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Local times of fractional Brownian sheets

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Abstract. Let $B_0^H = \{B_0^H(t),\ t \in \mathbb{R}_+^N\}$ be a real-valued fractional Brownian sheet. Consider the (N, d) Gaussian random field B^H defined by

$$B^H(t) = (B_1^H(t), \dots, B_d^H(t)) \ \ (t \in \mathbb{R}_+^N),$$

where $B_1^H, \dots, B_d^H$ are independent copies of B_0^H. In this paper, the existence and joint continuity of the local times of B^H are established.

1. Introduction

For a given vector $H = (H_1, \dots, H_N)$ $(0 < H_\ell < 1$ for $\ell = 1, \dots, N)$, a real valued fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}_+^N\}$ with Hurst index H is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbb{E}[B_0^H(s)B_0^H(t)] = \prod_{\ell=1}^N \frac{1}{2}\Big(s_\ell^{2H_\ell} + t_\ell^{2H_\ell} - |s_\ell - t_\ell|^{2H_\ell}\Big), \quad s, t \in \mathbb{R}_+^N. \tag{1.1}$$

It follows from (1.1) that $B_0^H(t) = 0$ a.s. for every $t \in \partial\mathbb{R}_+^N$, where $\partial\mathbb{R}_+^N$ denotes the boundary of $\mathbb{R}_+^N$.

Fractional Brownian sheet has the following stochastic integral representation

$$B_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} \prod_{\ell=1}^N g_{H_\ell}(t_\ell, s_\ell) W(ds), \tag{1.2}$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard Brownian sheet and

$$g_H(t, s) = ((t - s)_+)^{H-1/2} - ((-s)_+)^{H-1/2},$$

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with $s_+ = \max\{s, 0\}$, and κ_H is the normalizing constant given by

$$\kappa_H^2 = \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} \Big[\prod_{\ell=1}^{N} g_{H_\ell}(1, s_\ell)\Big]^2 ds.$$

We associate with B_0^H a Gaussian random field $B^H = \{B^H(t) : t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}^d$ by

$$B^H(t) = (B_1^H(t), \dots, B_d^H(t)), \tag{1.3}$$

where $B_1^H, \dots, B_d^H$ are independent copies of B_0^H. We call B^H the (N, d)-fractional Brownian sheet with Hurst index $H = (H_1, \dots, H_N)$.

Note that if $N = 1$, then B^H is a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H_1 \in (0, 1)$; if $N > 1$ and $H_1 = \cdots = H_N = 1/2$, then B^H is the (N, d)-Brownian sheet. Hence B^H can be regarded as a natural generalization of one parameter fractional Brownian motion in $\mathbb{R}^d$ to Gaussian random fields in $\mathbb{R}^d$, as well as a generalization of the Brownian sheet. Another well known generalization is the multiparameter fractional Brownian motion $X = \{X(t),\ t \in \mathbb{R}^N\}$, which is a centered Gaussian random field with covariance function

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{1}{2}\delta_{ij}\Big(|s|^{2H_1} + |t|^{2H_1} - |s-t|^{2H_1}\Big), \tag{1.4}$$

where $0 < H_1 < 1$ is a constant and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. Fractional Brownian sheets arise naturally in many areas, including in stochastic partial differential equations (cf. Øksendal and Zhang (2000), Hu, Øksendal and Zhang (2000)) and in studies of most visited sites of symmetric Markov processes (cf. Eisenbaum and Khoshnevisan (2001)).

Many authors have studied the sample path properties of the Brownian sheet and fractional Brownian motion. See Orey and Pruitt (1973), Adler (1978, 1981), Pitt (1978), Ehm (1981), Rosen (1984), Talagrand (1995), Xiao (1997), just mention a few. In all these papers, the independent increment property of the Brownian sheet and the local nondeterminism of fractional Brownian motion have played crucial rôles. Since, in general, the fractional Brownian sheet B_H has neither the property of independent increments nor local nondeterminism, it seems quite difficult to investigate fine properties of its sample paths. Recently, Dunker (2000) has studied the small ball probability of fractional Brownian sheet. For certain special class of fractional Brownian sheets, Mason and Shi (2001) have obtained the exact rate for small ball probability and have computed the Hausdorff dimension of some exceptional sets related to the oscillation of their sample paths. Stochastic partial differential equations driven by fractional Brownian sheet have been studied by Øksendal and Zhang (2000), Hu, Øksendal and Zhang (2000) and Duncan et al. (2000).

The main objective of this paper is to study the existence and joint continuity of the local times of fractional Brownian sheet B^H. Our existence theorem is sharp and its proof is quite different from the proofs for the Brownian sheet and fractional Brownian motion. However, for the joint continuity, we can only establish a sufficient condition; see Theorem 4.1. It is still an open problem to find the best possible condition for the joint continuity of the local times.

The rest of the paper is organized as follows. In Section 2, we collect some definitions and basic facts about fractional Brownian motion and local times that will be useful to our arguments. In Section 3, we prove the existence theorem (Theorem 3.6) and in Section 4, we give a sufficient condition for the joint continuity of the local times of B^H and list some open problems.

2. Preliminaries

In this section, we present some notations, and collect basic facts about Gaussian processes as well as local times.

2.1. General notations

The underlying parameter space is $\mathbb{R}_+^N = [0, \infty)^N$, throughout. A typical parameter, $t \in \mathbb{R}^N$ is written as $t = (t_1, \dots, t_N)$, coordinatewise. There is a natural partial order, "$\preccurlyeq$", on $\mathbb{R}^N$. Namely, $s \preccurlyeq t$ if and only if $s_\ell \le t_\ell$ for all $\ell = 1, \dots, N$. When $s \preccurlyeq t$, we define the *closed interval*,

$$[s, t] = \prod_{\ell=1}^{N} [s_\ell, t_\ell].$$

Throughout, we will let $\mathcal{A}$ denote the class of all N-dimensional closed intervals $I \subset (0, \infty)^N$ that are parallel to the axes. That is $I \in \mathcal{A}$ is of the form $I = [s, t]$, where $s \preccurlyeq t$ are both in $(0, \infty)^N$. We always write λ_m for Lebesgue's measure on $\mathbb{R}^m$, no matter the value of the integer m. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively.

We will use $K, \ K_1, \ K_2, \dots$, to denote unspecified positive and finite constants which may not be the same in each occurrence.

2.2. Fractional Brownian motion

Given a constant $\alpha \in (0, 1)$, a fractional Brownian motion in $\mathbb{R}$ with index α is a real-valued, centered Gaussian process $X^\alpha = \{X^\alpha(t), \ t \in \mathbb{R}\}$, with the covariance function

$$\mathbb{E}[X^\alpha(s) X^\alpha(t)] = \frac{1}{2}\Big(|s|^{2\alpha} + |t|^{2\alpha} - |s - t|^{2\alpha}\Big).$$

Fractional Brownian motion was introduced by Mandelbrot and van Ness (1968) as a moving average Gaussian process

$$X^\alpha(t) = \kappa_\alpha \int_{-\infty}^{t} [((t - s)_+)^{\alpha - 1/2} - ((-s)_+)^{\alpha - 1/2}] dB(s),$$

where B is the ordinary Brownian motion and κ_α is the normalizing constant.

From the covariance function, it is easy to verify that X^α is a self-similar process with self-similarity index α and it has stationary increments. Except in the Brownian motion case (i. e. $\alpha = 1/2$), X^α does not have the independent increment property. Instead, when $\alpha > 1/2$, X^α is a process with long range dependence. As such, fractional Brownian motion has become a popular model in many areas from telecommunication networks to mathematical finance. We refer to Samorodnitsky and Taqqu (1994) for more information on fractional Brownian motion and related properties.

The following property of strong local nondeterminism of fractional Brownian motion was discovered by Pitt (1978): there exists a constant $0 < K_1 < \infty$, depending on α only, such that for all $t \in \mathbb{R}$ and $0 \le r \le |t|$,

$$\mathrm{Var}\big(X^\alpha(t)|X^\alpha(s) : |s-t| \ge r\big) = K_1 r^{2\alpha}. \tag{2.1}$$

This property has been very useful in studying sample path properties of fractional Brownian motion. See, for example, Talagrand (1995) and Xiao (1997).

