Fractional Brownian Motion and Packing Dimension *

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Abstract

Let $X(t)$ $(t \in \mathbf{R}^N)$ be a fractional Brownian motion of index α in $\mathbf{R}^d$. For any compact set $E \subseteq \mathbf{R}^N$, we compute the packing dimension of $X(E)$.

Running head: Talagrand and Xiao, Fractional Brownian motion and Packing dimension.

AMS Classification numbers: Primary 60G15, 60G17.

KEY WORDS: Packing dimension, fractional Brownin motion, image .

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* *J. Theoret. Probab.* **9** (1996), 579 - 593.

† Partially supported by an NSF grant.

1 Introduction

Since packing dimension was introduced in the early 1980's, it has become a very useful tool in analyzing fractal sets and in studying sample path properties of stochastic preocesses (see [ST] [TT1] [TT2] [T1] [T2] [Tr] and references therein).

Let $X(t) = (X_1(t), \cdots, X_d(t))$ $(t \in \mathbf{R}^N)$ be a fractional Brownian motion in $\mathbf{R}^d$ of index α $(0 < \alpha < 1)$ (see [K], chapter 18). If $N = 1, \quad \alpha = \frac{1}{2}$, then $X(t)$ $(t \in \mathbf{R})$ is the ordinary Brownian motion in $\mathbf{R}^d$. If $N > 1, \ \alpha = \frac{1}{2}$, then $X(t)$ $(t \in \mathbf{R}^N)$ is Levy's Brownian motion with N parameters.

In [K], Kahane proved that for every Borel set $E \subseteq \mathbf{R}^N$,

$$\dim X(E) = \min(d, \quad \frac{1}{\alpha}\dim E) \quad a.\ s. \tag{1.1}$$

where $\dim E$ is the Hausdorff dimension of E. An excellent reference for Hausdorff measure and Hausdorff dimension is [F].

It is natural to ask whether an equality similar to (1.1) holds for packing dimension Dim. Since dim $\leq$ Dim ([TT1] [ST]), it is easy to deduce from (1.1) the following result: for every Borel set $E \subseteq \mathbf{R}^N$, which is a fractal in the sense of Taylor ([T2]), i.e. $\dim E = \mathrm{Dim} E$,

$$\mathrm{Dim} X(E) = \min(d, \ \frac{1}{\alpha}\mathrm{Dim} E) \ \ a.\ s. \tag{1.2}$$

For those sets E with $\dim E \neq \mathrm{Dim} E$, (1.2) is not trivial. The first progress in this direction was made by Perkins and Taylor ([PT]). Let $X(t)$ $(t \in R^+)$ be a Brownian motion in $\mathbf{R}^d$ with $d \geq 2$, Perkins and Taylor proved that with probability 1,

$$\mathrm{Dim} X(E) = 2\mathrm{Dim} E \quad \text{for every Borel set} \quad E \subseteq R^+. \tag{1.3}$$

This result is much stronger than (1.2) since the exceptional null set does not depend on E. The proof of (1.3) relies on the strong Markov property of Brownian motion and hence can not be used in the case of fractional Brownian motion. By using an idea from [MP], which goes back to Kaufman ([Ka]), Xiao ([X]) generalised (1.3) to fractional Brownian motion: if $N \leq \alpha d$, then with probability 1,

$$\begin{aligned} \mathrm{Dim} X(E) &= \min\{d, \quad \frac{1}{\alpha}\mathrm{Dim} E\} \qquad (1.4) \\ &= \frac{1}{\alpha}\mathrm{Dim} E \quad \text{for every Borel set } E \subseteq \mathbf{R}^N. \end{aligned}$$

If $N > \alpha d$, by a result of Xiao ([X]),

$$\mathrm{Dim} X^{-1}(0) = N - \alpha d \ . \quad a.\ s.$$

Take $E = X^{-1}(0)$, it is clear that (1.4) fails. In this case, it has been open whether for every Borel set $E \subseteq \mathbf{R}^N$, the following weaker result is true

$$\mathrm{Dim} X(E) = \min(d; \quad \frac{1}{\alpha}\mathrm{Dim} E\ ) \quad a.\ s. \qquad (1.5)$$

The objective of this paper is to study $\mathrm{Dim} X(E)$. In contrast to (1.1), we show in section 3 that, if $N > \alpha d$, then (1.5) does not hold in general and $\mathrm{Dim} X(E)$ can be strictly smaller than the right-hand side of (1.5). In section 4, we give the best lower bound for $\mathrm{Dim} X(E)$ in terms of α and $\mathrm{Dim} E$.

2 Preliminaries

First we recall briefly the definitions of packing measure and packing dimension. Let Φ be the class of functions $\phi : (0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \le K, \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

For $\phi \in \Phi$, Taylor and Tricot ([TT1]) defined the set function ϕ-$P(E)$ on R^N by

$$\phi\text{-}P(E) = \lim_{\epsilon \to 0} \sup\{\sum_i \phi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \epsilon\}\ ,$$

where $B(x, r)$ denotes the open ball of radius r centered at x. Observe that ϕ-P is not an outer measure because it fails to be countably subadditive.

