# The Hausdorff Dimension of the Level Sets of Stable Processes in Random Scenery*

Yimin Xiao †

**Abstract**

Let $X(t)$ $(t \in \mathbf{R}_+)$ be a stable process in a random scenery. The Hausdorff dimension of certain level sets is determined and the existence of the local time of $X(t)$ is proved.

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## 1 Introduction

Let $B_+ = \{B_+(t): \ t \geq 0\}$ and $B_- = \{B_-(t): \ t \geq 0\}$ denote two standard Brownian motions in $\mathbf{R}$ and define a two-sided Brownian motion $B = \{B(t), \ t \geq 0\}$ by

$$B(t) = \begin{cases} B_+(t) & \text{if } t \geq 0 \\ B_-(-t) & \text{if } t < 0\,. \end{cases}$$

Let $Z = \{Z(t): \ t \geq 0\}$ be a strictly stable Lévy process of index $\beta \in (1, 2]$ in $\mathbf{R}$ with characteristic function

$$E\left(\exp(i\xi Z(t))\right) = \exp\Big(-t|\xi|^\beta \frac{1 + i\nu sgn(\xi)\tan(\beta\pi/2)}{\chi}\Big),$$

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where $\chi$ and $\nu$ are real parameters such that $\chi > 0$ and $-1 \le \nu \le 1$. We assume that $B_+$, $B_-$ and $Z$ are defined on a common probability space and that they are mutually independent. Given a function $f : \mathbf{R} \to \mathbf{R}$, we let

$$\int_{\mathbf{R}} f(x) dB(x) = \int_0^\infty f(x) dB_+(x) + \int_0^\infty f(-x) dB_-(x)$$

provided that both of the Itô integrals on the right-hand side are defined.

Let $L = \{L_t^x : \ t \ge 0, x \in \mathbf{R}\}$ be the local time of $Z$. It is well known that (cf. Boylan [1]) $L$ has a version which is jointly continuous in $(t, x)$. Throughout this paper, we assume that $L$ is such a version. For each $t \ge 0$, let

$$X(t) = \int_{\mathbf{R}} L_t^x dB(x) \, .$$

Then the process $X = \{X(t) : \ t \ge 0\}$ is well defined. It is easy to verify that $X$ is self-similar with index

$$\alpha = 1 - \frac{1}{2\beta} \, ,$$

that is, for every $a > 0$, $\{X(at) : \ t \ge 0\} \stackrel{\mathrm{d}}{=} \{a^\alpha X(t) : \ t \ge 0\}$, where "$\stackrel{\mathrm{d}}{=}$" means the two processes have the same distribution, and $X$ has stationary increments. This class of self-similar processes was first introduced and studied by Kesten and Spitzer [9]. See also Lou [13]. Since each of these processes can be realized as the limit in distribution of a random walk in random scenery, following Khoshnevisan and Lewis [10], we will call them stable processes in random scenery. If $\beta = \chi = 2$, then $Z$ is a standard Brownian motion and $X$ is called a Brownian motion in random scenery. For each $t \ge 0$, let $Y(t) = B(Z(t))$. Then $Y = \{Y(t), \ t \ge 0\}$ is an iterated Brownian motion, which has received a lot of attention in the past several years; we refer to Khoshnevisan and Lewis [11] for an extensive list of references. Recently, Khoshnevisan and Lewis [12] have related the quadratic and quartic variations of iterated Brownian motion to Brownian motion in random scenery, which led them to expect a certain duality between iterated Brownian motion and Brownian motion in random scenery. This was supported by the results in Khoshnevisan and Lewis [10], where a law of the iterated logarithm for stable processes in random scenery was proved. Khoshnevisan and Lewis [11], motivated by "duality", conjectured that for Brownian motion in random scenery

$$\dim X^{-1}(0) = \frac{1}{4}\,, \tag{1.1}$$

where $X^{-1}(0) = \{s \geq 0 : \; X(s) = 0\}$ is the zero set of $X$ and dim denotes Hausdorff dimension. We refer to Falconer [6] for the definition and properties of Hausdorff measure and Hausdorff dimension.