Let $\mathrm{detCov}(Z_1, \cdots, Z_n)$ denote the determinant of the covariance matrix of a Gaussian random vector $(Z_1, \ldots, Z_n)$. It is well known that

$$\mathrm{detCov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{j=2}^{n} \mathrm{Var}(Z_j|Z_1, \ldots, Z_{j-1}). \tag{2.2}$$

Applying this fact, together with (2.1), we see that for all integer $n \ge 2$ and distinct $t_1, \ldots, t_n \in \mathbb{R}$,

$$\mathrm{detCov}(X^\alpha(t_1), \cdots, X^\alpha(t_n)) \ge K_1^{n-1} \prod_{j=1}^{n} \min\{|t_j - t_i|^{2\alpha} : 0 \le i \le j-1\}, \tag{2.3}$$

where $t_0 := 0$.

2.3. Local times

We end this section by briefly recalling aspects of the theory of local times. More information on local times of random, as well as non-random, functions can be found in Ref.'s (Geman and Horowitz 1980; Geman et al. 1984; Xiao 1997).

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of X on T is defined as the following measure on $\mathbb{R}^d$:

$$\mu_T(\bullet) = \lambda_N\big\{t \in T : X(t) \in \bullet\big\}.$$

If μ_T is absolutely continuous with respect to λ_d, we say that $X(t)$ has *local times* on T, and define its local times, $L(\bullet, T)$, as the Radon–Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and T is the *time* variable. Sometimes, we write $L(x,t)$ in place of $L(x,[0,t])$. It is clear that if X has local times on T, then for every Borel set $I \subseteq T$, $L(x,I)$ also exists.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula*: for every Borel set $T \subseteq \mathbb{R}^N$, and for every measurable function $f:\mathbb{R}^d \to \mathbb{R}$,

$$\int_T f(X(t))\,dt = \int_{\mathbb{R}^d} f(x)L(x,T)\,dx. \tag{2.4}$$

Suppose we fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i]$ in $\mathcal{A}$. Then, whenever we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, X is said to have *jointly continuous local times* on T. When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X_T^{-1}(x) = \{t \in T : X(t) = x\}; \tag{2.5}$$

see Adler (1981) for details. In other words, local times often act as a natural measure on the level sets of X. As such, they are useful in studying the various fractal properties of level sets and inverse images of the vector field X. In this regard, we refer to Berman (1972), Adler (1978), Ehm (1981), Monrad and Pitt (1986), Rosen (1984) and Xiao (1997).

It follows from (25.5) and (25.7) in Geman and Horowitz (1980) (see also Geman, Horowitz and Rosen (1984), Pitt (1978)) that for any $x, y \in \mathbb{R}^d$, $T \in \mathcal{A}$ and any integer $n \geq 1$,

$$\begin{aligned}\mathbb{E}\Big[L(x,T)^n\Big] = (2\pi)^{-nd} \int_{T^n}\int_{\mathbb{R}^{nd}} &\exp\Big(-i\sum_{j=1}^n \langle u^j, x\rangle\Big) \\ &\times \mathbb{E}\exp\Big(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) d\overline{u}\, d\overline{t}\end{aligned} \tag{2.6}$$

and for any even integer $n \geq 2$

$$\begin{aligned}&\mathbb{E}\Big[(L(x,T)-L(y,T))^n\Big] \\ &\quad = (2\pi)^{-nd}\int_{T^n}\int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[\exp(-i\langle u^j, x\rangle) - \exp(-i\langle u^j, y\rangle)\Big] \\ &\qquad \cdot \mathbb{E}\exp\Big(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) d\overline{u}\, d\overline{t},\end{aligned} \tag{2.7}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in (0,\infty)^N$. In the coordinate notation we then write $u^j = (u_1^j, \ldots, u_d^j)$.

3. Existence of the local times

In this section, we prove a sufficient condition for the existence of the local times of an (N, d)-fractional Brownian sheet on any rectangle $I \in \mathcal{A}$. Because of the complicated covariance, the proof of the existence is quite involved. Therefore, we split the proof into several lemmas, which are also of their own interest.

Lemma 3.1. *Let* $0 < \alpha \le 1$, $a \ge b \ge 0$. *Then*

$$(a+b)^\alpha \le a^\alpha + \alpha b^\alpha. \tag{3.1}$$

Remark 3.2. The above inequality is stronger than the well known inequality: $(a+b)^\alpha \le a^\alpha + b^\alpha$. □

Lemma 3.3. *(i). Let* $0 \le \alpha \le 1$ *be fixed. Then for all* $u \ge 0$, *it holds that*

$$(u+1)^\alpha \le \frac{u^{\alpha+1}-1}{u-1}, \tag{3.2}$$

where for $u = 1$ *the right hand sides of (3.2) is defined to be* $\alpha + 1$, *the limiting value as* $u \to 1$.
(ii). Assume $-1 < \alpha < 0$. *Then for all* $u \ge 0$,

$$(u+1)^\alpha \ge \frac{u^{\alpha+1}-1}{u-1}, \tag{3.3}$$

where for $u = 1$ *the right hand side of (3.3) is defined as* $\alpha + 1$.

Proof. (i). First we prove (3.2) for $0 \le \alpha \le 1$ and $u \ge 1$. Define

$$f(u) = \frac{(u-1)(u+1)^\alpha}{u^{\alpha+1}-1}.$$

Then, by Lemma 3.1

$$\lim_{u \to 1} f(u) = \frac{2^\alpha}{1+\alpha} \le 1, \qquad \lim_{u \to \infty} f(u) = 1.$$

To prove (3.2) holds for $u \ge 1$, it suffices to show that $f'(u) \ge 0$ for $u \ge 1$. By a simple calculation, we get that

$$f'(u) = \frac{(u+1)^{\alpha-1}}{(u^{\alpha+1}-1)^2}\Big[(1+\alpha)u^\alpha + (1-\alpha)u^{\alpha+1} - ((1+\alpha)u + (1-\alpha))\Big].$$

Thus, $f'(u) \ge 0$ if and only if

$$f_1(u) := (1+\alpha)u^\alpha + (1-\alpha)u^{\alpha+1} - [(1+\alpha)u + (1-\alpha)] \ge 0.$$

Observe that $f_1(1) = 0$ and

$$f_1'(u) = (1+\alpha)[\alpha u^{\alpha-1} + (1-\alpha)u^\alpha - 1].$$

To show that $f_1(u) \geq 0$ for all $u \geq 1$, it is sufficient to prove that $f_2(u) := \alpha u^{\alpha-1} + (1-\alpha)u^{\alpha} - 1 \geq 0$ for $u \geq 1$. This follows from the fact that $f_2(1) = 0$ and

$$f_2'(u) = \alpha(1-\alpha)u^{\alpha-2}(u-1) \geq 0, \quad \text{for all} \quad u \geq 1.$$

This completes the proof of (3.2) for $0 \leq \alpha \leq 1$ and $u \geq 1$.

Inequality (3.2) clearly holds for $u = 0$. The case of $0 \leq \alpha \leq 1$ and $0 < u \leq 1$ follows by applying the first part to $\hat{u} = \frac{1}{u}$.