However, ϕ-P is a premeasure, so one can obtain a metric outer measure ϕ-p on $\mathbf{R}^N$ by defining

$$\phi\text{-}p(E) = \inf\{\sum_n \phi\text{-}P(E_n) : \quad E \subseteq \cup_n E_n\} \ .$$

ϕ-$p(E)$ is called the ϕ-packing measure of E. If $\phi(s) = s^\alpha$, s^α-$p(E)$ is sometimes called the α-dimensional packing measure of E. The packing dimension of E is defined by

$$\begin{aligned} \mathrm{Dim}E &= \inf\{\alpha > 0 : \quad s^\alpha\text{-}p(E) = 0\} \\ &= \sup\{\alpha > 0 : \quad s^\alpha\text{-}p(E) = +\infty\} \ . \end{aligned}$$

It is well known that for any $E \subseteq \mathbf{R}^N$,

$$0 \leq \dim E \leq \mathrm{Dim}E \leq N.$$

There is an equivalent definition for $\mathrm{Dim}E$ which is sometimes more convenient. For any $\epsilon > 0$ and any bounded set $E \subseteq \mathbf{R}^N$, let

$$N(E, \epsilon) = \text{smallest number of balls of radius } \epsilon \text{ needed to cover } \ E \ .$$

Define

$$\Delta(E) = \limsup_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{-\log \epsilon} \ .$$

$\Delta(E)$ is called the upper box-counting dimension of E ([F]) or the upper entropy index of E. The index Δ is not σ-stable (cf [Tr] [F]). We can obtain a σ-stable index $\widehat{\Delta}$ by letting

$$\widehat{\Delta}(E) = \inf\{\sup \Delta(E_n) : \quad E \subseteq \cup_n E_n\} \ .$$

Tricot ([Tr]) proved that $\mathrm{Dim}E = \widehat{\Delta}(E)$.

We need the following lemmas from [ST] [Tr] [TT1]. Recall from [Tr] that Δ is uniform on E if there exists a constant c such that for any $x \in \overline{E}$,

$$\lim_{r \to 0} \Delta(E \cap B(x, r)) = c \ .$$

Lemma 2.1 *([Tr]) If E is compact and Δ is uniform on E, then $\Delta(E) = \mathrm{Dim}E$.*

Let μ be a Borel measure on $\mathbf{R}^N$. For any $\phi \in \Phi$ and any $x \in \mathbf{R}^N$, the upper and lower ϕ-densities of μ at x are defined by

$$\overline{D}_\mu^\phi(x) = \limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\phi(2r)} \ ,$$

$$\underline{D}_\mu^\phi(x) = \liminf_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\phi(2r)} \ .$$

If $\phi(s) = s^\alpha$, then we write $\overline{D}_\mu^\phi$ and $\underline{D}_\mu^\phi$ as $\overline{D}_\mu^\alpha$ and $\underline{D}_\mu^\alpha$ respectively.

The following lemma is a special case of the density theorems for packing measure ([ST] [TT1]).

Lemma 2.2 *Let μ be a probability Borel measure on $\mathbf{R}^N$. Consider $E \subseteq \mathbf{R}^N$ with $\mu(E) > 0$.*

(i) If there exists a constant $c_2 > 0$ such that

$$\sup_{x\in E} \underline{D}_\mu^\alpha(x) \le c_2,$$

then $\mathrm{Dim}E \ge \alpha$.

(ii) If $\mu(E) = 1$ and there exists a constant $c_3 > 0$ such that

$$\inf_{x\in E} \underline{D}_\mu^\alpha(x) \ge c_3,$$

then $\mathrm{Dim}E \le \alpha$.

We will use $c_1, c_2, \cdots, K_1, K_2, \cdots$ to denote unspecified positive constants. They may be different from line to line.

3 Packing dimension and Hölder functions

In this section, we first construct a compact set E with $\dim E < \mathrm{Dim}E$ and then show that the packing dimension of the image of E under any Hölder function of order α is strictly smaller than $\frac{1}{\alpha}\mathrm{Dim}E$. This proves that (1.5) does not hold in general. We start with the following lemma.

Lemma 3.1 *Given $0 < \alpha < 1$ and $0 < \beta < 1$, there exist two sequences of positive numbers $\{\delta_k\}$ and $\{\eta_k\}$ satisfying*

$$\delta_{k-1} = 2\eta_k^{1-\beta}, \tag{3.1}$$

$$m_1 m_2 \ldots m_k \leq \delta_k^{-2^{-(k+1)}}, \tag{3.2}$$

where $m_k = [(\frac{1}{\eta_k})^\beta]$ and $[x]$ is the integer part of x.

