A Packing Dimension Theorem for Gaussian Random Fields

Yimin Xiao
Michigan State University

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Abstract

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad \forall\, t \in \mathbb{R}^N,$$

where $X_1, \ldots, X_d$ are independent copies of a centered Gaussian random field X_0. Under certain general conditions, Xiao (2007a) defined an upper index α^* and a lower index α_* for X_0 and showed that the Hausdorff dimensions of the range $X\big([0,1]^N\big)$ and graph $\mathrm{Gr}X\big([0,1]^N\big)$ are determined by the upper index α^*. In this paper, we prove that the packing dimensions of $X\big([0,1]^N\big)$ and $\mathrm{Gr}X([0,1]^N)$ are determined by the lower index α_* of X_0. Namely,

$$\dim_{\rm P} X\big([0,1]^N\big) = \min\left\{d,\ \frac{N}{\alpha_*}\right\}, \qquad \text{a.s.}$$

and

$$\dim_{\rm P} \mathrm{Gr}X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha_*},\ N + (1-\alpha_*)d\right\}, \qquad \text{a.s.}$$

This verifies a conjecture in Xiao (2007a). Our method is based on the potential-theoretic approach to packing dimension due to Falconer and Howroyd (1997).

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1 Introduction

Fractal dimensions such as Hausdorff dimension, box-counting dimension and packing dimension are very useful in studying fractals [see, e.g., Falconer (1990)], as well as in characterizing roughness or irregularity of stochastic processes and random fields. We refer Taylor (1986) and Xiao (2004) for extensive surveys on results and techniques for Markov processes, and to Adler (1981), Kahane (1985) and Xiao (2007b) for geometric results for Gaussian random fields such as fractional Brownian motion and the Brownian sheet.

Compared with Lévy processes, however, there have not been many results on the packing dimensions of random fractals associated with Gaussian random fields. The main reason is that most studies on fractals properties of Gaussian random fields so far have been limited to fractional Brownian motion or the Brownian sheet, whose analytic and geometric properties are determined by a single parameter and are typical examples of random monofractals [see, e.g., Seuret (2008a, 2008b) for more information]. If X is such a Gaussian random field, then it is often true that the packing dimensions of its range $X\big([0,1]^N\big) = \{X(t), t \in [0,1]^N\}$ and graph $\mathrm{Gr}X\big([0,1]^N\big) = \big\{(t, X(t)) : t \in [0,1]^N\big\}$ coincide with their Hausdorff dimensions.

Significant difference between the Hausdorff and packing dimensions of the image $X(E)$ appears when $E \subseteq \mathbb{R}^N$ is an arbitrary Borel set. Talagrand and Xiao (1996) proved that, even for such "nice" Gaussian random fields as fractional Brownian motion and the Brownian sheet, the Hausdorff and packing dimensions of $X(E)$ can be different because they depend on different aspects of the fractal structure of E. Xiao (1997) further showed that the packing dimension of $X(E)$ is determined by the packing dimension profile introduced by Falconer and Howroyd (1997) [see Section 2 for its definition].

As noted in Xiao (2007a), fractal properties of the range $X\big([0,1]^N\big)$ and graph $\mathrm{Gr}X\big([0,1]^N\big)$ themselves become more involved when X is a general Gaussian random field. To be more specific, let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad \forall t \in \mathbb{R}^N, \tag{1.1}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ which belongs to a wide class of Gaussian random fields. In order to study sample path properties of X, Xiao (2007a) introduced an upper index α^* and a lower index α_* for X_0 [see Section 2 for their definitions] and proved that

$$\dim_{\mathrm{H}} X\big([0,1]^N\big) = \min\left\{d,\ \frac{N}{\alpha^*}\right\}, \qquad \text{a.s.} \tag{1.2}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha^*},\ N + (1-\alpha^*)d\right\}, \qquad \text{a.s.} \tag{1.3}$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. That is, the Hausdorff dimension of $X\big([0,1]^N\big)$ and $\mathrm{Gr}X\big([0,1]^N\big)$ are determined by the upper index α^* of X_0. Xiao (2007a, Remark 3.9) conjectured that the packing dimensions of $X\big([0,1]^N\big)$ and $\mathrm{Gr}X\big([0,1]^N\big)$ are determined by the lower index α_* of X_0.

The objective of this paper is to verify this conjecture by proving the following result: Under certain mild conditions on X_0,

$$\dim_{\mathrm{P}} X\big([0,1]^N\big) = \min\left\{d,\ \frac{N}{\alpha_*}\right\}, \qquad \text{a.s.} \tag{1.4}$$

and

$$\dim_{\mathrm{P}} \mathrm{Gr}X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha_*},\ N + (1-\alpha_*)d\right\}, \qquad \text{a.s.,} \tag{1.5}$$

where $\dim_{\mathrm{P}} E$ denotes the packing dimension of E. The results (1.2)– (1.5) show that, similar to the well-known cases of Lévy processes [see Pruitt and Taylor (1996)], the Hausdorff dimensions of $X\big([0,1]^N\big)$ and $\mathrm{Gr}X\big([0,1]^N\big)$ may be different from their packing dimensions.