The objective of the present paper is to study the Hausdorff dimension of the level sets of stable processes in random scenery. For each $t \geq 0$ fixed, consider the level set

$$M_t = \{s \geq 0 : \; X(s) = X(t)\}\,.$$

Here is the main result.

**Theorem 1.1** *Let $X = \{X(t),\; t \geq 0\}$ be a stable process in random scenery. Then for every $t > 0$,*

$$\dim M_t = \frac{1}{2\beta} \qquad \text{a.s.} \tag{1.2}$$

Remark. Our result is not about the Hausdorff dimension of the level set at a fixed level, so it does not prove (1.1) in the case of Brownian motion in random scenery. But the proof of Theorem 1.1 does show that the right hand side of (1.2) serves as an upper bound for $\dim X^{-1}(x)$, where $x \in \mathbf{R}$ is fixed. We believe that a result similar to (1.2) also holds for $\dim X^{-1}(x)$.

The rest of this paper is organized as follows. In Section 2 we establish an upper bound for the uniform modulus of continuity and prove the existence of square-integrable local times of $X$. The proof of Theorem 1.1 is given in Section 3. It is evident that our arguments rely heavily on the results of Khoshnevisan and Lewis [10] and Berman [4].

# 2 Modulus of Continuity and Local Times

**Lemma 2.1** (Khoshnevisan and Lewis [10]) *There exists a positive number $\gamma = \gamma(\beta)$ such that*

$$\lim_{\lambda \to \infty} \lambda^{-\frac{2\beta}{1+\beta}} \log P(X(1) \geq \lambda) = -\gamma\,.$$

The following result on the uniform modulus of continuity is an easy consequence of Lemma 2.1 and the first part of Lévy's uniform modulus of continuity argument for Brownian motion. Discussions for general, not necessarily Gaussian, Banach space valued processes can be found in Csáki and Csörgő [5].

**Proposition 2.1** *For any $-\infty < T_1 < T_2 < \infty$, we have*

$$\limsup_{h\to 0} \sup_{T_1\le t\le T_2-h} \sup_{0\le s\le h} \frac{|X(t+s)-X(t)|}{h^\alpha(\log(1/h))^{\frac{1+\beta}{2\beta}}} \le \Big(\frac{1}{\gamma}\Big)^{\frac{1+\beta}{2\beta}} \quad \text{a.s.} \tag{2.1}$$

**Proof.** Since $X = \{X(t),\ t\ge 0\}$ is $\alpha$-self-similar with stationary increments, it follows from Lemma 2.1 that for every $\epsilon > 0$, there exists $\lambda_0 > 0$ such that for every $t \ge 0$, $h > 0$ and $\lambda \ge \lambda_0$

$$P\Big(|X(t+h)-X(t)| \ge h^\alpha\lambda\Big) \le \exp\Big(-(\gamma-\epsilon)\lambda^{\frac{2\beta}{1+\beta}}\Big). \tag{2.2}$$

Hence (2.1) follows from (2.2) and a chaining argument. See Theorem 3.1 of Csáki and Csörgő [5].

Remark. An open problem is to find the uniform modulus of continuity for $X$. We do not know whether the constant in (2.1) is the right one.

In order to study the existence of the local times of $X$, we need some information about the density function of $X(t)$ relative to Lebesgue measure. For each $t\ge 0$, let

$$V_t = \int_{\mathbf{R}} (L_t^x)^2 dx\,.$$

Then $V_t$ is the conditional variance of $X(t)$ given $\sigma(Z(s):\ 0\le s\le t)$ and it is also self-similar: for every $a > 0$,

$$\{V_{at}:\ t\ge 0\} \overset{\mathrm{d}}{=} \{a^{2\alpha}V_t:\ t\ge 0\}\,. \tag{2.3}$$

As observed by Khoshnevisan and Lewis [10], there exist two positive constants $K_1$ and $K_2$, depending on $\beta$ only, such that for each $\lambda \ge 0$,

$$\exp(-K_1\lambda^\beta) \le P(V_1 > \lambda) \le \exp(-K_2\lambda^\beta)\,. \tag{2.4}$$