(ii). To prove (3.3) for $-1 < \alpha < 0$ and $u \geq 0$, we first note that it holds for $u = 1$ and $u = 0$. Now, we assume $u > 1$. Put $\hat{\alpha} = -\alpha$. Then it follows from (3.2) that

$$(u+1)^{\hat{\alpha}} \leq \frac{u^{\hat{\alpha}+1} - 1}{u-1}.$$

Consequently,

$$(u+1)^{-\hat{\alpha}} \geq \frac{u-1}{u^{\hat{\alpha}+1} - 1}. \tag{3.4}$$

On the other hand, it is seen that

$$2u = 2u^{\frac{\hat{\alpha}}{2}+\frac{1}{2}}u^{-\frac{\hat{\alpha}}{2}+\frac{1}{2}} \leq u^{\hat{\alpha}+1} + u^{-\hat{\alpha}+1}.$$

This implies that

$$(u-1)^2 \geq (u^{\hat{\alpha}+1} - 1)(u^{-\hat{\alpha}+1} - 1).$$

and hence,

$$\frac{u-1}{u^{\hat{\alpha}+1} - 1} \geq \frac{u^{-\hat{\alpha}+1} - 1}{u-1}. \tag{3.5}$$

Combining the inequalities (3.4) and (3.5) together, we get

$$(u+1)^{-\hat{\alpha}} \geq \frac{u^{-\hat{\alpha}+1} - 1}{u-1},$$

which is what we wanted. Finally, applying the result to $\hat{u} = \frac{1}{u}$, we easily see that (3.3) holds for $0 \leq u < 1$ as well. □

Fix a constant $0 < h < 1$, we consider the function

$$F(u) = \frac{1}{2}(1+u^{-1})^{2h} - \frac{1}{4}(1+u^{-1})^{2h}(1+u)^{2h} - \frac{1}{4}u^{-2h}$$
$$-\frac{1}{4}u^{2h} + \frac{1}{2}(1+u)^{2h} + \frac{1}{2}, \quad (0 < u < \infty). \tag{3.6}$$

Lemma 3.4. *Let $K_2 = 4^h(1 - 4^{h-1})$. Then,*

$$F(u) \geq K_2 \quad \textit{for all} \quad 0 < u < \infty.$$

Proof. As

$$F(1) = K_2 \leq 1 \quad \text{and} \quad \lim_{u \to 0} F(u) = \lim_{u \to \infty} F(u) = 1,$$

it suffices to show that $F(\cdot)$ reaches its global minimum at $u = 1$. To this end, we study the sign change of $F'(u)$ at $u = 1$. Differentiating F gives

$$\begin{aligned} F'(u) = & -h(1+u^{-1})^{2h-1}u^{-2} - \frac{h}{2}(1+u)^{2h-1}(1+u^{-1})^{2h} \\ & + \frac{h}{2}(1+u)^{2h}(1+u^{-1})^{2h-1}u^{-2} + \frac{h}{2}u^{-2h-1} \\ & - \frac{h}{2}u^{2h-1} + h(1+u)^{2h-1}. \end{aligned} \tag{3.7}$$

Clearly, $F'(1) = 0$. By rewriting and rearranging the terms in (3.7), we have

$$\begin{aligned} F'(u) &= -h(1+u)^{2h-1}u^{-2h-1} - \frac{h}{2}(1+u)^{4h-1}u^{-2h} \\ &\quad + \frac{h}{2}(1+u)^{4h-1}u^{-2h-1} + \frac{h}{2}u^{-2h-1} - \frac{h}{2}u^{2h-1} + h(1+u)^{2h-1} \\ &= \frac{h}{2}u^{-2h-1}\Big[1 + (1+u)^{4h-1} - 2(1+u)^{2h-1}\Big] \\ &\quad - \frac{h}{2}u^{2h-1}\Big[1 + (1+u)^{4h-1}u^{-4h+1} - 2(1+u)^{2h-1}u^{-2h+1}\Big] \\ &= \frac{h}{2}\, u^{-2h-1}\Big[((1+u)^{2h-1} - 1)^2 + (1+u)^{4h-1} - (1+u)^{4h-2}\Big] \\ &\quad - \frac{h}{2}\, u^{2h-1}\Big[((1+u)^{2h-1}u^{-2h+1} - 1)^2 + (1+u)^{4h-1}u^{-4h+1} \\ &\quad - (1+u)^{4h-2}u^{-4h+2}\Big] \\ &= \frac{h}{2}\, u^{-2h-1}\Big((1+u)^{2h-1} - 1\Big)^2 - \frac{h}{2}\, u^{2h-1}\Big((1+u)^{2h-1}u^{-2h+1} - 1\Big)^2. \end{aligned}$$

Thus $F'(u) \geq 0$ if and only if

$$\Big|(1+u)^{2h-1} - 1\Big|\, u^{-h-\frac{1}{2}} \geq \Big|(1+u)^{2h-1}u^{-2h+1} - 1\Big|\, u^{h-\frac{1}{2}}. \tag{3.8}$$

If $2h - 1 \geq 0$, this holds if and only if

$$g(u) := u^{2h} + (1+u)^{2h-1} - u(1+u)^{2h-1} - 1 \geq 0.$$

If $2h - 1 < 0$, (3.8) is equivalent to

$$g(u) := u^{2h} + (1+u)^{2h-1} - u(1+u)^{2h-1} - 1 \leq 0.$$

Applying Lemma 3.3 with $\alpha = 2h - 1$ to the two cases $0 \leq \alpha \leq 1$ and $-1 < \alpha < 0$, respectively, we see that (3.8) holds if and only if $u \geq 1$. Therefore, we have showed that $F'(u) \geq 0$ if and only if $u \geq 1$. This, together with $F'(1) = 0$, implies that $u = 1$ is the global minimum point of F. $\square$

Corollary 3.5. *For* $0 < h < 1$, $s \geq r > 0$, *define*

$$G(s,r) = s^{2h} r^{2h} - \frac{1}{4}\big[s^{2h} + r^{2h} - (s-r)^{2h}\big]^2.$$

Then $G(s,r) \geq K_2\, r^{2h}(s-r)^{2h}$, *where* K_2 *is the constant defined in the previous lemma.*

Proof. Set $u = \frac{s}{r} - 1$. We have

$$\begin{aligned} G(s,r) &= r^{2h}(s-r)^{2h}\Big[\frac{1}{2}\big(\frac{s}{s-r}\big)^{2h} - \frac{1}{4}\big(\frac{s}{r}\big)^{2h}\big(\frac{s}{s-r}\big)^{2h} \\ &\qquad -\frac{1}{4}\big(\frac{r}{s-r}\big)^{2h} - \frac{1}{4}\big(\frac{s-r}{r}\big)^{2h} + \frac{1}{2}\big(\frac{s}{r}\big)^{2h} + \frac{1}{2}\Big] \\ &= r^{2h}(s-r)^{2h} F(u), \end{aligned}$$

where $F(u)$ is the function defined in (3.6). Thus Corollary 3.5 follows from Lemma 3.4. □

The following is the main theorem of this section.

Theorem 3.6. *Let* $B^H = \{B^H(t),\ t \in \mathbb{R}_+^N\}$ *be an* (N,d) *fractional Brownian sheet with Hurst index* $H = (H_1, \ldots, H_N)$. *If* $d < \sum_{\ell=1}^N \frac{1}{H_\ell}$, *then for all* $I \in \mathcal{A}$, B^H *has local times* $\{L(x,I),\ x \in \mathbb{R}^d\}$ *on* I; *and* $L(x,I)$ *admits the following* L^2 *representation:*

$$L(x,I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_I e^{i\langle y, B^H(s)\rangle} ds\, dy, \tag{3.9}$$

where $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$.