The combined results of Xiao (2007a) and this paper suggest that, unlike fractional Brownian motion which is a monofractal, general Gaussian processes and random fields may exhibit interesting multifractal structures. This would make general Gaussian random fields attractive for stochastic modeling in various areas including turbulence and image processing. It would be of significance to pursue this line of research [see Remark 3.7].

The rest of this paper is organized as follows. In Section 2 we recall the definitions and some basic properties of Gaussian random fields, packing dimension and packing dimension profiles. In Section 3 we state and prove our main result. The method for proving the lower bounds in (1.4) and (1.5) is potential-theoretic. It can be viewed as an analogue of the classical and powerful "capacity argument" [based on the Frostman theorem] for Hausdorff dimension computation. It will be clear from the proof of Theorem 3.1 that this potential-theoretic method is applicable to stochastic processes which are not necessarily Gaussian. In particular, it can be applied to the locally self-similar processes considered by Benassi, Cohen and Istas (2003).

We will use K to denote a positive constant which may differ in each occurrence.

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2 Preliminaries

In this section, we recall briefly the definitions and some basic properties of Gaussian random fields, packing dimension and packing dimension profiles.

2.1 Upper and lower indices of Gaussian random fields

Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a real-valued Gaussian random field with $X_0(0) = 0$ almost surely. We assume that X_0 satisfies the following Condition (C):

(C) There exist positive constants δ_0, $K \geq 1$ and a right continuous function $\phi : [0, \delta_0) \to [0, \infty)$ such that $\phi(0) = 0$ and for all $t \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$ with $\|h\| \leq \delta_0$,

$$K^{-1}\, \phi^2(\|h\|) \leq \mathbb{E}\big[\big(X_0(t+h) - X_0(t)\big)^2\big] \leq K\, \phi^2(\|h\|). \tag{2.1}$$

Note that the function $\phi^2(\|h\|)$ depends only on $\|h\|$, so we can say that Condition (C) requires the increments of X_0 to be *approximately stationary and isotropic.*

The upper index of ϕ at 0 is defined by

$$\alpha^* = \inf\left\{\beta \geq 0 : \lim_{r \downarrow 0} \frac{\phi(r)}{r^\beta} = \infty\right\} \tag{2.2}$$

with the convention $\inf \emptyset = \infty$. Analogously, the lower index of ϕ at 0 is defined by

$$\alpha_* = \sup\left\{\beta \geq 0 : \lim_{r \downarrow 0} \frac{\phi(r)}{r^\beta} = 0\right\}. \tag{2.3}$$

For convenience, we simply call α^* and α_* the upper and lower indices of X_0, respectively. It is well known that, if $\alpha_* > 0$, then X_0 has almost surely continuous sample paths; see Lemma 3.3 below.

When the real-valued Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ has stationary, isotropic increments and a continuous covariance function, the above upper and lower indices α^* and α_* coincide with the upper and lower indices of $\sigma(h)$, where

$$\sigma^2(h) = \mathbb{E}\big[\big(X_0(t+h) - X_0(t)\big)^2\big], \qquad \forall\, h \in \mathbb{R}^N. \tag{2.4}$$

When $\alpha^* = \alpha_* = \alpha$, X_0 is called an index-α Gaussian field; see Adler (1981).

Since most interesting examples of Gaussian random fields satisfying Condition (C) are those with stationary increments, we collect some basic facts about them.

Suppose the Gaussian field X_0 has stationary increments and continuous covariance function $R(s,t) = \mathbb{E}\big[X_0(s)X_0(t)\big]$. If follows from Yaglom (1957) that $R(s,t)$ can be written as

$$R(s,t) = \int_{\mathbb{R}^N} \big(e^{i\langle s,\lambda\rangle} - 1\big)\big(e^{-i\langle t,\lambda\rangle} - 1\big)\,\Delta(d\lambda) + \langle s, Qt\rangle, \tag{2.5}$$

where $\langle x, y\rangle$ is the ordinary scalar product in $\mathbb{R}^N$, Q is an $N \times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1 + \|\lambda\|^2}\,\Delta(d\lambda) < \infty. \tag{2.6}$$

The measure Δ is called the *spectral measure* of X_0. It follows from (2.5) that X_0 has the following stochastic integral representation:

$$\big\{X_0(t),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \left\{\int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\, W(d\lambda) + \langle \mathbf{Y}, t\rangle,\, t \in \mathbb{R}^N\right\}, \tag{2.7}$$

where $\stackrel{d}{=}$ means equality in all finite-dimensional distributions. In (2.7), $\mathbf{Y}$ is an N-dimensional Gaussian random vector with mean 0 and covariance matrix Q, $W(d\lambda)$ is a centered complex-valued Gaussian random measure which is independent of $\mathbf{Y}$ and satisfies $\mathbb{E}\big(W(A)\overline{W(B)}\big) = \Delta(A \cap B)$ and $W(-A) = \overline{W(A)}$ for all Borel sets $A,\ B \subseteq \mathbb{R}^N$ with finite Δ measure. Since the effect of the linear term $\langle \mathbf{Y}, t\rangle$ in (2.7) on the problems we consider is trivial, we will assume $\mathbf{Y} = 0$. Consequently, we have

$$\sigma^2(h) = \mathbb{E}\big[\big(X_0(t+h) - X_0(t)\big)^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\,\Delta(d\lambda). \tag{2.8}$$