It follows from Fubini's theorem and (2.4) that

$$E\Big(\frac{1}{\sqrt{V_1}}\Big) = \frac{1}{2}\int_0^\infty x^{-\frac{3}{2}} P(V_1 \le x)dx = K_3 < \infty\,. \tag{2.5}$$

**Lemma 2.2** *For every $t > 0$, $X(t)$ has a bounded continuous density function $p_t(y)$ and there exists a positive constant $r_0$ such that for every $0 < r \le r_0$ and every $x \in \mathbf{R}$,*

$$K_4 \min\{1, r\} \le P(|X(1) - x| \le r) \le K_5 \min\{1, r\}\,, \tag{2.6}$$

*where $K_4$ is a positive constant depending on $x$ and $\beta$ only and $K_5$ is a positive constant depending on $\beta$ only.*

**Proof.** For each $t > 0$, consider the characteristic function of $X(t)$:

$$E\exp(iuX(t)) = E\exp\Big(-\frac{u^2}{2}\int_{\mathbf{R}}(L_t^x)^2dx\Big)\,, \quad u \in \mathbf{R}.$$

By Fubini's theorem, (2.3) and (2.5),

$$\int_{\mathbf{R}} |E\exp(iuX(t))|\,du = E\Big(\sqrt{\frac{2\pi}{\int_{\mathbf{R}}(L_t^x)^2dx}}\Big) = \frac{\sqrt{2\pi}K_3}{t^\alpha}\,.$$

It follows that $X(t)$ has a bounded continuous density function $p_t(y)$ and, by the inversion formula for Fourier transformations,

$$\begin{aligned} p_t(y) &= E\Big[\frac{1}{\sqrt{2\pi\int_{\mathbf{R}}(L_t^x)^2dx}}\exp\Big(-\frac{y^2}{2\int_{\mathbf{R}}(L_t^x)^2dx}\Big)\Big] \\ &= \frac{1}{\sqrt{2\pi}t^\alpha}E\Big[\frac{1}{\sqrt{V_1}}\ \exp\Big(-\frac{y^2}{2t^{2\alpha}V_1}\Big)\Big]\,. \end{aligned} \tag{2.7}$$

Finally, (2.6) follows from (2.7) and (2.5) directly.

We recall briefly the definition of local time. For a comprehensive survey on the study of local times of both random and non-random vector fields before the early 80's, we refer to Geman and Horowitz [7]. For some new developments on the local times of Gaussian random fields and related processes, see Xiao [15]. Let $X(t)$ be any Borel function on $\mathbf{R}_+$ with values in $\mathbf{R}$. For any Borel set $I \subset \mathbf{R}_+$, the occupation measure of $X$ is defined by

$$\mu_I(A) = \lambda_1\{t \in I:\ X(t) \in A\}$$

for every Borel set $A \subset \mathbf{R}$, where $\lambda_1$ is the one-dimensional Lebesgue measure. If $\mu_I$ is absolutely continuous with respect to Lebesgue measure, we say that $X(t)$ has a local time on $I$ and define its local time $\phi(x, I)$ to be the Radon-Nikodym derivative of $\mu_I$. If

$I = [0, t]$, we will simply write $\phi(x, I)$ as $\phi(x, t)$. It is known that local time satisfies the following occupation density formula: for every Borel set $I \subset \mathbf{R}_+$ and every measurable function $f : \mathbf{R} \to \mathbf{R}$,

$$\int_I f(X(t))dt = \int_{\mathbf{R}} f(x)\phi(x, I)dx \,. \tag{2.8}$$

**Proposition 2.2** *For each finite interval $I \subset [0, \infty)$, $X(t)$ has a local time $\phi(x, I)$ on $I$ satisfying*

$$\int_{\mathbf{R}} \phi(x, I)^2 dx < \infty \qquad \text{a.s.} \tag{2.9}$$

**Proof.** By the fact that $X$ is $\alpha$-self-similar and has stationary increments, and (2.6), we have for every $\epsilon > 0$,

$$\begin{aligned} \frac{1}{2\epsilon} \int_I \int_I P(|X(t) - X(s)| \le \epsilon) ds dt &= \frac{1}{2\epsilon} \int_I \int_I P\Big(|X(1)| \le \frac{\epsilon}{|t - s|^\alpha}\Big) dt ds \\ &\le K_5 \int_I \int_I \frac{1}{|t - s|^\alpha} dt ds = K_6 < \infty \,. \end{aligned}$$