Proof. We choose δ_k and η_k inductively. We take $\delta_0 \in (0, \frac{1}{2})$ and $\eta_1 = (\frac{1}{2}\delta_0)^{\frac{1}{1-\beta}}$; then we choose $0 < \delta_1 < \eta_1$ such that

$$m_1 \leq \delta_1^{-2^{-2}}.$$

Suppose that $\eta_1, \cdots, \eta_{n-1}; \delta_1, \cdots, \delta_{n-1}$ have been chosen and that they satisfy (3.1) and (3.2). We first take

$$\eta_n = \left(\frac{1}{2}\delta_{n-1}\right)^{\frac{1}{1-\beta}}$$

and then we can choose $\delta_n < \eta_n$ such that

$$m_1 m_2 \cdots m_n \leq \delta_n^{-2^{-(n+1)}}.$$

By induction, we obtain the two sequences $\{\delta_k\}$ and $\{\eta_k\}$. $\square$

Let $\{\delta_k\}, \{\eta_k\}$ and $\{m_k\}$ be the sequences in Lemma 3.1. We construct a generalized Cantor set $E \subseteq [0, 1]$ in the following way.

Let $E_0 = [0, 1]$ and let E_1 be the union of m_1 closed subintervals of $[0, 1]$ of length δ_1, which are arranged in such a way that the distance (or gap) between any two of them is at least η_1. This is possible since $m_1(\eta_1 + \delta_1) < 1$. At the second stage, each interval I_{i_1} of E_1 contains m_2 closed subintervals $I_{i_1 i_2}$ of length δ_2 with gaps at least η_2. This is possible since $m_2(\eta_2 + \delta_2) \leq \delta_1$. Let $E_2 = \cup_{i_1=1}^{m_1} \cup_{i_2=1}^{m_2} I_{i_1 i_2}$. Suppose now that E_{n-1} has been constructed, $E_{n-1} = \cup_{i_1=1}^{m_1} \cdots \cup_{i_{n-1}}^{m_{n-1}} I_{i_1 \cdots i_{n-1}}$. Since $|I_{i_1 \cdots i_{n-1}}| = \delta_{n-1}$ and by (3.1), we have $m_n(\eta_n + \delta_n) \leq \delta_{n-1}$, so we can construct m_n closed subintervals $I_{i_1 \cdots i_n}$ $(i_n = 1, \cdots, m_n)$ of length δ_n and gaps at least η_n in $I_{i_1 \cdots i_{n-1}}$. We set $E_n = \cup_{i_1=1}^{m_1} \cdots \cup_{i_n=1}^{m_n} I_{i_1 \cdots i_n}$. Continuing this process, we obtain a decreasing sequence $\{E_n\}$ of compact subsets of $[0, 1]$. Let $E = \cap_{n=1}^{\infty} E_n$; then $E \subseteq [0, 1]$ is compact.

Lemma 3.2 *Let E be the compact set constructed above. Then*

$$\dim E = 0, \qquad \operatorname{Dim} E \geq \beta.$$

If for each n and each $I_{i_1 \cdots i_{n-1}}$, the gap between each consecutive interval $I_{i_1 \cdots i_n}$ equals to η_n, then $\operatorname{Dim} E = \beta$.

Proof. For any $k \geq 1$, E can be covered by $m_1 m_2 \cdots m_k$ intervals of length δ_k. By (3.2), for any $\gamma > 0$,

$$m_1 \cdots m_k \delta_k^{\gamma} \leq \delta_k^{-2^{-(k+1)}+\gamma} \to 0, \quad \text{as } k \to \infty.$$

This implies that $\dim E \leq \gamma$. Since $\gamma > 0$ is arbitrary, we have $\dim E = 0$.

On the other hand, for any $\delta_k \leq \epsilon < \eta_k$, we have

$$N(E, \epsilon) = m_1 m_2 \cdots m_k,$$

Thus by Lemma 3.1, we have

$$\begin{aligned} \Delta(E) &\geq \limsup_{k \to \infty} \frac{\log(m_1 \cdots m_k)}{-\log \eta_k} \\ &= \limsup_{k \to \infty} \frac{\log m_1 + \cdots + \log m_k}{-\log \eta_k} \\ &= \beta \, . \end{aligned}$$

Since Δ is uniform on E, it follows from Lemma 2.1 that $\operatorname{Dim} E \geq \beta$.

To prove the second conclusion, we define a measure σ on E by distributing mass to E_n. We start with $n = 1$. For each i_1, we define $\sigma(I_{i_1}) = m_1^{-1}$. In general, for each interval $I_{i_1 \cdots i_n}$ in E_n, we define

$$\sigma(I_{i_1 \cdots i_n}) = (m_1 m_2 \cdots m_n)^{-1}$$

and $\sigma(\mathbf{R} \setminus E_n) = 0$ for $n \geq 1$. Then by the mass distribution principle ([F]), σ can be extended to a Borel measure on $\mathbf{R}$ with $\sigma(E) = 1$. For any $x \in E$ and any $r \in (0, \frac{1}{2})$, there exists an $n \geq 1$ such that $\delta_n \leq r < \delta_{n-1}$. Consider first the case $\delta_n \leq r < \eta_n$. Then $B(x, r)$ contains exactly one interval of E_n. By (3.1) and (3.2) we know that for any $\epsilon > 0$, if $r > 0$ is small enough (that is, if n is large enough),

$$\begin{aligned}\sigma(B(x,r)) &= (m_1 m_2 \cdots m_n)^{-1} \\ &\geq \delta_{n-1}^{2^{-n}} \cdot \eta_n^{\beta} \\ &= 2^{2^{-n}} \eta_n^{\beta+(1-\beta)2^{-n}} \\ &\geq \eta_n^{\beta+\epsilon} \\ &\geq r^{\beta+\epsilon} \ . \end{aligned} \tag{3.3}$$