Remark 3.7. The result in Theorem 3.6 is sharp. When $H_1 = \cdots = H_N = \frac{1}{2}$, it recovers the corresponding result for the Brownian sheet. See Ehm (1981) for additional information about local times of the Brownian sheet. □

Proof. Let $I = [s,t] \in \mathcal{A}$ be fixed. Without loss of generality, we assume $I = [\epsilon, 1]^N$. By the method in Orey (1970) and Pitt (1978), (see also (21.3) in Geman and Horowitz (1980)), it is sufficient to prove that

$$\mathcal{J}(I) := \int_I ds \int_I dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} \Big| E \exp\Big(i\langle y, B^H(s)\rangle + i\langle z, B^H(r)\rangle\Big)\Big| dz < \infty. \tag{3.10}$$

For this purpose, we first establish the following estimate: let $0 < h < 1$ be a constant, then for any $\delta > 2h$, $M > 0$ and $p > 0$, there exist positive and finite constants K_3 and K_4, depending on $\delta,\ \epsilon,\ p$ and M only, such that for $0 < a \leq M$,

$$\begin{aligned} \mathcal{I}(a) &:= \int_\epsilon^1 dr \int_\epsilon^1 \Big[a + |s-r|^{2h}\Big]^{-p} ds \\ &\leq K_3(a^{-p+\frac{1}{\delta}} + K_4). \end{aligned} \tag{3.11}$$

Indeed, by the symmetry of the integrand, we see that

$$\mathcal{I}(a) = 2\int_{\epsilon}^{1} r^{-2ph} dr \int_{r}^{1} \Big[\frac{a}{r^{2h}} + (\frac{s}{r} - 1)^{2h}\Big]^{-p} ds.$$

Putting $u = s/r$ and using the fact that $r \geq \varepsilon,\ s \geq \varepsilon$, we see that the above integral is bounded by

$$\begin{aligned} K\int_{\epsilon}^{1} r dr \int_{1}^{1/r} \Big[a + (u-1)^{2h}\Big]^{-p} du &\leq K \int_{1}^{1/\epsilon} \Big[a + (u-1)^{2h}\Big]^{-p} du \\ &= K \int_{a}^{a+(\frac{1}{\epsilon}-1)^{2h}} v^{-p}(v-a)^{\frac{1-2h}{2h}} dv, \end{aligned} \tag{3.12}$$

where we have used the substitution $v := a + (u-1)^{2h}$ and $K > 0$ is a finite constant depending on ϵ, h and p.

Let $b_\epsilon = a + (\frac{1}{\epsilon} - 1)^{2h}$. We prove (3.11) for $h > \frac{1}{2}$ and $0 < h \leq \frac{1}{2}$ separately. If $h > \frac{1}{2}$, then for $\delta > 2h$, (3.12) and Hölder's inequality imply

$$\begin{aligned} \mathcal{I}(a) &\leq K\Big(\int_{a}^{b_\epsilon} v^{-p\delta} dv\Big)^{\frac{1}{\delta}} \Big(\int_{a}^{b_\epsilon} (v-a)^{(-\frac{2h-1}{2h}\cdot\frac{\delta}{\delta-1})} dv\Big)^{\frac{\delta-1}{\delta}} \\ &\leq K_3(a^{-p+\frac{1}{\delta}} + K_4), \end{aligned}$$

where we have used the fact that $\frac{2h-1}{2h} \cdot \frac{\delta}{\delta-1} < 1$. This proves (3.11) for $h > 1/2$.

If $0 < h \leq 1/2$, elementary calculation shows that for all $0 < a \leq M$

$$\begin{aligned} \mathcal{I}(a) &\leq K \int_{a}^{b_\epsilon} v^{-p}(v-a)^{\frac{1-2h}{2h}} dv \\ &\leq K \int_{a}^{b_\epsilon} v^{-p-1+\frac{1}{2h}} dv \\ &\leq K_3(a^{-p+\frac{1}{\delta}} + K_4) \end{aligned}$$

for some positive and finite constants K_3 and K_4 depending on $\delta,\ \epsilon,\ M$ only.

Let us now go back to show (3.10). As B^H is Gaussian, using the expression for the characteristic functions of Gaussian random variables it turns out that

$$\begin{aligned} \mathcal{J}(I) = (2\pi)^d \int_I ds \int_I \Big[\prod_{\ell=1}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell} \\ - \prod_{\ell=1}^{N} \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\Big]^{-\frac{d}{2}} dr. \end{aligned} \tag{3.13}$$

We claim that there exists a constant $K_5 = K_{\epsilon,H} > 0$ such that for all $s,\ r \in I = [\epsilon, 1]^N$,

$$\prod_{\ell=1}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=1}^{N} \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2 \geq K_5 \sum_{\ell=1}^{N} |s_\ell - r_\ell|^{2H_\ell}. \tag{3.14}$$

To see (3.14), we observe that

$$
\begin{aligned}
&\prod_{\ell=1}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=1}^{N} \frac{1}{4}\big[s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell}\big]^2 \\
&\quad = \Big(\prod_{\ell=2}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell}\Big)\Big(s_1^{2H_1} r_1^{2H_1} - \frac{1}{4}(s_1^{2H_1} + r_1^{2H_1} - |s_1 - r_1|^{2H_1})^2\Big) \\
&\qquad\quad + \frac{1}{4}\big[s_1^{2H_1} + r_1^{2H_1} - |s_1 - r_1|^{2H_1}\big]^2 \times \Big[\prod_{\ell=2}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell} \\
&\qquad\quad - \prod_{\ell=2}^{N} \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\Big] \\
&\quad \geq K_2\, \epsilon^{\sum_{\ell=1}^{N} 4H_\ell} |s_1 - r_1|^{2H_1} \\
&\qquad\quad + \frac{1}{4}\epsilon^{2H_1}\Big[\prod_{\ell=2}^{N} s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=2}^{N} \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\Big],
\end{aligned}
$$

where we have used Corollary 3.5 to obtain the last inequality.

Applying the above procedure repeatedly we finally can find a positive constant K_5 depending on ϵ, N and H only such that (3.14) holds.

Choose $\delta_2, ..., \delta_N$ such that $\delta_\ell > 2H_\ell$ and

$$
\frac{d}{2} < \Big(\frac{1}{2H_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_N}\Big).
$$

This is possible because $d < (\frac{1}{H_1} + \cdots + \frac{1}{H_N})$. Applying the estimate (3.11) for

$$
a = \sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell} \quad \text{and} \quad p = d/2,
$$

we obtain from (3.13) and (3.14) that $\mathcal{J}(I)$ is at most

$$
\begin{aligned}
&K_6 \int_\epsilon^1 ds_1 \int_\epsilon^1 dr_1 \cdots \int_\epsilon^1 ds_{N-1} \int_\epsilon^1 dr_{N-1} \\
&\quad \Big[\int_\epsilon^1 ds_N \int_\epsilon^1 \Big(\sum_{\ell=1}^{N} |s_\ell - r_\ell|^{2H_\ell}\Big)^{-d/2} dr_N\Big] \\
&\leq K_7 \int_\epsilon^1 ds_1 \int_\epsilon^1 dr_1 \cdots \int_\epsilon^1 ds_{N-1} \int_\epsilon^1 \Big[\Big(\sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell}\Big)^{-d/2+\frac{1}{\delta_N}} + K_4\Big] dr_{N-1} \\
&\leq K_8 + K_9 \int_\epsilon^1 ds_1 \int_\epsilon^1 dr_1 \cdots \int_\epsilon^1 ds_{N-1} \int_\epsilon^1 \Big[\sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell}\Big]^{-\frac{d}{2}+\frac{1}{\delta_N}} dr_{N-1},
\end{aligned}
\tag{3.15}
$$

where K_8 and K_9 are positive and finite constants depending on ϵN, δ_N and H only. By repeatedly using the estimate (3.11) as in (3.15), after $N-1$ steps, we obtain that

$$\mathcal{J}(I) \leq K_{10} + K_{11} \int_{\epsilon}^{1} ds_1 \int_{\epsilon}^{1} \Big[|s_1 - r_1|^{2H_1}\Big]^{-\frac{d}{2}+\frac{1}{\delta_N}+\cdots+\frac{1}{\delta_2}} dr_1. \qquad (3.16)$$

Since $(\frac{d}{2} - \frac{1}{\delta_N} - \cdots - \frac{1}{\delta_2})2H_1 < 1$ by the choices of δ_ℓ, the integral on the right hand side of (3.16) is finite. This proves (3.10), and hence Theorem 3.6. □

4. Joint continuity

When the local times of B^H exist, it is natural to ask whether there exists a version of the local times which is jointly continuous in both space and time variables. The answers are affirmative for the Brownian sheet and (N, d)-fractional Brownian motion; see Pitt (1978), Ehm (1981) for details. The question for (N, d)-fractional Brownian sheets is significantly harder due to the fact that B^H does not have the independent increment property nor local nondeterminism. In this section, we prove a sufficient condition for the joint continuity of the local times of B^H.