Hence

$$\liminf_{\epsilon \to 0} \frac{1}{2\epsilon} \int_I \int_I P(|X(t) - X(s)| \le \epsilon) ds dt \le K_6 < \infty \,.$$

By Theorem 21.15 in Geman and Horowitz [7], $X$ has a local time $\phi(x, I)$ on $I$ such that (2.9) is satisfied.

Remark. Proposition 2.2 can also be proved by using Fourier analysis as in Berman [2] and Kahane [8]. We notice that for each fixed Borel set $I$, the Fourier transform of the local time $\phi(x, I)$ is

$$\hat{\phi}_I(u) = \int_I \exp(iuX(t)) dt \,. \tag{2.10}$$

# 3 Proof of Theorem 1.1

In order to prove the lower bound in (1.2), we need to construct a random Borel measure with support in $M_t$ and use a capacity argument. The following lemma, which is a combination of Proposition 2.2 and the results of Berman ([3], [4]), asserts that a version of the local time of $X$ will meet our requirements.

**Lemma 3.1** *There exists a version of the local time $\phi(x,t)$, still denoted by $\phi(x,t)$, such that*

*(i) For each $t > 0$, $\phi(x,t)$ is square-integrable in $x$; and for each $x \in \mathbf{R}$, $\phi(x,\cdot)$ is a measure on the Borel subsets of $[0,\infty)$;*

*(ii) For each $x \in \mathbf{R}$, the measure $\phi(x,\cdot)$ has its support contained in $X^{-1}(x)$ or else is equal to 0;*

*(iii) $\phi(X(t), I) > 0$ for almost all $t \in I$, almost surely.*

**Proof.** Statement (i) is a consequence of Proposition 2.2, Lemmas 1.1 and 1.4 in Berman [4]. (ii) follows from Lemma 1.5 of Berman [4] and (iii) follows from (i) and Lemma 1.1 of Berman [3] respectively.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We first prove that for every $x \in \mathbf{R}$,

$$\dim X^{-1}(x) \leq \frac{1}{2\beta} \qquad \text{a.s.}$$

By the $\sigma$-stability of Hausdorff dimension, it is sufficient to show that for every interval $I = [\epsilon, M]$, with $\epsilon > 0$,

$$\dim(X^{-1}(x) \cap I) \leq \frac{1}{2\beta} \qquad \text{a.s.} \tag{3.1}$$

The proof of (3.1) is very similar to the first part of the proof of Theorem 2.1 of Berman [4]. Even though the stable process in random scenery considered here is not a Gaussian process, the two main ingredients needed in Berman's proof are provided by Proposition 2.1 and Lemma 2.2. Hence we omit the details. It follows from (3.1) and Fubini's theorem that

$$\dim X^{-1}(x) \leq \frac{1}{2\beta} \quad \text{for almost all } x \quad \text{a.s.} \tag{3.2}$$

Now since $X(t)$ has a local time $\phi(x, I)$, by (3.2) and the occupation density formula (2.8) we have

$$\dim M_t \leq \frac{1}{2\beta} \quad \text{for almost all } t \quad \text{a.s.} \tag{3.3}$$

To prove the reverse inequality, we adapt the argument of Berman [4]. Fix a finite interval $I \subset [0,\infty)$. For simplicity, let $I = [0,1]$. Put

$$H(s,t) = \int_{\mathbf{R}} \phi(x,s)\phi(x,t)dx\,.$$

Using (2.10) and Parseval's identity, we can write

$$\begin{aligned} H(s,t) &= \int_{\mathbf{R}} \hat{\phi}_{[0,s]}(u)\overline{\hat{\phi}_{[0,t]}(u)}\;du \\ &= \int_{\mathbf{R}}\int_0^s\int_0^t \exp\Big(iu(X(s')-X(t'))\Big)\;ds'dt'du\,. \end{aligned}$$