Consider the second case $\eta_n \leq r < \delta_{n-1}$. Then $B(x,r)$ contains at least $\frac{r}{\eta_n+\delta_n}$ intervals from E_n, so that

$$\begin{aligned}\sigma(B(x,r)) &\geq \frac{r}{\eta_n+\delta_n} \cdot (m_1 m_2 \cdots m_n)^{-1} \\ &\geq \frac{1}{2}(m_1 \cdots m_{n-1})^{-1} \cdot (\frac{r}{\eta_n})^{1-\beta} \cdot r^{\beta} \\ &\geq \frac{1}{2} r^{\beta+\epsilon} \ . \end{aligned} \tag{3.4}$$

where the last inequality follows from (3.2). Combining (3.3) and (3.4), we have

$$\underline{D}_{\sigma}^{\beta+\epsilon}(x) \geq (\frac{1}{2})^{1+\beta+\epsilon} \qquad \text{for every } x \in E \ .$$

By Lemma 2.2, Dim$E \leq \beta+\epsilon$. Since $\epsilon > 0$ is arbitrary, we have Dim$E \leq \beta$. Therefore Dim$E = \beta$. □

Proposition 3.1 *Let E be the compact set in Lemma 3.2. Let $0 < \alpha < 1$ and $f : [0,1] \to \mathbf{R}^d$ satisfy a uniform Hölder condition of order α. Then*

$$\mathrm{Dim} f(E) \leq \frac{\beta d}{\alpha d + \beta(1-\alpha d)}. \tag{3.5}$$

Proof. For any $\epsilon > 0$, there exists an n such that $\delta_n \leq \epsilon < \delta_{n-1}$. Since $E \subseteq E_n$, by the Hölder continuity of f, $f(E)$ can be covered by $m_1 m_2 \cdots m_n$ balls of radius ϵ^{α}. On the other hand, $E \subseteq E_{n-1}$ and $f(E_{n-1})$ is contained in the union of $m_1 \cdots m_{n-1}$ balls of radius δ_{n-1}^{α}. Hence

$$\begin{aligned} N(f(E), \epsilon^{\alpha}) &\leq \min\{m_1 m_2 \cdots m_n, \quad m_1 \cdots m_{n-1} K_1 (\frac{\delta_{n-1}}{\epsilon})^{\alpha d}\} \\ &\leq m_1 \cdots m_{n-1} \min\{(\frac{1}{\eta_n})^{\beta}, \quad K_1 \epsilon^{-\alpha d} \eta_n^{(1-\beta)\alpha d}\}, \end{aligned} \tag{3.6}$$

where K_1 is a constant depending only on d. For any $0 < \theta < 1$, we have

$$\min\{(\frac{1}{\eta_n})^{\beta}; K_1\epsilon^{-\alpha d}\eta_n^{(1-\beta)\alpha d}\} \le (\frac{1}{\eta_n})^{\beta\theta} \cdot (K_1\epsilon^{-\alpha d}\eta_n^{(1-\beta)\alpha d})^{1-\theta}$$
$$\le K_2\epsilon^{-(1-\theta)\alpha d}\eta_n^{(1-\beta)\alpha d-\theta(\alpha d+\beta(1-\alpha d))}. \quad (3.7)$$

If we take

$$\theta = \frac{(1-\beta)\alpha d}{\alpha d+\beta(1-\alpha d)},$$

then by (3.7) and (3.2), (3.6) is less than

$$\begin{aligned} &\le K_2 m_1\cdots m_{n-1}\epsilon^{-(1-\theta)\alpha d} \\ &\le K_2\delta_{n-1}^{-2^{-n}}\cdot\epsilon^{-(1-\theta)\alpha d} \\ &\le K_3(\epsilon^{\alpha})^{-((1-\theta)d+2^{-n})}. \end{aligned}$$

This proves that

$$\Delta(f(E)) \le \frac{\beta d}{\alpha d+\beta(1-\alpha d)}.$$

and hence (3.5). □

Remark. If for any $\epsilon > 0$, $f : [0,1] \to \mathbf{R}^d$ satisfies a uniform Hölder condition of order $\alpha - \epsilon$, (3.5) still holds. Applying this to fractional Brownian motion $X(t)$ $(t \in \mathbf{R})$ of index α in $\mathbf{R}^d$, we see that, if $1 > \alpha d$, then $\mathrm{Dim} f(E) < \frac{1}{\alpha}\mathrm{Dim}E$. Hence (1.5) does not hold in general.

4 Packing dimension of fractional Brownian motion

Let $X(t)(t \in \mathbf{R})$ be a fractional Brownian motion in $\mathbf{R}^d$ of index α. In this section, we prove that, if $1 > \alpha d$, then for every compact set $E \subseteq \mathbf{R}$, we have

$$\mathrm{Dim}X(E) \ge \frac{\mathrm{Dim}E\cdot d}{\alpha d+\mathrm{Dim}E\cdot(1-\alpha d)} \qquad a.\ s.$$

The result can be extended to the case of $N > 1$.

