $s,\ t \in T$

$$\big|X^2(s) - X^2(t)\big| \leq K|s - t|^{\gamma}|X(t_0)| \tag{4.8}$$

where $\gamma = 2\alpha$ *if* $\alpha \leq 1/2$, $\gamma = 1$ *if* $\alpha > 1/2$, *and* $K > 0$ *is a finite constant depending on* N, σ, d, t_0 *and* η *only.*

Proof. It is sufficient to consider the case of $d = 1$. Then

$$\begin{aligned}\big|Y^2(s) - Y^2(t)\big| &= \frac{|\mathrm{E}[(Y(s) - Y(t))Y(t_0)]|}{E\Big(Y(t_0)^2\Big)}\ |Y(t_0)| \\ &= \frac{\big|\sigma^2(|s|) - \sigma^2(|t|) + \sigma^2(|t - t_0|) - \sigma^2(|s - t_0|)\big|}{2\sigma^2(|t_0|)}\ |Y(t_0)| \ . \end{aligned} \tag{4.9}$$

Since $L(s)$ is smooth, we have that for any $s, t \in T$

$$\begin{aligned}\big|\sigma^2(|s|) - \sigma^2(|t|)\big| &\leq \Big||s|^{2\alpha} - |t|^{2\alpha}\Big|L^2(|s|) + |t|^{2\alpha}\big|L^2(|s|) - L^2(|t|)\big| \\ &\leq K|s - t|^{\gamma}\end{aligned} \tag{4.10}$$

where $\gamma = 2\alpha$ if $\alpha \leq 1/2$, $\gamma = 1$ if $\alpha > 1/2$, and $K > 0$ is a finite constant depending on N, σ, d, t_0 and η only. Combining (4.9) and (4.10) gives (4.8).

The following lemma is a generalization of Proposition 4.1 in Talagrand (1995), see Xiao (1996) for a proof.

Lemma 4.3 *There exists a constant* $\delta_1 > 0$ *such that for any* $0 < r_0 \leq \delta_1$, *we have*

$$\begin{aligned}P\Bigg\{\exists\, r \in [r_0^2,\ r_0]\ \textit{such that} \sup_{|t| \leq 2\sqrt{N}r} |X(t)| \leq K\sigma\Bigg(r\bigg(\log\log\frac{1}{r}\bigg)^{-\frac{1}{N}}\Bigg)\Bigg\} \\ \geq 1 - \exp\Bigg(-\bigg(\log\frac{1}{r_0}\bigg)^{\frac{1}{2}}\Bigg) \ .\end{aligned}$$

Theorem 4.2 *Let* $X(t)(t \in \mathbb{R}^N)$ *be the Gaussian random field defined by* (1.7) *with* $Y(t)$ *satisfying* (4.7) *and* $N > \alpha d$. *Then for every fixed* $x \in \mathbb{R}^d$, *with probability* 1

$$\phi_1\text{-}m\big(X^{-1}(x) \cap T\big) < \infty \ . \tag{4.11}$$

Proof. It is sufficient to prove that

$$E\Big(\phi_1\text{-}m\big(X^{-1}(x) \cap T\big)\Big) \le K \tag{4.12}$$

for some finite constant $K = K(N, d, \sigma, t_0, \eta) > 0$. For $k \ge 1$, let

$$R_k = \Bigg\{ t \in T : \exists\, r \in [2^{-2k}, 2^{-k}] \ \text{ such that} \\ \sup_{|s-t| \le 2\sqrt{N} r} |X(s) - X(t)| \le K\sigma\left(r\left(\log\log\frac{1}{r}\right)^{-1/N}\right)\Bigg\} \ .$$

By Lemma 4.3 we have

$$P\{t \in R_k\} \ge 1 - \exp\Big(-\sqrt{k/2}\Big) \ .$$

It follows from Fubini's theorem that

$$\sum_{k=1}^{\infty} P(\Omega_{k,1}^c) < \infty \ ,$$

where

$$\Omega_{k,1} = \Big\{\omega : \ \lambda_N(R_k) \ge \lambda_N(T)\Big(1 - \exp(-\sqrt{k}/4)\Big)\Big\} \ .$$

Let

$$\Omega_{k,2} = \Bigg\{\omega : \ \text{for every dyadic cube } C \text{ of order } k \text{ with } C \cap T \ne \emptyset \\ \sup_{s,t \in C} |X(s) - X(t)| \le K\sigma(2^{-k})\sqrt{k} \ \Bigg\} \ .$$

It is known (e.g. it follows directly from Lemma 2.1 in Talagrand (1995)) that we can choose K large enough such that

$$\sum_{k=1}^{\infty} P(\Omega_{k,2}^c) < \infty \ .$$

Now we choose $\beta > 0$ such that $\alpha + \beta < \gamma$ and let

$$\Omega_{k,3} = \big\{\omega : \ |X(t_0)| \le 2^{k\beta}\big\} \ .$$

Then

$$\sum_{k=1}^{\infty} P(\Omega_{k,3}^c) < \infty \ .$$

We also introduce the following event. Let

$$R'_k = \Bigg\{ t \in T : \exists\, r \in [2^{-2k}, 2^{-k}] \ \text{ such that}$$

$$\sup_{|s-t| \le 2\sqrt{N} r} |X^1(s) - X^1(t)| \le K\sigma\left(r\left(\log\log\frac{1}{r}\right)^{-1/N}\right)\Bigg\} .$$

and let

$$\Omega_{k,4} = \Big\{\omega : \ \lambda_N(R'_k) \ge \lambda_N(T)\Big(1 - \exp\Big(-\sqrt{k}/4\Big)\Big)\Big\} .$$