Theorem 4.1. *Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N_+\}$ be a fractional Brownian sheet in $\mathbb{R}^d$ with Hurst index $H = (H_1, \ldots, H_N)$. If $H_\ell d < 1$ for all $\ell = 1, \ldots, N$, then for all closed intervals $I \in \mathcal{A}$, B^H has a jointly continuous local time on I. In particular, a real-valued fractional Brownian sheet has jointly continuous local times.*

Our proof of Theorem 4.1 is based on an moment argument and the continuity lemma of Garsia (1971). The basic estimates that are required for the proof are contained in Lemmas 4.8 and 4.10. For their proofs, we also need some results from Cuzick and Du Peez (1982) and Xiao (1997).

Lemma 4.2. *Let $Z_1, \ldots, Z_n$ be the mean zero Gaussian variables which are linearly independent and assume that*

$$\int_{-\infty}^{\infty} g(v) e^{-\epsilon v^2} dv < \infty$$

for all $\epsilon > 0$. Then

$$\int_{\mathbb{R}^n} g(v_1) \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{n} v_j Z_j\Big)\Big] dv_1 \cdots dv_n$$
$$= \frac{(2\pi)^{(n-1)/2}}{(\mathrm{detCov}(Z_1, \cdots, Z_n))^{1/2}} \int_{-\infty}^{\infty} g\Big(\frac{v}{\sigma_1}\Big) e^{-v^2/2} dv,$$

where $\sigma_1^2 = \mathrm{Var}(Z_1|Z_2, \ldots, Z_n)$ is the conditional variance of Z_1 given $Z_2, \ldots, Z_n$.

Lemma 4.3. *Assume $p(y)$ is positive and non-decreasing on $(0, \infty)$, $p(0) = 0$, $y^n/p^n(y)$ is non-decreasing on $[0, 1]$, and $\int_1^\infty p^{-2}(y)dy < \infty$. Then there exists a constant K_{12} such that for all $n \geq 1$*

$$\int_0^\infty \frac{|\exp(ivy) - 1|^n}{p^n(y)} dy \leq K_{12}^n \, p_+^{-n}\Big(\frac{1}{v}\Big),$$

where $p_+(y) = \min\{1, p(y)\}$.

Lemma 4.4. *For $\alpha \geq e^2/2$,*

$$\int_1^\infty (\log x)^\alpha \exp(-x^2/2)dx \leq \sqrt{\pi}\, (\log \alpha)^\alpha.$$

In the above, Lemmas 4.2 and 4.4 are due to Cuzick and Du Peez (1982), and Lemma 4.3 is a slight modification of their Lemma 3.

Lemma 4.5 below will be important for our purpose. It connects the determinants of the covariance matrices of B^H with those of fractional Brownian motions, hence makes it possible for us to use the arguments in Xiao (1997) to prove the joint continuity of the local times of fractional Brownian sheet.

Lemma 4.5. *For any integer $n \geq 2$ and $t^1, \ldots, t^n \in \mathbb{R}_+^N$, we have*

$$\mathrm{detCov}(B_0^H(t^1), \cdots, B_0^H(t^n)) \geq \prod_{\ell=1}^N \mathrm{detCov}(X^{H_\ell}(t_\ell^1), \cdots, X^{H_\ell}(t_\ell^n)), \qquad (4.1)$$

where for $\ell = 1, \ldots, N$, X^{H_ℓ} is the one parameter fractional Brownian motion in $\mathbb{R}$ with Hurst index H_ℓ.

Proof. Recall that the Hadamard product of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is an $n \times n$ matrix defined as $A \circ B = (a_{ij}b_{ij})$. A classical theorem of Oppenheim (cf. Horn and Johnson (1999, p.480) or Bapat and Raghavan (1996, p.137)) asserts that if A and B are positive semidefinite Hermitian matrices, then

$$\det(A \circ B) \geq \det(A) \cdot \det(B), \qquad (4.2)$$

where $\det(A)$ denotes the determinant of A.

By Eq. (1.1), we see that the covariance matrix

$$\mathrm{Cov}(B_0^H(t^1), \cdots, B_0^H(t^n))$$

is the Hadamard product of the covariance matrices

$$\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \cdots, X^{H_\ell}(t_\ell^n)), \quad (\ell = 1, \ldots, N).$$

Hence Inequality (4.1) follows from (4.2) and induction. □

Lemma 4.6. *Let* $0 < \gamma < 1$. *Then for all* $r > 0$, $a \in \mathbb{R}$, *all integers* $n \geq 1$ *and all distinct* $t_1, \cdots, t_n \in [a, a+r]$, *we have*

$$\int_{[a,a+r]} \frac{dt}{\min\Big\{|t - t_j|^{\gamma},\ j = 1, \cdots, n\Big\}} \leq K_{13}\, r^{1-\gamma} n^{\gamma}.$$

where $K_{13} > 0$ *is a finite constant depending on* γ *only.*

Proof. This is a special case of Lemma 2.3 in Xiao (1997). □

For completeness, we also state the following basic result of Garsia (1971).

Lemma 4.7. *Assume that* $p(u)$ *and* $\Psi(u)$ *are two positive increasing functions on* $[0, \infty)$, $p(u) \downarrow 0$ *as* $u \downarrow 0$, $\Psi(u)$ *is convex and* $\Psi(u) \uparrow \infty$ *as* $u \uparrow \infty$. *Let D denote an open hypercube in* $\mathbb{R}^d$. *If the function* $f(x)$ *is measurable in D and*

$$A := A(D, f) = \int_D \int_D \Psi\Big(\frac{|f(x) - f(y)|}{p(|x - y|/\sqrt{d})}\Big) dx dy < \infty, \tag{4.3}$$

then after modifying $f(x)$ *on a set of Lebesgue measure 0, we have*

$$|f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A}{u^{2d}}\Big)\, dp(u) \quad \textit{for all } x, y \in D.$$

As we mentioned earlier, the following Lemmas 4.8 and 4.10 give the required moment estimates for proving the joint continuity of local times.

Lemma 4.8. *Under the assumptions of Theorem 4.1, there exists a positive and finite constant* K_{14}, *depending on N, d, H and I only, such that for all hypercubes* $T \subseteq I$ *with edge length* $r > 0$ *and* $T \in \mathcal{A}$, *all* $x \in \mathbb{R}^d$ *and all integers* $n \geq 1$,

$$\mathbb{E}\big[L(x, T)^n\big] \leq K_{14}^n\, r^{(N - \sum_{\ell=1}^N H_\ell d)n} (n!)^{\sum_{\ell=1}^N H_\ell d}. \tag{4.4}$$

Proof. Similar to the argument in Xiao (1997, pp 137–138), we apply (2.6) and the independence of $B_1^H, \ldots, B_d^H$ to deduce that for all $T \in \mathcal{A}$ and all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}\big[L(x, T)^n\big] &\leq (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^d \Big\{\int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u_k^j B_0^H(t^j)\Big)\Big] dU^k\Big\} d\bar{t} \\ &= (2\pi)^{-nd/2} \int_{T^n} \Big[\mathrm{detCov}(B_0^H(t^1), \ldots, B_0^H(t^n))\Big]^{-d/2} d\bar{t}. \end{aligned} \tag{4.5}$$

where $U^k = (u_k^1, \ldots, u_k^n) \in \mathbb{R}^n$, and we have used the fact that

$$\int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u_k^j B_0^H(t^j)\Big)\Big] dU^k = \frac{(2\pi)^{n/2}}{\Big[\mathrm{detCov}(B_0^H(t^1), \ldots, B_0^H(t^n))\Big]^{1/2}}$$

for λ_{Nn}-almost all $(t^1, \ldots, t^n) \in T^n$.