We need the following lemma. A compact set E_γ is called a Cantor-type set if $E_\gamma = \cap_{n=1}^{\infty} E_n$, where

$$E_n = \cup_{i_1=1}^{N_0}\cdots\cup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n} \quad (n = 1, 2, \cdots)$$

is a decreasing sequence of compact sets and for each $n \geq 1$, $I_{i_1 \cdots i_n}$ ($i_n = 1, \cdots, N_{i_1 \cdots i_{n-1}}$) are disjoint closed subintervals of $I_{i_1 \cdots i_{n-1}}$.

Our next lemma may be known, but we can not locate a proof: we believe it is of independent interest.

Lemma 4.1 *Let $E \subseteq \mathbf{R}$ be compact. For any $\gamma < \mathrm{Dim}E$, there exists a Cantor-type set $E_\gamma = \cap_{n=1}^{\infty} E_n$ such that $E_\gamma \subseteq E$ and $\mathrm{Dim}E_\gamma \geq \gamma$.*

Proof. Fix $\gamma < \gamma' < \mathrm{Dim}E$ and let

$$\mathcal{F} = \{I : I \text{ is a rational closed interval with } \mathrm{Dim}(I \cap E) \leq \gamma'\}.$$

Then $\mathcal{F}$ is countable and by the σ-stability of Dim, we have

$$\mathrm{Dim}F = \mathrm{Dim}E,$$

where $F = E \setminus \cup_{I \in \mathcal{F}} I$. Observe that for any $x \in F$ and for any $r > 0$,

$$\mathrm{Dim}(F \cap [x - r, x + r]) > \gamma'. \tag{4.1}$$

Since $\Delta(F) \geq \mathrm{Dim}F > \gamma'$, there exists an $0 < \eta_0 < (\frac{1}{2})^{\frac{1}{\gamma}}$ such that

$$N(F, \eta_0) > (\frac{1}{\eta_0})^\gamma.$$

Let $x_1, \cdots, x_{N_0}$ be $N_0 = [(\frac{1}{\eta_0})^\gamma] + 1$ points in F with $|x_j - x_k| \geq 2\eta_0$ for $j \neq k$. If we choose $\delta_0 < \frac{1}{4}\eta_0$, then the intervals $I_{i_1} = [x_{i_1} - \delta_0, x_{i_1} + \delta_0]$ ($i_1 = 1, \cdots, N_0$) are disjoint and by (4.1)

$$\mathrm{Dim}(F \cap I_{i_1}) > \gamma' \quad \text{for every} \quad 1 \leq i_1 \leq N_0. \tag{4.2}$$

Let $E_1 = \cup_{i_1=1}^{N_0} I_{i_1}$; by (4.2), for each i_1 we can choose $0 < \eta_{i_1} < \eta_0$ such that

$$N(F \cap I_{i_1}, \eta_{i_1}) > (\frac{1}{\eta_{i_1}})^\gamma, \tag{4.3}$$

Let $N_{i_1} = [(\frac{1}{\eta_{i_1}})^\gamma] + 1$; there exist N_{i_1} points $x_{i_1 i_2}$ ($i_2 = 1, \cdots, N_{i_1}$) in $F \cap I_{i_1}$ such that $|x_{i_1 j} - x_{i_1 k}| \geq 2\eta_{i_1}$ for $j \neq k$. For each i_1, if we choose $\delta_{i_1} < \frac{1}{4}\eta_{i_1}$, then the intervals

$$I_{i_1 i_2} = [x_{i_1 i_2} - \delta_{i_1}, x_{i_1 i_2} + \delta_{i_1}]$$

are disjoint (in order to have $I_{i_1i_2} \subseteq I_{i_1}$ for every $1 \le i_2 \le N_{i_1}$, we may have to delete two intervals which intersect $\mathbf{R}\backslash I_{i_1}$. Since this ambiguity does not affect the conclusion of Lemma 4.1, we will not clarify it) and

$$\mathrm{Dim}(F \cap I_{i_1i_2}) > \gamma'. \tag{4.4}$$

Let $E_2 = \cup_{i_1=1}^{N_0} \cup_{i_2=1}^{N_{i_1}} I_{i_1i_2}$ be the union of $\tilde{N}_2 = \prod_{i_1=1}^{N_0} N_{i_1}$ disjoint closed intervals. Suppose now that E_{n-1} has been defined as a union of $\tilde{N}_{n-1}$ disjoint closed intervals $I_{i_1\cdots i_{n-1}}$ of length $2\delta_{i_1\cdots i_{n-2}}$ with gaps at least $\eta_{i_1\cdots i_{n-2}}$, and

$$\mathrm{Dim}(F \cap I_{i_1\cdots i_{n-1}}) > \gamma'. \tag{4.5}$$

Now fix an interval $I_{i_1\cdots i_{n-1}}$; by (4.5), we can choose $0 < \eta_{i_1\cdots i_{n-1}} < \eta_{i_1\cdots i_{n-2}}$ such that

$$N(F \cap I_{i_1\cdots i_{n-1}}, \eta_{i_1\cdots i_{n-1}}) > \Big(\frac{1}{\eta_{i_1\cdots i_{n-1}}}\Big)^{\gamma}.$$