It is important to observe that the incremental-variance function $\sigma^2(h)$ in (2.8) is a negative definite function in the sense of I. J. Schoenberg and can be viewed as the characteristic exponent of a symmetric infinitely divisible distribution; see Berg and Forst (1975) for more information on negative definite functions. This connection suggests that there may be some duality between the properties of X_0 and those of symmetric Lévy processes. Indeed, the upper and lower indices of X_0 defined above are reminiscent to the upper and lower indices for Lévy processes introduced by Blumenthal and Getoor (1961) [Even though for studying local properties of a Gaussian random field one is interested in the behavior of $\sigma^2(h)$ near

$h = 0$, while Blumenthal and Getoor's indices are concerned with the asymptotic behavior of σ^2 at infinity]. To large extent, our work in this paper is inspired by studies on sample path properties of Lévy processes.

Note that, in general, one can also define upper and lower indices β^* and β_* for $\sigma(h)$ in a way similar to (2.2) and (2.3). For example,

$$\beta^* = \inf\left\{\beta \geq 0 : \lim_{\|h\|\to 0} \frac{\sigma(h)}{\|h\|^\beta} = \infty\right\} \tag{2.9}$$

with the convention $\inf \emptyset = \infty$, and the lower index β_* is defined analogously. However, compared to the indices α^* and α_* for ϕ, the indices β^* and β_* may behave more wildly as shown by the following example.

Example 2.1 Let $N \geq 2$ and let Δ be a Borel measure on $\mathbb{R}^N$ with support in a linear subspace L of $\mathbb{R}^N$ and satisfying (2.6). If Y is a Gaussian random field with stationary increments and spectrum measure Δ, then for all h in the linear subspace of $\mathbb{R}^N$ that is orthogonal to L, we have $\sigma^2(h) = 0$. Thus $\beta^* = \infty$. □

Lemma 2.2 below, which is taken from Xiao (2007a), provides a sufficient condition for the inequality $\beta^* \leq 1$ to hold.

Lemma 2.2 *Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ with stationary increments and spectrum measure Δ. If either $N = 1$ or, $N \geq 2$ and Δ has an absolutely continuous part with density $f(\lambda)$. Then $\beta^* \leq 1$.*

In this paper, we only consider Gaussian random fields satisfying Condition (C) and leave general anisotropic random fields to be treated elsewhere [see Remark 3.6]. We remark that the class of Gaussian random fields satisfying Condition (C) is large. It includes not only fractional Brownian motion, the Brownian sheet and Gaussian processes with regularly varying incremental variance functions [all of them satisfy $\alpha_* = \alpha^*$], but more importantly also Gaussian random fields with stationary increments and different upper and lower indices. To be more concrete, given any constants $0 < a < b < 1$ and a measurable function $H(\lambda)$: $\mathbb{R}^N \backslash \{0\} \to [a, b]$, let $\Delta := \Delta_H$ be the Borel measure on $\mathbb{R}^N \backslash \{0\}$ with density function

$$f_H(\lambda) = \frac{1}{\|\lambda\|^{2H(\lambda)+N}}. \tag{2.10}$$

Then the stochastic integral in (2.7) defines a centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ with stationary increments. If $H(\lambda) \equiv H \in (0, 1)$, then X_0 is the N-parameter fractional Brownian motion (FBM) with index H. If $N = 1$ and

$$H(\lambda) = \sum_{j=0}^{K} H_j \mathbb{1}_{[\omega_j, \omega_{j+1})}(|\lambda|),$$

where $H_j \in (0, 1)$ $(0 \leq j \leq K)$ and $\omega_0 = 0 < \omega_1 < \cdots < \omega_K < \omega_{K+1} = \infty$ are constants, then X_0 is the (M_K) multiscale FBM studied by Bardet and Bertrand (2007a, 2007b). It is not difficult to choose a function $H(\lambda)$ so that X_0 satisfies Condition (C) with different upper and lower indices. Such examples have been constructed by Xiao (2007a) and many more can be provided by modifying the constructions of Lévy processes with different upper and lower Blumenthal-Getoor indices [see Pruitt and Taylor (1996) and the references therein for more information].

2.2 Packing dimension and packing dimension profile

Packing dimension and packing measure were introduced in the early 1980s by Tricot (1982) and Taylor and Tricot (1985) as dual concepts to Hausdorff dimension and Hausdorff measure. Since then they have become useful tools in analyzing fractal sets. Packing dimension profile was introduced by Falconer and Howroyd (1997) for computing the packing dimension of orthogonal projections. Their definition of packing dimension profiles is based on potential-theoretic approach. Later Howroyd (2001) defined another packing dimension profile from the point of view of box-counting dimension. Recently, Khoshnevisan and Xiao (2006) proved that the packing dimension profiles of Falconer and Howroyd (1997) and Howroyd (2001) are the same.