It follows from a standard approximation that for any nonnegative Borel function $g(s,t)$,

$$\int_I\int_I g(s,t)H(ds,dt) = \int_I\int_I\int_{\mathbf{R}} g(s,t)\exp\Big(iu(X(s)-X(t))\Big)dudsdt\,.$$

Hence by Fubini's theorem for every $0<\eta<1/(2\beta)$ we have

$$\begin{aligned} E\Big(\int_I\int_I \frac{1}{|s-t|^\eta}H(ds,dt)\Big) &= \int_I\int_I\int_{\mathbf{R}} \frac{1}{|s-t|^\eta}E\exp\Big(iu(X(s)-X(t))\Big)dudsdt \\ &= \int_0^1\int_s^1\int_{\mathbf{R}} \frac{1}{|s-t|^\eta}E\exp\Big(-\frac{u^2}{2}\int_{\mathbf{R}}(L_t^x-L_s^x)^2dx\Big)dudsdt \\ &= \sqrt{2\pi}\int_0^1\int_s^1 \frac{1}{|s-t|^\eta}E\Big(\frac{1}{\sqrt{\int_{\mathbf{R}}(L_t^x-L_s^x)^2dx}}\Big)dsdt\,. \qquad (3.4) \end{aligned}$$

Now by using the facts that

$$L_t^x - L_s^x = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_s^t 1_{\{|Z(s')-x|\le\epsilon\}}ds' \qquad \text{a.s.}$$

and $Z$ is $1/\beta$-self-similar with stationary increments, for every fixed pair $0\le s<t$,

$$\int_{\mathbf{R}}(L_t^x-L_s^x)^2dx \stackrel{\mathrm{d}}{=} (t-s)^{2-\frac{1}{\beta}}V_1\,. \qquad (3.5)$$

It follows from (3.5) and (2.5) that (3.4) is at most

$$K_3\sqrt{2\pi}\int_0^1\int_s^1 \frac{1}{|s-t|^{\eta+1-1/(2\beta)}}\;dsdt = K(\eta,\beta)<\infty\,,$$

where $K(\eta,\beta)>0$ is a constant depending on $\eta$ and $\beta$ only. Hence by Fubini's theorem

$$\int_I\int_I \frac{1}{|s-t|^\eta}\;\phi(x,ds)\phi(x,dt)<\infty$$

for almost all $x \in \mathbf{R}$, almost surely. This implies that almost surely

$$\int_I \int_I \frac{1}{|u-v|^\eta} \phi(X(t), du)\phi(X(t), dv) < \infty \quad \text{for almost all} \quad t \in I\,. \tag{3.6}$$

By Lemma 3.1 (i) and (iii), we see that the measure $\phi(X(t), \cdot)$ is a positive measure on $I$ for almost all $t \in I$, almost surely. It follows from Lemma 3.1 (ii) that the random measure $\phi(X(t), \cdot)$ is supported on $M_t$. Hence by (3.6) and Frostman's theorem (see e.g. Kahane [8], p. 133) we have almost surely

$$\dim M_t \geq \frac{1}{2\beta} \quad \text{for almost all} \quad t \in I\,. \tag{3.7}$$

Since $\mathbf{R}_+$ is a countable union of finite intervals, (3.7) holds for almost all $t \geq 0$, almost surely. Combining this with (3.3), we have

$$\dim M_t = \frac{1}{2\beta} \quad \text{for almost all} \quad t \geq 0, \quad \text{almost surely}\;.$$

Finally by the self-similarity of $X$, the distribution of $\dim M_t$ does not depend on $t > 0$. This completes the proof of Theorem 1.1.