Let

$$N_{i_1\cdots i_{n-1}} = \Big[(\frac{1}{\eta_{i_1\cdots i_{n-1}}})^{\gamma}\Big] + 1;$$

then there exist $N_{i_1\cdots i_{n-1}}$ points $x_{i_1\cdots i_{n-1}i_n}$ $(i_n = 1, \cdots, N_{i_1\cdots i_{n-1}})$ in $F \cap I_{i_1\cdots i_{n-1}}$ such that for $j \ne k$,

$$|x_{i_1\cdots i_{n-1}j} - x_{i_1\cdots i_{n-1}k}| \ge 2\eta_{i_1\cdots i_{n-1}}.$$

For each $i_1 \cdots i_{n-1}$, we choose $\delta_{i_1\cdots i_{n-1}} < \frac{1}{4}\eta_{i_1\cdots i_{n-1}}$ so that the intervals

$$I_{i_1\cdots i_{n-1}i_n} = [x_{i_1\cdots i_{n-1}i_n} - \delta_{i_1\cdots i_{n-1}}, \quad x_{i_1\cdots i_{n-1}i_n} + \delta_{i_1\cdots i_{n-1}}]$$

are disjoint and

$$\mathrm{Dim}(F \cap I_{i_1\cdots i_{n-1}i_n}) > \gamma'.$$

Let

$$E_n = \cup_{i_1=1}^{N_0} \cup_{i_2=1}^{N_{i_1}} \cdots \cup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n}.$$

In this way, we obtain a sequence of decreasing closed sets $\{E_n\}$. Let $E_\gamma = \cap_{n=1}^{\infty} E_n$, then $E_\gamma \subseteq E$ is a Cantor-type compact set.

To estimate the lower bound of $\mathrm{Dim}E_\gamma$, we define a Borel measure σ on $\mathbf{R}$ with $\sigma(E_\gamma) = 1$. For each interval I_{i_1} $(i_1 = 1, \cdots, N_0)$, we define

$$\sigma(I_{i_1}) = N_0^{-1}.$$

Supposing that $\sigma(I_{i_1\cdots i_{n-1}})$ has been defined for each interval in E_{n-1}, we define

$$\sigma(I_{i_1\cdots i_n}) = \sigma(I_{i_1\cdots i_{n-1}})N^{-1}_{i_1\cdots i_{n-1}} \quad (i_n = 1, \cdots, N_{i_1\cdots i_{n-1}}) \ .$$

Finally for each $n \geq 1$, define $\sigma(\mathbf{R}\backslash E_n) = 0$. Then σ can be extended to a Borel measure on $\mathbf{R}$ with $\sigma(E_\gamma) = 1$. Now for each $t \in E_\gamma$, there exists a sequence $\mathbf{i} = i_1 i_2 \cdots i_n \cdots$ such that

$$\{t\} = \cap_{n=1}^\infty I_{i_1\cdots i_n}$$

and

$$\sigma(B(t, \frac{1}{2}\eta_{i_1\cdots i_{n-1}})) \leq \sigma(I_{i_1\cdots i_{n-1}}) \cdot \eta^\gamma_{i_1\cdots i_{n-1}} \ .$$

Thus

$$\liminf_{r\to 0} \frac{\sigma(B(t,r))}{(2r)^\gamma} \leq \liminf_{n\to\infty} \frac{\sigma(I_{i_1\cdots i_{n-1}}) \cdot \eta^\gamma_{i_1\cdots i_{n-1}}}{\eta^\gamma_{i_1\cdots i_{n-1}}} = 0 \ .$$

By Lemma 2.2, we have $\mathrm{Dim}E_\gamma \geq \gamma$. □

Remark. By the construction of E_γ, we can choose $\eta's$ satisfying

$$\eta^{-\alpha d}_{i_1\cdots i_{n-1}} \cdot \eta^{\alpha d(1-\gamma)}_{i_1\cdots i_n} \leq \frac{1}{(n+1)^2} \ , \tag{4.6}$$

Now we are in a position to prove the main result of this section.

Theorem 4.1 *Let $X(t)(t \in \mathbf{R})$ be a d-dimensional fractional Brownian motion of index α $(0 < \alpha < 1)$ and $1 > \alpha d$. Then for any compact set $E \subseteq \mathbf{R}$, with probability 1,*

$$\mathrm{Dim}X(E) \geq \frac{\mathrm{Dim}E \cdot d}{\alpha d + \mathrm{Dim}E \cdot (1 - \alpha d)} \ . \tag{4.7}$$

Proof. For any $\gamma < \mathrm{Dim}E$, let $E_\gamma \subseteq E$ be the Cantor-type set in Lemma 4.1 with $\eta's$ satisfying (4.6). It suffices to prove that with probability 1,

$$\mathrm{Dim}X(E_\gamma) \geq \frac{\gamma d}{\alpha d + \gamma(1 - \alpha d)} \ .$$