For any $\varepsilon > 0$ and any bounded set $E \subset \mathbb{R}^N$, let $N(E, \varepsilon)$ be the smallest number of balls of radius ε needed to cover E. The upper box-counting dimension of E is defined as

$$\overline{\dim}_{\rm B} E = \limsup_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}$$

and the packing dimension of E is defined as

$$\dim_{\rm P} E = \inf \left\{ \sup_n \overline{\dim}_{\rm B} E_n : \quad E \subset \bigcup_{n=1}^{\infty} E_n \right\}, \tag{2.11}$$

see Tricot (1982) or Falconer (1990, p.45). It is well known that $0 \le \dim_{\rm H} E \le \dim_{\rm P} E \le \overline{\dim}_{\rm B} E \le N$ for every set $E \subset \mathbb{R}^N$.

For a finite Borel measure μ on $\mathbb{R}^N$, its packing dimension is defined by

$$\dim_{\rm P} \mu = \inf\{\dim_{\rm P} E : \ \mu(E) > 0 \text{ and } E \subset \mathbb{R}^N \text{ is a Borel set}\}. \tag{2.12}$$

Falconer and Howroyd (1997) defined the s-dimensional packing dimension profile of μ as

$$\mathrm{Dim}_s \mu = \sup \left\{ \beta \ge 0 : \ \liminf_{r \to 0} \ \frac{F_s^{\mu}(x, r)}{r^{\beta}} = 0 \ \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}, \tag{2.13}$$

where, for any $s > 0$, $F_s^{\mu}(x, r)$ is the s-dimensional potential of μ defined by

$$F_s^{\mu}(x, r) = \int_{\mathbb{R}^N} \min\{1, \ r^s \, \|y - x\|^{-s}\} \, d\mu(y). \tag{2.14}$$

Falconer and Howroyd (1997) showed that

$$0 \le \mathrm{Dim}_s \mu \le s \quad \text{ and } \ \mathrm{Dim}_s \mu = \dim_{\rm P} \mu \ \text{ if } s \ge N, \tag{2.15}$$

Note that the identity in (2.15) provides the following equivalent characterization of $\dim_{\rm P} \mu$ in terms of the potential $F_N^{\mu}(x, r)$:

$$\dim_{\rm P} \mu = \sup \left\{ \beta \ge 0 : \ \liminf_{r \to 0} \ \frac{F_N^{\mu}(x, r)}{r^{\beta}} = 0 \ \text{ for } \mu\text{-a.a. } \ x \in \mathbb{R}^N \right\}. \tag{2.16}$$

For any Borel set $E \subseteq \mathbb{R}^N$, the s-dimensional packing dimension profile of E is defined by

$$\mathrm{Dim}_s E = \sup \left\{ \mathrm{Dim}_s \mu : \ \mu \in \mathcal{M}_c^{+}(E) \right\}, \tag{2.17}$$

where $\mathcal{M}_c^{+}(E)$ denotes the family of finite Borel measures with compact support in E. It follows from (2.15) that $0 \le \mathrm{Dim}_s E \le s$ and $\mathrm{Dim}_s E = \dim_{\rm P} E$ if $s \ge N$.

3 Main result

Now we consider the packing dimensions of the range and graph of an (N, d) Gaussian random field. The following is our main result.

Theorem 3.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be the Gaussian random field in $\mathbb{R}^d$ defined by (1.1). We assume that the associated random field X_0 satisfies Condition (C) and $0 < \alpha_* \le \alpha^* \le 1$. If ϕ satisfies either one of the following conditions: For any $\varepsilon > 0$ small enough, there exists a constant K such that either*

$$\int_0^1 \left(\frac{\phi(a)}{\phi(ax)}\right)^d x^{N-1}\, dx \le K\, a^{-\varepsilon} \quad \text{for all } \ a \in (0, 1], \tag{3.1}$$

or

$$\int_1^{1/a} \left(\frac{\phi(a)}{\phi(ax)}\right)^d x^{N-1}\, dx \le K\, a^{-\varepsilon} \quad \text{for all } \ a \in (0, 1]. \tag{3.2}$$

Then

$$\dim_{\rm P} X([0,1]^N) = \min\left\{d,\ \frac{N}{\alpha_*}\right\}, \qquad \text{a.s.} \tag{3.3}$$

and

$$\dim_{\rm P} {\rm Gr} X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha_*},\ N + (1-\alpha_*)d\right\}, \qquad \text{a.s.} \tag{3.4}$$

Remark 3.2 The conditions (3.1) and (3.2) correspond to, roughly speaking, whether X hits points or not. They appeared in Xiao (2007a) for studying respectively the exact Hausdorff measure of the range and the regularity of local times of X. If ϕ is regularly varying at the origin with index α, then (3.1) is satisfied whenever $N > \alpha d$, and (3.2) is satisfied whenever $N \le \alpha d$. Thus in the regularly varying case, at least one of these conditions are automatically satisfied. In general, (3.1) holds if $N \ge \alpha^* d$, and (3.2) holds provided $N \le \alpha_* d$. □

The proof of Theorem 3.1 will be divided into proving the upper and lower bounds for $\dim_{\rm P} X([0,1]^N)$ and $\dim_{\rm P} {\rm Gr} X\big([0,1]^N\big)$ separately. The upper bounds are proved by using the modulus of continuity of X and a covering argument, and the proof of the lower bounds is based on the potential-theoretic approach to packing dimension [see (2.16)] of finite Borel measures.