It follows from Lemma 4.5 and Inequality (2.3) that

$$\begin{aligned}\mathrm{detCov}(B_0^H(t^1),\ldots,B_0^H(t_n)) &\ge \prod_{\ell=1}^N \mathrm{detCov}(X^{H_\ell}(t_\ell^1),\cdots,X^{H_\ell}(t_\ell^n))\\ &\ge K_{15}^n \prod_{\ell=1}^N\prod_{j=1}^n \Big[\min\{|t_\ell^j-t_\ell^i|^{2H_\ell}, 0\le i\le j-1\}\Big].\end{aligned} \tag{4.6}$$

Denote $T=\prod_{\ell=1}^N[\epsilon_\ell,\epsilon_\ell+r]$. Putting together (4.5), (4.6) and applying Lemma 4.6 n times, we obtain

$$\begin{aligned}\mathbb{E}\big[L(x,T)^n\big] &\le K_{16}^n\int_{T^n}\prod_{\ell=1}^N\prod_{j=1}^n\Big[\min\{|t_\ell^j-t_\ell^i|^{2H_\ell}, 0\le i\le j-1\}\Big]^{-d/2}\,d\bar{t}\\ &\le K_{16}^n\prod_{\ell=1}^N\int_{[\epsilon_\ell,\epsilon_\ell+r]^n}\prod_{j=1}^n\Big[\min\{|t_\ell^j-t_\ell^i|^{2H_\ell}, 0\le i\le j-1\}\Big]^{-d/2}\\ &\quad\times dt_\ell^1\ldots dt_\ell^n\\ &\le K_{14}^n\, r^{(N-\sum_{\ell=1}^N H_\ell d)n}\,(n!)^{\sum_{\ell=1}^N H_\ell d},\end{aligned}$$

for some positive and finite constant K_{14}. This proves (4.4). □

Remark 4.9. From the proof of Lemma 4.8, it is clear that if $T=\prod_{\ell=1}^N[s_\ell,t_\ell]$ is any closed subinterval of I with at least one side length $\le r$, say, $|s_1-t_1|\le r$, then the following inequality holds

$$\mathbb{E}\big[L(x,T)^n\big]\le K_{14}^n\, r^{(1-H_1 d)n}\,(n!)^{\sum_{\ell=1}^N H_\ell d}.$$

We will apply this inequality in the proof of Theorem 4.1 below. □

Lemma 4.8 implies that for all $n\ge 1$, $L(x,T)\in L^n(\mathbb{R}^d)$ a.s. (see Geman and Horowitz (1980, page 42). Our next lemma estimates the moments of the increments of $L(x,T)$ in x.

Let $\gamma>0$ be a constant whose value will be determined later. Define

$$p(y)=\begin{cases}0, & \text{if } y=0\\ \log^{-\gamma}\big(e/|y|\big), & \text{if } 0<|y|\le 1\\ \gamma|y|-\gamma+1, & \text{if } |y|>1.\end{cases}$$

Clearly, the function p is symmetric, strictly increasing on $[0,\infty)$ and $p(u)\downarrow 0$ as $u\downarrow 0$.

Lemma 4.10. *Let $I\in\mathcal{A}$ be a fixed closed interval and let $D\subset\mathbb{R}^d$ be a hypercube. Then there exists a finite constant $K_{17}>0$, depending on N, d, H, I and D only,* *such that for all even integers $n \geq 2$ and all hypercubes $T \subseteq I$ with side length r and $T \in \mathcal{A}$,*

$$\begin{aligned}\mathbb{E} \int_D \int_D \Big(\frac{L(x,T) - L(y,T)}{p(|x-y|/\sqrt{d})} \Big)^n \, dx dy \\ \leq K_{17}^n (n!)^N \, (\log n)^{n\gamma} \, r^{n(N - \sum_{\ell=1}^N H_\ell d)} \log_+^{nN\gamma} \Big(\frac{e}{r}\Big). \end{aligned} \tag{4.7}$$

Proof. First we note that for $u^1, \dots, u^n, \ y \in \mathbb{R}^d$,

$$\begin{aligned}
\prod_{j=1}^n \Big| \exp(-i \langle u^j, y\rangle) - 1 \Big| &= \prod_{j=1}^n \Big| \exp\Big(-i \sum_{\ell=1}^d u_\ell^j y_\ell \Big) - 1 \Big| \\
&\leq \prod_{j=1}^n \Big| \sum_{k=1}^d \Big[\exp\Big(-i \sum_{\ell=0}^k u_\ell^j y_\ell \Big) - \exp\Big(-i \sum_{\ell=0}^{k-1} u_\ell^j y_\ell \Big) \Big] \Big| \quad (y_0 = 0, \ u_0^j = 0) \\
&\leq \prod_{j=1}^n \Big[\sum_{k=1}^d \Big| \exp(-i u_k^j y_k) - 1 \Big| \Big] \\
&= \sum{}' \prod_{j=1}^n \Big| \exp(-i u_{k_j}^j y_{k_j}) - 1 \Big|,
\end{aligned} \tag{4.8}$$

where the summation $\sum'$ is taken over all sequences $(k_1, \cdots, k_n) \in \{1, \dots, d\}^n$.

Secondly, without loss of generality, we may and will assume the edge length of T is sufficiently small such that for all $s, t \in T$

$$\mathbb{E}\Big(|B_0(s) - B_0(t)|^2 \Big) \leq \min\{1, [e \operatorname{Var}(B_0^H(u))]^{-1}, u \in T\}. \tag{4.9}$$

Otherwise, we can divide T into, say, m small subintervals $T_1, \dots, T_m$ such that $T_j \in \mathcal{A} \, (j = 1, \dots, m)$ and (4.9) holds on each T_j. Then Inequality (4.7) will only be affected by a factor that is at most m^n.

It follows from (2.7) and (4.8) that

$$\begin{aligned}
\mathbb{E}\Big[(L(x,T) - L(y,T))^n \Big] &\leq (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big| \exp(-i \langle u^j, y - x \rangle) - 1 \Big| \\
&\quad \cdot \exp\Big[-\frac{1}{2} \operatorname{Var}\Big(\sum_{j=1}^n \langle u^j, \ B^H(t^j) \rangle \Big) \Big] d\overline{u} \, d\overline{t} \\
&\leq (2\pi)^{-nd} \sum{}' \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big| \exp(i u_{k_j}^j (y_{k_j} - x_{k_j})) - 1 \Big| \\
&\quad \cdot \exp\Big[-\frac{1}{2} \operatorname{Var}\Big(\sum_{j=1}^n \langle u^j, \ B^H(t^j) \rangle \Big) \Big] d\overline{u} \, d\overline{t}.
\end{aligned} \tag{4.10}$$

Hence for any fixed hypercube $D \subset \mathbb{R}^d$ and any even integer $n \geq 2$, we have

$$\mathbb{E}\int_D\int_D\Big[\Big(\frac{L(x,T)-L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n\Big]dxdy \leq (2\pi)^{-nd}\sum{}' \int_D\int_D\int_{T^n}\int_{\mathbb{R}^{nd}}$$
$$\prod_{j=1}^n \frac{\big|\exp(iu^j_{k_j}(y_{k_j}-x_{k_j}))-1\big|}{p(|y-x|/\sqrt{d})}\cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\,d\overline{t}\,dx\,dy$$
$$\leq (2\pi)^{-nd}\lambda_d(D)\sum{}'\int_{D\ominus D}\int_{T^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\frac{\big|\exp(iu^j_{k_j}y_{k_j})-1\big|}{p(|y_{k_j}|/\sqrt{d})}$$
$$\cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\,d\overline{t}\,dy. \tag{4.11}$$

In the above, $D\ominus D=\{x-y: x, y\in D\}$ and we have made a change of variables and have used the fact that $p(|y|/\sqrt{d})\geq p(|y_k|/\sqrt{d})$ for all $k=1,\dots,d$, here $y=(y_1,\dots,y_d)$.