By Lemma 4.2, we see that for k large enough, $\Omega_{k,1} \cap \Omega_{k,3} \subseteq \Omega_{k,4}$, this implies

$$\sum_{k=1}^{\infty} P(\Omega^c_{k,4}) < \infty .$$

Now for any fixed $x \in \mathbb{R}^d$, we start to construct a random covering of $X^{-1}(x) \cap T$ in the following way. For any integer $n \ge 1$ and $t \in T$, denote by $C_n(t)$ the unique dyadic cube of order n containing t. We call $C_n(t)$ a good dyadic cube of order n if it has the following property

$$\sup_{u,v \in C_n(t) \cap T} |X^1(u) - X^1(v)| \le K\sigma\Big(2^{-n}(\log\log 2^n)^{-1/N}\Big) .$$

We see that each $t \in R'_k$ is contained in a good dyadic cube of order n with $k \le n \le 2k$. Thus we have

$$R'_k \subseteq V = \bigcup_{n=k}^{2k} V_n$$

and each V_n is a union of good dyadic cubes C_n of order n. Let $\mathscr{H}_1(k)$ be the family of the dyadic cubes in V. Now $T \backslash V$ is contained in a union of dyadic cubes of order $q = 2k$, none of which meets R'_k. When the event $\Omega_{k,4}$ occurs, there can be at most

$$2^{Nq}\lambda_N(T \backslash V) \le K\lambda_N(T)2^{Nq}\exp\Big(-\sqrt{k}/4\Big)$$

such cubes. We denote the family of such dyadic cubes of order q by $\mathscr{H}_2(k)$. Let $\mathscr{H}(k) = \mathscr{H}_1(k) \cup \mathscr{H}_2(k)$. Then $\mathscr{H}(k)$ depends only upon the random field $X^1(t)$ $(t \in T)$. For every $A \in \mathscr{H}(k)$, we pick a distinguished point $v_A \in A \cap T$, let

$$\Omega_A = \{|X(v_A) - x| \le 2r_A\}$$

where if A is a dyadic cube of order n,

$$r_A = \begin{cases} K\sigma\Big(2^{-n}(\log\log 2^n)^{-1/N}\Big) & \text{if } A \in \mathscr{H}_1(k) \\ K\sigma(2^{-n})\sqrt{n} & \text{if } A \in \mathscr{H}_2(k) \end{cases}$$

Denote by $\mathscr{F}(k)$ the subfamily of $\mathscr{H}(k)$ defined by

$$\mathscr{F}(k) = \{A \in \mathscr{H}(k) : \Omega_A \text{ occurs}\} .$$

Let $\Omega_k \hat{=} \Omega_{k,2} \cap \Omega_{k,3} \cap \Omega_{k,4}$. Then

$$\sum_{k=1}^{\infty} P(\Omega_k^c) < \infty \ .$$

Hence, with probability 1, for k large enough, the event Ω_k occurs. We denote $D = \liminf_{k\to\infty} \Omega_k$.

Claim 1. For k large enough, on Ω_k, $\mathscr{F}(k)$ covers $X^{-1}(x) \cap T$.

For each $t \in X^{-1}(x) \cap T$, we have $X(t) = x$. Assume that t belongs to some dyadic cube A of order n in $\mathscr{H}_1(k)$ (the case of A in $\mathscr{H}_2(k)$ is simpler). Recall that $k \le n \le 2k$. Hence for $\omega \in \Omega_k$, by Lemma 4.2 we have

$$\begin{aligned} |X(v_A) - x| &\le \left|X^1(v_A) - X^1(t)\right| + \left|X^2(v_A) - X^2(t)\right| \\ &\le r_A + K2^{-n\gamma}2^{\beta k} \\ &\le 2r_A \ . \end{aligned}$$

Therefore $A \in \mathscr{F}(k)$.

Denote

$$\mathscr{H} = \bigcup_{k=1}^{\infty} \mathscr{H}(k) \ .$$

Let Σ_1 be the σ-algebra generated by $X^1(t)$ $(t \in T)$. Then $\mathscr{H}$ depends on Σ_1 only.

Claim 2. For any $A \in \mathscr{H}$,

$$P(\Omega_A|\Sigma_1) \le K r_A^d \ , \tag{4.13}$$

where $K > 0$ is a finite constant depending on N, d, σ, η and t_0 only.