We will make use of the following lemmas. Lemma 3.3 is reminiscent to Corollary 2.3 or Theorem 2.10 in Dudley (1973). It can be proved by using the Gaussian isoperimetric inequality.

Lemma 3.3 *Assume the real-valued Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ satisfies Conditions (C) and $0 < \alpha_* \le \alpha^* \le 1$. Let*

$$\omega_{X_0}(\delta) = \sup_{\substack{t,\ t+s \in [0,1]^N \\ \|s\| \le \delta}} |X_0(t+s) - X_0(t)|$$

be the uniform modulus of continuity of $X_0(t)$ on $[0,1]^N$. Then there exists a finite constant $K > 0$ such that

$$\limsup_{\delta \to 0} \frac{\omega_{X_0}(\delta)}{\phi(\delta)\sqrt{\log \frac{1}{\delta}}} \le K, \qquad \text{a.s.} \tag{3.5}$$

For any Borel measure μ on $\mathbb{R}^N$, the image measure of μ under the mapping $t \mapsto f(t)$ is defined by

$$\big(\mu \circ f^{-1}\big)(B) := \mu\big\{t \in \mathbb{R}^N : \; f(t) \in B\big\} \quad \text{for all Borel sets } \; B \subset \mathbb{R}^d.$$

The following lemma was proved in Xiao (1997), which relates $\dim_{\rm P} f(E)$ with the packing dimensions of the images measures.

Lemma 3.4 *Let $E \subset \mathbb{R}^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \to \mathbb{R}^d$*

$$\dim_{\rm P} f(E) = \sup\big\{\dim_{\rm P}\big(\mu \circ f^{-1}\big) : \; \mu \in \mathcal{M}_c^+(E)\big\}. \tag{3.6}$$

Proof of Theorem 3.1 We first prove the upper bound in (3.3). Since $\dim_{\rm P} X([0,1]^N) \le d$ a.s., it is sufficient to show that $\dim_{\rm P} X([0,1]^N) \le N/\alpha_*$ a.s. For any $\gamma < \alpha_*$, Lemma 3.3 implies that $X(t)$ satisfies almost surely a uniform Hölder condition of order γ on $[0,1]^N$. Hence a standard covering argument [cf. Falconer (1990), Kahane (1985)] shows that $\overline{\dim}_{\rm B} X([0,1]^N) \le N/\gamma$ a.s. This and (2.11) imply $\dim_{\rm P} X([0,1]^N) \le N/\gamma$ a.s. Letting $\gamma \uparrow \alpha_*$ along the sequence of rational numbers yields the desired upper bound.

Similarly, by using Lemma 3.3 and a covering argument, we can verify that for any $\gamma < \alpha_*$,

$$\overline{\dim}_{\rm B} {\rm Gr} X\big([0,1]^N\big) \le \min\left\{\frac{N}{\gamma}, \; N + (1-\gamma)d\right\}, \qquad \text{a.s.},$$

which implies the upper bound in (3.4).

Now we proceed to prove the lower bounds in (3.3). Let λ_N be the Lebesgue measure on $[0,1]^N$. By Lemma 3.4, we have $\dim_{\rm P} X([0,1]^N) \ge \dim_{\rm P}\big(\lambda_N \circ X^{-1}\big)$ almost surely. Hence it is sufficient to show that

$$\dim_{\rm P}\big(\lambda_N \circ X^{-1}\big) \ge \min\left\{d, \; \frac{N}{\alpha_*}\right\}, \qquad \text{a.s.} \tag{3.7}$$

For simplicity of notation, we will, from now on, denote the image measure $\lambda_N \circ X^{-1}$ by μ_X.

Note that, for every fixed $s \in \mathbb{R}^N$, Fubini's theorem implies

$$\begin{aligned} \mathbb{E} F_d^{\mu_X}\big(X(s), r\big) &= \mathbb{E} \int_{\mathbb{R}^d} \min\big\{1, \; r^d \|v - X(s)\|^{-d}\big\}\, d\mu_X(v) \\ &= \int_{[0,1]^N} \mathbb{E} \min\big\{1, \; r^d \|X(t) - X(s)\|^{-d}\big\}\, dt. \end{aligned} \tag{3.8}$$

The last integrand in (3.8) can be written as

$$\begin{aligned} &\mathbb{E} \min\big\{1, r^d \|X(t) - X(s)\|^{-d}\big\} \\ &= \mathbb{P}\big\{\|X(t) - X(s)\| \le r\big\} + \mathbb{E}\big\{r^d \|X(t) - X(s)\|^{-d} \cdot \mathbb{1}_{\{\|X(t)-X(s)\| \ge r\}}\big\}. \end{aligned} \tag{3.9}$$

By Condition (C), we obtain that for all $s, t \in [0,1]^N$ and $r > 0$,

$$\mathbb{P}\big\{\|X(t) - X(s)\| \le r\big\} \le K \, \min\left\{1, \; \frac{r^d}{\phi(\|t-s\|)^d}\right\}. \tag{3.10}$$

Denote the distribution of $X(t)-X(s)$ by $\Gamma_{s,t}(\cdot)$. Let ν be the image measure of $\Gamma_{s,t}(\cdot)$ under the mapping $T: z\mapsto \|z\|$ from $\mathbb{R}^d$ to $\mathbb{R}_+$. Then the second term in (3.9) can be written as

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{r^d}{\|z\|^d}\,\mathbb{1}_{\{\|z\|\ge r\}}\,\Gamma_{s,t}(dz) &= \int_r^\infty \frac{r^d}{\rho^d}\,\nu(d\rho)\\ &\le d\int_r^\infty \frac{r^d}{\rho^{d+1}}\,\mathbb{P}\big\{\|X(t)-X(s)\|\le\rho\big\}\,d\rho,\end{aligned}\tag{3.11}$$

where the last inequality follows from an integration-by-parts formula.