**Comments**. This paper raises many open questions about stable processes in random scenery. The most obvious one is to prove that for every $x \in \mathbf{R}$

$$\dim X^{-1}(x) = \frac{1}{2\beta} \qquad \text{a.s.}$$

The upper bound has already been established in the proof of Theorem 1.1. The lower bound would follow from a capacity argument similar to those of Kahane [8] and Marcus [14] if we had the following inequality: there exists a positive constant $K > 0$ such that for every $s < t$ and $r > 0$ small enough

$$P\Big(|X(s) - x| \leq |s|^\alpha r,\ |X(t) - X(s)| \leq 2|t-s|^\alpha r\Big) \leq K r^2\,.$$

It seems that this can not be derived from the fact that $X(s)$ and $X(t) - X(s)$ are quasi-associated as proved by Khoshnevisan and Lewis [10]. Another problem is to study the joint continuity of the local time $\phi(x,t)$ and the Hölder conditions in the set variable. Results of the later type will shed light on the exact Hausdorff measure of the level sets of $X$.

More generally it would also be interesting to study sample path properties of the stable process $X$ in a stable scenery defined by

$$X(t) = \int_{\mathbf{R}} L_t^x dU(x)\,,$$

where $U = \{U(x),\ -\infty < x < \infty\}$ is a strictly stable Lévy process of index $\gamma \in (0, 2]$ and $L = \{L_t^x :\ t \geq 0, x \in \mathbf{R}\}$ is the local time of $Z$ as in the introduction, and $U$ and $Z$ are independent. Then $X$ is a self-similar process with index

$$\alpha = 1 - \frac{1}{\beta} + \frac{1}{\gamma\beta}$$

and has stationary increments. For details see Kesten and Spitzer [9]. The method of the present paper can be applied to prove the existence of local time and the lower bound for the Hausdorff dimension of $M_t$ for stable processes in stable scenery. However our method for proving the upper bound for $\dim M_t$ does not work, because we have not been able to produce a desired modulus of continuity for $X$. In fact for stable processes in stable scenery, neither local nor uniform moduli are known.

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# References

[1] E. S. Boylan, Local times for a class of Markov processes, *Illinois J. Math.* **8** (1964), 19 - 39.

[2] S. M. Berman, Local times and sample function properties of stationary Gaussian processes, *Trans. Amer. Math. Soc.* **137** (1969), 277 - 299.

[3] S. M. Berman, Harmonic analysis of local times and sample functions of Gaussian processes, *Trans. Amer. Math. Soc.* **143** (1969), 269 - 281.

[4] S. M. Berman, Gaussian processes with stationary increments: local times and sample function properties, *Ann. Math. Statist.* **41** (1970), 1260 - 1272.

[5] E. Csáki and M. Csörgő, Inequalities for increments of stochastic processes and moduli of continuity, *Ann. Probab.* **20** (1992), 1031 - 1052.

[6] K. J. Falconer, *Fractal Geometry – Mathematical Foundations And Applications,* John Wiley & Sons, Chichester, 1990.

[7] D. Geman and J. Horowitz, Occupation densities, *Ann. Probab.* **8** (1980), 1 - 67.

[8] J-P. Kahane, *Some Random Series of Functions*, 2nd edition, Cambridge University Press, 1985.

[9] H. Kesten and F. Spitzer, A limit theorem related to a new class of self-similar processes, *Z. Wahrsch. verw. Gebiete* **50** (1979), 327 - 340.

[10] D. Khoshnevisan and T. M. Lewis, A law of the iterated logarithm for stable processes in random scenery, *Stoch. Proc. Appl.* **74** (1998), 89 - 122.

[11] D. Khoshnevisan and T. M. Lewis, Iterated Brownian motion and its intrinsic skeletal structure, Preprint, 1997.

[12] D. Khoshnevisan and T. M. Lewis, Stochastic calculus for Brownian motion on a Brownian fracture, *Ann. Applied Probab.* to appear.

[13] J. H. Lou, Some properties of a special class of self-similar processes, *Z. Wahrsch. verw. Gebiete* **68** (1985), 493 - 502.

[14] M. B. Marcus, Capacity of level sets of certain stochastic processes. *Z. Wahrsch. verw. Gebiete* **34** (1976), 279 - 284.

[15] Yimin Xiao, Fractal measures of the sets associated to Gaussian random fields, In: *Trends in Probability and Related Analysis: Proceedings of the Symposium on Analysis and Probability 1996* (N. Kôno and N-R. Shieh, Editors), pp. 311 - 324, World Scientific, 1997.

Yimin Xiao, Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, U. S. A. E-mail address: xiao@math.utah.edu