Now we fix a sequence $(k_1,\dots,k_n)\in\{1,\dots,d\}^n$ and consider the integral

$$\mathcal{M}=\int_{D\ominus D}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\frac{\big|\exp(iu^j_{k_j}y_{k_j})-1\big|}{p(|y_{k_j}|/\sqrt{d})}\ \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\,dy.$$

For any fixed n points $t^1,\dots,t^n\in T$ such that $t^1_\ell,\dots,t^n_\ell$ are all distinct for $1\leq\ell\leq N$ (the set of such points has full (nN)-dimensional Lebesgue measure), Lemma 4.5 and Inequality (2.3) imply that the Gaussian random variables $B^H_k(t^j)$ $(k=1,\dots,d,\ j=1,\dots,n)$ are linearly independent. Hence by applying the generalized Hölder's inequality, Lemma 4.2 and Lemma 4.3, we have

$$\mathcal{M}\leq K\prod_{j=1}^n\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}^{nd}}\Big[\frac{|\exp(iu^j_{k_j}y_{k_j})-1|}{p(|y_{k_j}|/\sqrt{d})}\Big]^n$$
$$\cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\sum_{\ell=1}^d u^j_\ell B^H_\ell(t^j)\Big)\Big]\,d\overline{u}\,dy_{k_j}\Big\}^{1/n}$$
$$=\frac{K(2\pi)^{n(d-1)/2}}{[\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n))]^{d/2}}$$
$$\cdot\prod_{j=1}^n\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\big|\exp\big(iu^j_{k_j}y_{k_j}/\sigma_j\big)-1\big|^n}{p^n(y_{k_j}/\sqrt{d})}\exp\Big(-\frac{(u^j_{k_j})^2}{2}\Big)dy_{k_j}\,du^j_{k_j}\Big\}^{1/n}$$
$$\leq\frac{K^n}{[\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n))]^{d/2}}\prod_{j=1}^n\Big[\int_{\mathbb{R}}p_+^{-n}\Big(\frac{\sigma_j}{v}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv\Big]^{1/n}, \tag{4.12}$$

where $K > 0$ is a constant depending on D and K_{12} in Lemma 4.3, and σ_j^2 is the conditional variance of $B_{k_j}^H(t^j)$ given $B_\ell^H(t^i)$ $(\ell \neq k_j$ or $\ell = k_j,\ i \neq j)$.

Since

$$p_+^{-n}(x) = \begin{cases} \log^{n\gamma}\big(e/x\big), & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

and $\log_+^\alpha(xy) \leq 2^\alpha(\log_+^\alpha x + \log_+^\alpha y)$ for all $\alpha \geq 0$, where $\log_+ x = \max\{1, \log x\}$, we deduce

$$\begin{aligned}\int_{\mathbb{R}} p_+^{-n}\Big(\frac{\sigma_j}{v}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv \leq \int_{|\sigma_j/v|\geq 1} &\exp(-v^2/2)dv \\ +2^{n\gamma}\int_{|\sigma_j/v|<1} &\log_+^{n\gamma}(v)\exp\Big(-\frac{v^2}{2}\Big)dv \\ +2^{n\gamma}\int_{|\sigma_j/v|<1} &\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv. \end{aligned} \tag{4.13}$$

By Lemma 4.4, for n large, the above is bounded by

$$K^n\Big[\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big) + (\log(n\gamma))^{n\gamma}\Big] \leq K^n\Big[\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big)\Big][\log n]^{n\gamma}. \tag{4.14}$$

It follows from (4.11), (4.12), (4.13) and (4.14) that

$$\begin{aligned}\mathbb{E}\int_D\int_D\Big(\frac{L(x,T)-L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n dxdy &\leq K^n\lambda_d(D)[\log n]^{n\gamma} \\ \cdot\int_{T^n}\frac{1}{(\det\mathrm{Cov}(B_0^H(t^1),\dots,B_0^H(t^n)))^{d/2}}&\prod_{j=1}^n\log_+^\gamma\Big(\frac{e}{\sigma_j}\Big)d\bar{t}. \end{aligned}\tag{4.15}$$

By the independence of $B_1^H, \dots, B_d^H$, we deduce that

$$\begin{aligned}\sigma_j^2 &= \mathrm{Var}\Big(B_{k_j}^H(t^j)\Big|B_\ell^H(t^i),\ \ell \neq k_j \text{ or } \ell = k_j,\ i \neq j\Big) \\ &= \mathrm{Var}\Big(B_{k_j}^H(t^j)\Big|B_{k_j}^H(t^i),\ i \neq j\Big) \\ &= \frac{\det\mathrm{Cov}(B_0^H(t^1),\dots,B_0^H(t^n))}{\det\mathrm{Cov}(B_0^H(t^i), i \neq j)} \\ &\geq e\,\det\mathrm{Cov}(B_0^H(t^1),\dots,B_0^H(t^n)). \end{aligned}\tag{4.16}$$

Notice that in obtaining the last inequality in (4.16), we have used (2.2) and the assumption that the edge length of T is very small (so that (4.9) holds) to derive that for any $s^1, \dots, s^n \in T$

$$\begin{aligned}&\det\mathrm{Cov}(B_0^H(s^1),\dots,B_0^H(s^n)) \\ &\quad= \mathrm{Var}(B_0^H(s^1))\prod_{j=2}^n\mathrm{Var}\Big(B_0^H(s^j)\big|B_0^H(s^i), i = 1,\dots,j-1\Big) \\ &\quad\leq \mathrm{Var}(B_0^H(s^1))\prod_{j=2}^n\mathbb{E}\Big(|B_0^H(s^j)-B_0^H(s^{j-1})|^2\Big) \leq e^{-1}.\end{aligned}$$

By (4.16), we have

$$\prod_{j=1}^{n} \log_+^{\gamma}\left(\frac{e}{\sigma_j}\right) \leq 2^{-n\gamma} \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n))}. \tag{4.17}$$

It follows from (4.15), (4.17), Lemma 4.5 and (2.2) that

$$\begin{aligned}
&\mathbb{E}\int_D\int_D \Big(\frac{L(x,T)-L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n\,dxdy \\
&\quad\leq K^n\lambda_d(D)[\log n]^{n\gamma}\int_{T^n}\frac{1}{(\det\mathrm{Cov}(B_0^H(t^1),\ldots,B_0^H(t^n)))^{d/2}} \\
&\qquad\qquad\cdot\log_+^{n\gamma}\frac{e}{\det\mathrm{Cov}(B_0^H(t^1),\ldots,B_0^H(t^n))}\,d\bar{t} \\
&\quad\leq K^n\lambda_d(D)[\log n]^{n\gamma}\sum_{k=1}^{N}\int_{T^n}\prod_{\ell=1}^{N}\frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad\qquad\cdot\log_+^{n\gamma}\frac{e}{\det\mathrm{Cov}(X^{H_k}(t_k^1),\ldots,X^{H_k}(t_k^n))}\,d\bar{t} \\
&\quad\leq K^n\lambda_d(D)[\log n]^{n\gamma}\prod_{\ell=1}^{N}\int_{[\epsilon,\epsilon+r]^n}\frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad\qquad\cdot\log_+^{n\gamma}\frac{e}{\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n))}\,d\bar{t}_\ell.
\end{aligned} \tag{4.18}$$

In the last inequality, we have written $T = [\epsilon, \epsilon + r]^N$ and $d\bar{t}_\ell$ for $dt_\ell^1 \cdots dt_\ell^n$ to simplify the notation.