Hence, by (3.10) and (3.11) we derive that the second term in (3.9) can be bounded by

$$\begin{aligned}&K\,r^d\int_r^\infty \frac{1}{\rho^{d+1}}\,\min\left\{1,\left(\frac{\rho}{\phi(\|t-s\|)}\right)^d\right\}d\rho\\ &\le K\begin{cases}1 & \text{if } r\ge\phi(\|t-s\|),\\ \left(\frac{r}{\phi(\|t-s\|)}\right)^d\log\left(\frac{\phi(\|t-s\|)}{r}\right) & \text{if } r<\phi(\|t-s\|).\end{cases}\end{aligned}\tag{3.12}$$

It follows from (3.9), (3.10), (3.11) and (3.12) that for any $0<\varepsilon<1$ and $s,t\in[0,1]^N$,

$$\mathbb{E}\min\big\{1,\ r^d\|X(t)-X(s)\|^{-d}\big\}\le K\,\min\left\{1,\ \left(\frac{r}{\phi(\|t-s\|)}\right)^{d-\varepsilon}\right\}.\tag{3.13}$$

Combining (3.8) and (3.13) we derive

$$\begin{aligned}\mathbb{E}F_d^{\mu_X}\big(X(s),r\big) &\le K\int_{[0,1]^N}\min\left\{1,\ \left(\frac{r}{\phi(\|t-s\|)}\right)^{d-\varepsilon}\right\}dt\\ &\le K\int_0^1\min\left\{1,\ \left(\frac{r}{\phi(x)}\right)^{d-\varepsilon}\right\}x^{N-1}\,dx\\ &= K\left\{\int_0^{\phi^{-1}(r)}x^{N-1}\,dx+\int_{\phi^{-1}(r)}^1\left(\frac{r}{\phi(x)}\right)^{d-\varepsilon}x^{N-1}\,dx\right\}\\ &\hat{=} I_1+I_2.\end{aligned}\tag{3.14}$$

In the above, $\phi^{-1}(x)=\inf\{y:\phi(y)>x\}$ is the right-continuous inverse function of ϕ. It can be seen that ϕ^{-1} is non-decreasing and satisfies $\phi\big(\phi^{-1}(x)\big)=x$ and $\lim_{x\to0}\phi^{-1}(x)=0$.

Clearly, we have $I_1=K\,\big[\phi^{-1}(r)\big]^N$. In order to estimate I_2, we distinguish two cases. If ϕ satisfies (3.1), then for all $r>0$ small enough, we derive

$$I_2\le K\,r^{d-\varepsilon}\int_0^1\left(\frac{1}{\phi(x)}\right)^{d-\varepsilon}x^{N-1}\,dx\le K\,r^{d-\varepsilon}.\tag{3.15}$$

On the other hand, if ϕ satisfies (3.2), then we make a change of variables to derive that for all $r>0$ small enough,

$$I_2\le K\,r^{d-\varepsilon}\,\big[\phi^{-1}(r)\big]^N\int_1^{1/\phi^{-1}(r)}\frac{x^{N-1}}{\phi\big(\phi^{-1}(r)x\big)^{d-\varepsilon}}\,dx\le K\big[\phi^{-1}(r)\big]^{N-\varepsilon}.\tag{3.16}$$

It follows from the above that for all $r > 0$ small enough,

$$\mathbb{E}F_d^{\mu_X}\big(X(s), r\big) \le K\Big\{\big[\phi^{-1}(r)\big]^{N-\varepsilon} + r^{d-\varepsilon}\Big\}. \tag{3.17}$$

Now for any $0 < \gamma < \min\big\{d,\ N/\alpha_*\big\}$, we choose $\varepsilon > 0$ small such that

$$\gamma < \frac{N - 2\varepsilon}{\alpha_*} \qquad \text{and} \qquad \gamma < d - \varepsilon. \tag{3.18}$$

By the first inequality in (3.18), we see that there exists a sequence $\rho_n \to 0$ such that

$$\phi(\rho_n) \ge \rho_n^{(N-2\varepsilon)/\gamma} \qquad \text{for all integers } \ n \ge 1. \tag{3.19}$$

We choose a sequence $\{r_n, n \ge 1\}$ of positive numbers such that $\phi^{-1}(r_n) = \rho_n$. Then $\phi(\rho_n) = r_n$ and $\lim_{n\to\infty} r_n = 0$.