Let σ be the Borel measure on $\mathbf{R}$ constructed in Lemma 4.1 with $\sigma(E_\gamma) = 1$ and let μ be the image of σ under $X(t)(t \in \mathbf{R})$. Then μ is a random Borel measure in $\mathbf{R}^d$ and $\mu(X(E_\gamma)) = 1$. Set $\theta = \frac{\alpha d + \gamma(1-\alpha d)}{d}$. For any $t \in E_\gamma$, there exists an index $\mathbf{i} = i_1 i_2 \cdots i_n \cdots$ such that for any $n \geq 1, t \in I_{i_1\cdots i_n}$. For a fixed $n \geq 2$, let $\mathbf{j} = j_1 \cdots j_{n-1}$ be the sequence of indices of the intervals in the (n-1)th stage of the construction of E_γ. Then

$$\mu\Big(B(X(t), \eta^{\theta}_{i_1\cdots i_{n-1}})\Big) = \sum_{\mathbf{j}} \sigma\{s \in I_{\mathbf{j}} : \quad |X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}$$

$$= \sigma\{s \in I_{i_1\cdots i_{n-1}} : |X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}$$
$$+ \sum_{\mathbf{j} \neq i_1\cdots i_{n-1}} \sigma\{s \in I_{\mathbf{j}} : |X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}$$

$$= \sum_{j_n=1}^{N_{i_1\cdots i_{n-1}}} \sigma\{s \in I_{i_1\cdots i_{n-1}j_n} : |X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}$$
$$+ \sum_{\mathbf{j} \neq i_1\cdots i_{n-1}} \sigma\{s \in I_{\mathbf{j}} : |X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\} .$$

Observe now that

$$P\{|X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\} \leq c_1 \frac{\eta^{\theta d}_{i_1\cdots i_{n-1}}}{|s-t|^{\alpha d}} .$$

Hence

$$E\mu\Big(B(X(t), \eta^{\theta}_{i_1\cdots i_{n-1}})\Big)$$
$$= \sum_{j_n=1}^{N_{i_1\cdots i_{n-1}}} \int_{I_{i_1\cdots i_{n-1}j_n}} P\{|X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}\sigma(ds)$$
$$+ \sum_{\mathbf{j} \neq i_1\cdots i_{n-1}} \int_{I_{\mathbf{j}}} P\{|X(s) - X(t)| \leq \eta^{\theta}_{i_1\cdots i_{n-1}}\}\sigma(ds)$$
$$\leq \sigma(I_{i_1\cdots i_n}) + \sum_{j_n \neq i_n} \int_{I_{i_1\cdots i_{n-1}j_n}} \frac{c_1 \eta^{\theta d}_{i_1\cdots i_{n-1}}}{|s-t|^{\alpha d}} \sigma(ds)$$
$$+ \sum_{\mathbf{j} \neq i_1\cdots i_{n-1}} \int_{I_{\mathbf{j}}} \frac{c_1 \eta^{\theta d}_{i_1\cdots i_{n-1}}}{|s-t|^{\alpha d}} \sigma(ds) . \tag{4.8}$$

By using the lower bounds for $|s-t|$, we get upper bounds for the integrals in (4.8), so that (4.8) is less than

$$\eta^{\gamma}_{i_1\cdots i_{n-1}} + 2c_1 \sum_{j_n=1}^{N_{i_1\cdots i_{n-1}}} \eta^{\gamma(1-\alpha d)}_{i_1\cdots i_{n-1}} \cdot \frac{1}{j_n^{\alpha d}} \cdot \sigma(I_{i_1\cdots i_{n-1}j_n})$$
$$+ c_1 \sum_{\mathbf{j} \neq i_1\cdots i_{n-1}} \frac{\eta^{\alpha d + \gamma(1-\alpha d)}_{i_1\cdots i_{n-1}}}{\eta^{\alpha d}_{i_1\cdots i_{n-2}}} \cdot \sigma(I_{\mathbf{j}})$$

$$\le \eta_{i_1\cdots i_{n-1}}^{\gamma}\Big[1+c_2\eta_{i_1\cdots i_{n-1}}^{\gamma(1-\alpha d)}\cdot N_{i_1\cdots i_{n-1}}^{1-\alpha d}+c_1\sum_{\mathbf{j}\ne i_1\cdots i_{n-1}}\frac{\eta_{i_1\cdots i_{n-1}}^{\alpha d(1-\gamma)}}{\eta_{i_1\cdots i_{n-2}}^{\alpha d}}\cdot\sigma(I_{\mathbf{j}})\Big]$$
$$\le \quad c_3\eta_{i_1\cdots i_{n-1}}^{\gamma}\ .$$

where (4.6) is used to get the last inequality. By Fatou's Lemma, we have

$$E\Big(\liminf_{r\to 0}\frac{\mu(B(X(t),r^{\theta}))}{(2r)^{\gamma}}\Big)\le c_3.$$