To prove (4.13), it is sufficient to show that for every $v \in T$ and every $y \in \mathbb{R}^d$,

$$P\big(|X^2(v) - y| \le r\big) \le K r^d \ .$$

This follows from the fact that $Y^2(v)$ is a Gaussian random variable with mean 0 and variance

$$\begin{aligned} \mathrm{E}(Y^2(v)) &= \frac{\big(\sigma^2(|v|) + \sigma^2(|t_0|) - \sigma^2(|v - t_0|)\big)^2}{4\sigma^2(t_0)} \\ &\ge K > 0 \ . \end{aligned}$$

Let $|A|$ denote the diameter of A. It follows from (4.13) that for k large enough, we have

$$\begin{aligned} &\mathrm{E}\left[1_{\Omega_{k,4}} \sum_{A\in\mathscr{F}(k)} \phi_1(|A|)\right] \\ &= \mathrm{E}\left[\mathrm{E}\left(1_{\Omega_{k,4}} \sum_{A\in\mathscr{F}(k)} \phi_1(|A|)|\Sigma_1\right)\right] \\ &= \mathrm{E}\left[1_{\Omega_{k,4}} \sum_{A\in\mathscr{H}(k)} \mathrm{E}\big(1_{\{A\in\mathscr{F}(k)\}}|\Sigma_1\big)\phi_1(|A|)\right] \end{aligned}$$

$$\leq K\mathrm{E}\left[1_{\Omega_{k,4}} \sum_{A\in\mathscr{H}(k)} |A|^N\right] \tag{4.14}$$
$$\leq K\lambda_N(T) \ .$$

Hence by Fatou's lemma and (4.14) we have

$$\begin{aligned}\mathrm{E}\left[\phi_1\text{-}m\left(X^{-1}(x)\cap T\right)\right] &= \mathrm{E}\left[1_D\cdot\phi_1\text{-}m\left(X^{-1}(x)\cap T\right)\right] \\ &\leq \liminf_{k\to\infty}\ \mathrm{E}\left(1_{\Omega_{k,4}}\sum_{A\in\mathscr{F}(k)}\phi_1(|A|)\right) \\ &\leq K\lambda_N(T) \ .\end{aligned}$$

This proves (4.12) and hence (4.11).

Proof of Theorem 1.3. Combining Theorems 4.1 and 4.2 completes the proof of Theorem 1.3.

We end this section with the following result on the Hausdorff measure of the graph of Gaussian random fields. The exact Hausdorff measure function for the graph of fractional Brownian motion has been obtained by Xiao (1997). With the help of Theorem 1.1, we can give a different proof for the lower bound and extend the result to more general Gaussian random fields.

Theorem 4.3 *Let $X(t)(t\in\mathbb{R}^N)$ be the Gaussian random field defined by* (1.7) *and $N>\alpha d$. Then almost surely*

$$K_3\leq\phi_3\text{-}m\left(GrX\left([0,1]^N\right)\right)\leq K_4 \ , \tag{4.15}$$

where $K_3,\ K_4$ are positive finite constants depending on N, d and $\sigma(s)$ only and where

$$\phi_3(r)=\frac{r^{N+d}}{\left(\sigma\left(r/(\log\log 1/r)^{1/N}\right)\right)^d} \ .$$

Proof. We define a random Borel measure μ on $\mathrm{Gr}X[0,1]^N\subseteq\mathbb{R}^{N+d}$ by

$$\mu(B)=\lambda_N\{t\in[0,1]^N:(t,X(t))\in B\}\quad\text{for any}\quad B\subseteq\mathbb{R}^{N+d} \ .$$

Then $\mu(\mathbb{R}^{N+d})=\mu(\mathrm{Gr}X[0,1]^N)=1$. It follows from Theorem 1.1 that for any fixed $t_0\in[0,1]^N$ almost surely

$$\begin{aligned}&\mu(B((t_0,X(t_0)),\ r)) \\ &\leq\int_{B(X(t_0),\ r)}L(x,\ B(t_0,r))\,dx \\ &\leq K\phi_3(r) \ .\end{aligned} \tag{4.16}$$

By Fubini's theorem, we see that (4.16) holds almost surely for λ_N *a.e.* $t_0 \in [0,1]^N$. Then the lower bound in (4.15) follows from (4.16) and an upper density theorem for Hausdorff measure due to Rogers and Taylor (1961). The proof of the upper bound in (4.15), using Lemma 4.3, is similar to that of Theorem 3.1 in Xiao (1997).

Remark. Let $Y_i(t)$ $(t \in \mathbb{R}^N)$ be a centered Gaussian random field satisfying conditions (1.5) and (1.6) with $\sigma_i(s) = s^{\alpha_i} L_i(s)$ $(i = 1, 2, \ldots, d)$. Define $Z(t) = (Y_1(t), \ldots, Y_d(t))$ $(t \in \mathbb{R}^N)$. If each Y_i is a fractional Brownian motion of index α_i in $\mathbb{R}$, then $Z(t)$ is called a Gaussian random field with fractional Brownian motion components. Such Gaussian random fields have been studied by Cuzick (1978, 1982b), Adler (1981) and Xiao (1995, 1997). With a careful modification of the arguments in this paper, results analogous to Theorems 1.1, 1.2 and 1.3 can also be proved for $Z(t)$. We omit the details.

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