By Fatou's lemma and (3.17) we obtain that for every $s \in [0,1]^N$,

$$\begin{aligned}\mathbb{E}\bigg(\liminf_{r\to 0} \frac{F_d^{\mu_X}\big(X(s), r\big)}{r^\gamma}\bigg) &\le K \liminf_{n\to\infty} \frac{\big[\phi^{-1}(r_n)\big]^{N-\varepsilon} + r_n^{d-\varepsilon}}{r_n^\gamma} \\ &\le K \liminf_{n\to\infty} \bigg\{\frac{\rho_n^{N-\varepsilon}}{\phi(\rho_n)^\gamma} + \phi(\rho_n)^{d-\gamma-\varepsilon}\bigg\} = 0.\end{aligned} \tag{3.20}$$

In deriving the last equality, we have made use of (3.18) and (3.19).

By using Fubini's theorem again, we see that almost surely,

$$\liminf_{r\to 0} \frac{F_d^{\mu_X}\big(X(s), r\big)}{r^\gamma} = 0 \qquad \text{for } \lambda_N\text{-a.a. } s \in \mathbb{R}^N.$$

This and (2.16) together imply $\dim_{\rm P} \mu_X \ge \gamma$ almost surely. Since γ can be arbitrarily close to $\min\big\{d,\ N/\alpha_*\big\}$, we have proved (3.7) and, consequently, the lower bound in (3.3).

Finally, we prove the lower bound in (3.4). Denote by ν_X the image measure of λ_N under the mapping $t \mapsto (t, X(t))$. Then ν_X is a Borel probability measure on $\mathbb{R}^{N+d}$ with support in ${\rm Gr}X\big([0,1]^N\big)$. Because of Lemma 3.4, it is sufficient to show that for all γ satisfying

$$0 < \gamma < \min\bigg\{\frac{N}{\alpha_*},\ N + (1-\alpha_*)d\bigg\}, \tag{3.21}$$

we have $\dim_{\rm P} \nu_X \ge \gamma$ almost surely.

For this purpose, let us fix an $s \in [0,1]^N$ and consider

$$\mathbb{E}F_{d+N}^{\nu_X}\Big(\big(s, X(s)\big), r\Big) = \int_{[0,1]^N} \mathbb{E}\min\bigg\{1,\ \frac{r^{d+N}}{(\|t-s\| + \|X(t)-X(s)\|)^{d+N}}\bigg\}\, dt. \tag{3.22}$$

We split the last integral over the regions $\{t \in [0,1]^N : \|t-s\| \le r\}$ and $\{t \in [0,1]^N : \|t-s\| > r\}$. Then, in the first region, we can bound the integrand in (3.22) as follows:

$$\begin{aligned}\mathbb{E}\min\bigg\{1,\ \frac{r^{d+N}}{\big(\|t-s\| + \|X(t)-X(s)\|\big)^{d+N}}\bigg\} &\le \mathbb{P}\Big\{\|X(t)-X(s)\| \le r\Big\} \\ &\quad + \mathbb{E}\bigg(\frac{r^{d+N}}{\big(\|t-s\| + \|X(t)-X(s)\|\big)^{d+N}}\, \mathbb{1}_{\{\|X(t)-X(s)\|>r\}}\bigg).\end{aligned} \tag{3.23}$$

The first term is bounded in (3.10). Some simple computation shows that the second term in (3.23) is bounded from above by $K \min\{1, (\frac{r}{\phi(\|t-s\|)})^d\}$. Hence for all $t \in [0,1]^N$ with $\|t-s\| \le r$,

$$\mathbb{E} \min\left\{1, \frac{r^{d+N}}{\big(\|t-s\| + \|X(t)-X(s)\|\big)^{d+N}}\right\} \le K \min\left\{1, \left(\frac{r}{\phi(\|t-s\|)}\right)^d\right\}. \tag{3.24}$$

On the region $\{t \in [0,1]^N : \|t-s\| > r\}$ we have

$$\begin{aligned} \mathbb{E} \min\left\{1, \frac{r^{d+N}}{\big(\|t-s\| + \|X(t)-X(s)\|\big)^{d+N}}\right\} &= \mathbb{E}\left(\frac{r^{d+N}}{\big(\|t-s\| + \|X(t)-X(s)\|\big)^{d+N}}\right) \\ &\le K \frac{r^{d+N}}{\|t-s\|^N \phi(\|t-s\|)^d}, \end{aligned} \tag{3.25}$$

where, in deriving the last inequality, we have used the following verifiable fact: If Ξ is a standard normal vector in $\mathbb{R}^d$ and $a \in \mathbb{R}$, then for all $\beta > d$,

$$\mathbb{E}\left[\frac{1}{\big(a^2 + \|\Xi\|^2\big)^{\beta/2}}\right] \le K\, a^{-(\beta-d)}.$$

See, e.g. Kahane (1985, p.279).