Hence by Fubini's theorem, with probability 1,

$$\underline{D}_{\mu}^{\frac{\gamma}{\theta}}(X(t))<\infty,\quad \sigma\ a.\ e.\ \ t\in E_{\gamma}.$$

For $k\ge 1$, let $E_{\gamma k}(\omega)=\{t\in E_{\gamma}:\underline{D}_{\mu}^{\frac{\gamma}{\theta}}(X(t))\le k\}$, then with probability 1, for k large enough, $\sigma(E_{\gamma k}(\omega))>0$. By Lemma 2.2, we have

$$\mathrm{Dim}X(E_{\gamma k}(\omega))\ge\frac{\gamma}{\theta},\qquad a.\ s.$$

Hence

$$\mathrm{Dim}X(E)\ge\frac{\gamma d}{\alpha d+\gamma(1-\alpha d)}.\qquad a.\ s.$$

Since $\gamma<\mathrm{Dim}E$ is arbitrary, this proves (4.7). □

Combining Proposition 3.1 and Theorem 4.1, we obtain the following corollary.

Corollary 4.1 *Let $X(t)(t\in\mathbf{R})$ be a d-dimensional fractional Brownian motion of index α $(0<\alpha<1)$ and $1>\alpha d$. Let E be the compact set in Lemma 3.2. Then with probability 1,*

$$\mathrm{Dim}X(E)=\frac{\mathrm{Dim}E\cdot d}{\alpha d+\mathrm{Dim}E\cdot(1-\alpha d)}\ .$$

Remark. It follows from (1.1) and Theorem 4.1 that if $1>\alpha d$, then for any Borel set $E\subseteq\mathbf{R}$, with probability 1,

$$\mathrm{Dim}X(E)\ge\max\Big\{\min\{d,\ \ \frac{1}{\alpha}\mathrm{dim}E\};\ \ \frac{\mathrm{Dim}E\cdot d}{\alpha d+\mathrm{Dim}E\cdot(1-\alpha d)}\Big\}\ .\qquad(4.9)$$

The lower bound in (4.9) reduces to (1.2) when $\mathrm{dim}E=\mathrm{Dim}E$, to Theorem 4.1 when $\mathrm{dim}E=0$. It is natural to ask whether the lower bound in (4.9) can be improved when $0<\mathrm{dim}E<\mathrm{Dim}E$. The following example shows that the answer is negative. Let E_1 be the compace set in Lemma 3.2 and let E_2 be a subset of $\mathbf{R}$ with

$$\dim E_2 = \mathrm{Dim} E_2 \leq \frac{\mathrm{Dim} E_1 \cdot \alpha d}{\alpha d + \mathrm{Dim} E_1 \cdot (1 - \alpha d)} \ .$$

Let $E = E_1 \cup E_2$, then $\dim E = \dim E_2$, $\mathrm{Dim} E = \mathrm{Dim} E_1$ and with probability 1,

$$\begin{aligned} \mathrm{Dim} X(E) &= \max\{\mathrm{Dim} X(E_1);\ \mathrm{Dim} X(E_2)\} \\ &= \frac{\mathrm{Dim} E \cdot d}{\alpha d + \mathrm{Dim} E \cdot (1 - \alpha d)} \end{aligned}$$

References

[F] Falconer, K. J. (1990) *Fractal Geometry – Mathematical Foundations And Applications.* Wiley & Sons.

[K] Kahane, J-P. (1985) *Some Random Series of Functions.* 2nd edition. Cambridge University Press.

[Ka] Kaufman, R. (1969) Une propriété métrique du mouvement brownien. *C. R. Acad. Sci. Paris* **268** 727 -728.

[MP] Monrad, D. and Pitt, L. D. (1986) Local nondeterminism and Hausdorff dimension. *Progress in Probability and Statistics. Seminar on Stochastic Processes.* Birkhauser, Boston.

[PT] Perkins, Ed. A. and Taylor, S. J. (1987) Uniform measure results for the image of subsets under Brownian motion. *Probab. Th. Rel. Fields* **76** 257 - 289.

[ST] Saint Raymond, X. and Tricot, C. (1988) Packing regularity of sets in n-space. *Math. Proc. Camb. Phil. Soc.* **103** 133 -145.

[T1] Taylor, S. J. (1986) The use of packing measure in the analysis of random sets. *Lecture Notes in Math.* **1203** 214 - 222.

[T2] Taylor, S. J. (1986) The measure theory of random fractals. *Math. Proc. Camb. Phil. Soc.* **100** 383 - 406.

[Tr] Tricot, C. (1982) Two definitions of fractional dimension. *Math. Proc. Camb. Phil. Soc.* **91** 57 - 74.

[TT1] Taylor, S. J. and Tricot, C. (1985) Packing measure and its evalution for a Brownian path. *Trans. Amer. Math. Soc.* **288** 679 - 699.

[TT2] Taylor, S. J. and Tricot, C. (1986) The packing measure of rectifiable subsets of the plane. *Math. Proc. Phil. Soc.* **99** 285 - 296.

[X] Xiao, Yimin. (1990) Uniform packing dimension results for fractional Brownian motion. *Proceeding of 1st China-France Symposium on Probability and Statistics.*