For each fixed $\ell \in \{1, \ldots, N\}$, we use $\mathcal{N}_\ell$ to denote the integral

$$\begin{aligned}
&\int_{[\epsilon,\epsilon+r]^n}\frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad\cdot\log_+^{n\gamma}\frac{e}{\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n))}\,d\bar{t}_\ell.
\end{aligned}$$

Since the integrand in $\mathcal{N}_\ell$ is invariant under permutations of the t_ℓ^j $(j = 1, \ldots, n)$, we have

$$\begin{aligned}
\mathcal{N}_\ell &= n!\int_{\{\epsilon\leq t_\ell^1\leq\ldots\leq t_\ell^n\leq\epsilon+r\}}\frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad\qquad\cdot\log_+^{n\gamma}\frac{e}{\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1),\ldots,X^{H_\ell}(t_\ell^n))}d\bar{t_\ell} \\
&\leq K^n n!\sum_{k=1}^{n}\int_{\{\epsilon\leq t_\ell^1\leq\ldots\leq t_\ell^n\leq\epsilon+r\}}\prod_{j=1}^{n}\frac{1}{|t_\ell^j-t_\ell^{j-1}|^{H_\ell d}}\cdot\log_+^{n\gamma}\frac{e}{|t_\ell^k-t_\ell^{k-1}|}\,d\bar{t}_\ell \\
&\leq K^n n!\,r^{n(1-H_\ell d)}\log_+^{n\gamma}\Big(\frac{e}{r}\Big).
\end{aligned} \tag{4.19}$$

The last inequality follows from the fact that $H_\ell d < 1$ and

$$\int_0^r \frac{1}{s^{H_\ell d}} \log^{n\gamma}\Big(\frac{e}{s}\Big) ds \le K 2^{n\gamma-1}\, r^{1-H_\ell d} \log^{n\gamma}\Big(\frac{e}{r}\Big)$$

for all $\gamma \ge 0$.

Combining (4.18) and (4.19), we have

$$\begin{aligned}
&\mathbb{E} \int_D \int_D \Big(\frac{L(x,T) - L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n dx dy \\
&\qquad \le K^n (n!)^N (\log n)^{n\gamma}\, r^{n(N - \sum_{\ell=1}^N H_\ell d)} \log_+^{nN\gamma}\Big(\frac{e}{r}\Big).
\end{aligned}$$

This finishes the proof of Lemma 4.10. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $I \in \mathcal{A}$ be fixed. For simplicity, again we assume $I = [\epsilon, 1]^N$.

Let $\Psi(u) = u \exp(u^\theta)$, where $\theta \in (0, 1/N)$ is a constant. Then Ψ is increasing and convex on $(0, \infty)$. First we fix an interval $T \in \mathcal{A}$ with side length r such that $T \subseteq I$. It follows from Jensen's inequality and Lemma 4.10 that for all closed hypercubes $D \subset \mathbb{R}^d$ and all n with $\theta + 1/n < 1$,

$$\begin{aligned}
&\mathbb{E} \int_D \int_D \Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big)^{n\theta+1} dx dy \\
&\qquad \le K \Big\{\mathbb{E} \int_D \int_D \Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big)^n dx dy\Big\}^{\theta+1/n} \\
&\qquad \le K^n (n!)^{N(\theta+1/n)} (\log n)^{n\gamma(\theta+1/n)},
\end{aligned}$$

where K is a finite constant depending on N, d, H, θ, I and D only.

Expanding $\Psi(s)$ into a power series and applying the above inequality, we derive

$$\begin{aligned}
&\mathbb{E} \int_D \int_D \Psi\Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big) dx dy \\
&\quad = \sum_{n=0}^\infty \frac{1}{n!}\, \mathbb{E} \int_D \int_D \Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big)^{n\theta+1} dx dy < \infty,
\end{aligned} \tag{4.20}$$

the last inequality follows from the fact that $N\theta < 1$. Hence Garsia's lemma implies that there are positive and finite random variables A_1 and A_2 such that for almost all $x,\ y \in D$ with $|x-y| \le e^{-1}$

$$\begin{aligned}
|L(x,T) - L(y,T)| &\le \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A_1}{u^{2d}}\Big) dp(u) \\
&\le A_2 \Big[\log\big(1/|x-y|\big)\Big]^{-(\gamma - 1/\theta)}.
\end{aligned}$$

By choosing $\gamma > 1/\theta$, we see that B^H has almost surely a local time $L(x, T)$ that is continuous for all $x \in D$. By taking an increasing sequence of closed hypercubes $\{D_n,\ n \geq 1\}$ such that $\mathbb{R}^d = \cup_{n=1}^{\infty} D_n$, we have proved that almost surely $L(x, T)$ is continuous for all $x \in \mathbb{R}^d$.

The proof of the joint continuity is similar to the above. By the easily verifiable inequality $\Psi(u+v) \leq \Psi(2u) + \Psi(2v)$ and the monotonicity of the functions Ψ and p, we see that for all $s, t \in I$,

$$\begin{aligned}\Psi\Big(\frac{|L(x,[\epsilon,t]) - L(y,[\epsilon,s])|}{p(\max\{|s-t|,|x-y|\}/\sqrt{N+d})}\Big) &\leq \Psi\Big(\frac{2|L(x,[\epsilon,t]) - L(x,[\epsilon,s])|}{p(|s-t|/\sqrt{N+d})}\Big)\\ &\quad + \Psi\Big(\frac{2|L(x,[\epsilon,s]) - L(y,[\epsilon,s]|)}{p(|x-y|/\sqrt{N+d})}\Big).\end{aligned} \tag{4.21}$$

Note that the difference $L(x, [\epsilon, t]) - L(x, [\epsilon, s])$ can be written as a sum of finite number (only depends on N) of terms of the form $L(x, T_j)$, where each $T_j \in \mathcal{A}$ is a closed subinterval of I with at least one edge length $\leq |s-t|$, we can use Lemma 4.8 and Remark 4.9, together with a power series expansion of Ψ and Jensen's inequality, to bound the corresponding integral below. On the other hand, the integral involving the last term in (4.21) can be dealt with using Lemma 4.10 as above. Consequently,

$$\begin{aligned}&\mathbb{E}\int_I\int_I\int_D\int_D \Psi\Big(\frac{|L(x,[\epsilon,t]) - L(y,[\epsilon,s])|}{p(\max\{|s-t|,|x-y|\}/\sqrt{N+d})}\Big)\,dxdy\,dsdt\\ &\quad\leq \mathbb{E}\int_I\int_I\int_D\int_D \Psi\Big(\frac{2|L(x,[\epsilon,t]) - L(x,[\epsilon,s])|}{p(|s-t|/\sqrt{N+d})}\Big)\,dxdy\,dsdt\\ &\qquad + \mathbb{E}\int_I\int_I\int_D\int_D \Psi\Big(\frac{2|L(x,[\epsilon,s]) - L(y,[\epsilon,s]|)}{p(|x-y|/\sqrt{N+d})}\Big)\,dxdy\,dsdt < \infty.\end{aligned}$$

Therefore the joint continuity of the local times follows again from Garsia's lemma. This finishes the proof of Theorem 4.1. □

Remark 4.11. We conjecture that B^H has jointly continuous local times whenever the condition $d < \sum_{\ell=1}^{N} \frac{1}{H_\ell}$ is satisfied. However, we have not been able to prove this by using the method in the present paper.

Furthermore, it would be interesting to investigate the local behavior of the random Borel measure $L(x, \bullet)$. More specifically, we state the following

Problem 4.12 Let $L^*(I) = \sup_{x\in\mathbb{R}^d} L(x, I)$ be the maximum local time of B^H on I. Find Hausdorff measure functions $\varphi_1(r)$ and $\varphi_2(r)$ such that

(i). for every $t \in (0,\infty)^N$ almost surely

$$\limsup_{r\to 0} \frac{L^*(I(t,r))}{\varphi_1(r)} \leq K,$$

where $I(t, r)$ is the (open) ball (or cube) centered at t with radius (edge length) r and $K > 0$ is a constant; and (ii). for any rectangle $T \in \mathcal{A}$, there exists a positive finite constant K such that almost surely

$$\limsup_{r \to 0} \sup_{t \in T} \frac{L^*(I(t,r))}{\varphi_2(r)} \le K.$$

This amounts to derive sharp bounds for the moments of $L(x, T)$ and its increments considered in Lemmas 4.8 and 4.10. Solutions to Problem 4.12 will have implications on the fractal properties of the level sets of B^H. For results on the Brownian sheet and fractional Brownian motion, we refer to Ehm (1981) and Xiao (1997), respectively. Similar results were obtained by Cuzick (1982) for a class of stationary Gaussian processes using a different approach.

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