Combining (3.22), (3.24), (3.25) and making a change of variables, we obtain

$$\begin{aligned} \mathbb{E} F_{d+N}^{\nu_X}\Big(\big(s, X(s)\big), r\Big) &\le K \int_0^r \min\left\{1, \left(\frac{r}{\phi(x)}\right)^d\right\} x^{N-1}\, dx + K\, r^{d+N} \int_r^1 \frac{dx}{x\phi(x)^d} \\ &\hat{=} J_1 + J_2. \end{aligned} \tag{3.26}$$

In order to bound J_1, once again we distinguish the two cases separately. If (3.1) holds, then similar to (3.15) we can verify that for $\varepsilon > 0$ and $r > 0$ small,

$$\begin{aligned} J_1 &\le K \left[\phi^{-1}(r)\right]^N + K \frac{r^{d+N}}{\phi(r)^d} \int_0^1 \left(\frac{\phi(r)}{\phi(rx)}\right)^d x^{N-1}\, dx \\ &\le K\left(\left[\phi^{-1}(r)\right]^N + \frac{r^{d+N-\varepsilon}}{\phi(r)^d}\right). \end{aligned} \tag{3.27}$$

On the other hand, if (3.2) holds, then we make a change of variables to get

$$J_1 \le K \left[\phi^{-1}(r)\right]^N \int_1^{1/\phi^{-1}(r)} \left(\frac{\phi\big(\phi^{-1}(r)\big)}{\phi\big(\phi^{-1}(r)x\big)}\right)^d x^{N-1}\, dx \le K \left[\phi^{-1}(r)\right]^{N-\varepsilon}. \tag{3.28}$$

Next, we note that ϕ is non-decreasing and this yields

$$J_2 = K\, r^{d+N} \int_r^1 \frac{dx}{x\phi(x)^d} \le K\, r^{d+N} \frac{\log 1/r}{\phi(r)^d}. \tag{3.29}$$

Combining (3.26)–(3.29), we obtain

$$\mathbb{E} F_{d+N}^{\nu_X}\Big(\big(s, X(s)\big), r\Big) \le K \left(\left[\phi^{-1}(r)\right]^{N-\varepsilon} + \frac{r^{d+N-\varepsilon}}{\phi(r)^d}\right). \tag{3.30}$$

Because of (3.30), we can use the same argument as in the proof of (3.7) to show that, for all γ satisfying (3.21), the following holds almost surely:

$$\liminf_{r\to 0} \frac{F^{\nu_X}_{d+N}\big((s, X(s)), r\big)}{r^\gamma} = 0 \quad \text{ for } \lambda_N\text{-a.a. } s \in \mathbb{R}^N.$$

Hence we have $\dim_{\rm P} \nu_X \ge \gamma$ almost surely. This finishes the proof of Theorem 3.1. □

We conclude this paper with some comments and open questions.

Remark 3.5 In general, the problems of determining the Hausdorff and packing dimensions of $X(E)$, where $E \subseteq \mathbb{R}^N$ is a Borel set, remain open. It seems possible to establish a zero-one law for the β-dimensional Hausdorff and packing measures of $X(E)$ and prove that there exist two constants $c_1(E, X)$ and $c_2(E, X)$ such that $\dim_{\rm H} X(E) = c_1(E, X)$ and $\dim_{\rm P} X(E) = c_2(E, X)$ almost surely. It is more difficult to identify the constants $c_1(E, X)$ and $c_2(E, X)$, which may require more information on the geometry of E than its packing dimension or Hausdorff dimension. In this regard, the results of Khoshnevisan and Xiao (2005) for Lévy processes may be instructive. □

Remark 3.6 In this paper we have only considered Gaussian random fields which are approximately isotropic in both time and space-variables. It would be interesting to determine the packing dimensions of the range and graph of anisotropic Gaussian random fields [cf. Xiao (2007b)]. Since the packing dimension profiles of Falconer and Howroyd (1997) relies on isotropy, new geometric tools will have to be developed. These problems are currently under investigation and will appear elsewhere. □

Remark 3.7 As we mentioned in the Introduction, a general Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ satisfying Condition (C) with different upper and lower indices may exhibit non-trivial multifractal structures. I believe that almost surely the local regularity of $X_0(t)$ is determined by the asymptotic properties of ϕ at 0, for all $t \in \mathbb{R}^N$. Since $\phi(r)$ oscillates between two power functions as $r \to 0$, it would be more appropriate to introduce the notion of *upper* and *lower local Hölder exponents* for X_0 and to determine the corresponding singularity spectra. The second question is to determine the Hausdorff and packing dimensions of the set of λ-fast points defined by

$$F(\lambda) = \left\{ t \in [0,1]^N : \limsup_{\|h\|\to 0} \frac{|X_0(t+h) - X_0(t)|}{\phi(\|h\|)\sqrt{\log\big(1/\|h\|\big)}} \ge \lambda \right\}$$

at least for a subclass of Gaussian random fields with stationary increments and different upper and lower indices. It may be possible to modify and extend the general results in Khoshnevisan, Peres and Xiao (2000) on limsup random fractals to solve this problem.

Finally, we remark that Seuret (2008a, 2008b) has recently characterized an interesting class of continuous multifractal functions, each of which is the composition of a monofractal (such as a fractional Brownian motion) with a time subordination. In light of this work, it would be interesting, even in the special case of $N = 1$, to study the multifractal structure of functions obtained by composing a general Gaussian process X_0 with a time subordination. □

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Yimin Xiao: Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.
E-mail: `xiao@stt.msu.edu`
URL: http://www.stt.msu.edu/~xiaoyimi/