FRACTAL MEASURES OF THE SETS ASSOCIATED TO GAUSSIAN RANDOM FIELDS

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ABSTRACT

This paper summarizes recent results about the Hausdorff measure of the image, graph and level sets of Gaussian random fields, the packing dimension and packing measure of the image of fractional Brownian motion, the local times and multiple points of Gaussian random fields. Some open problems are also pointed out.

§1. Introduction

Let $X(t)$ ($t \in \mathbf{R}$ or $\mathbf{R}^N$) be a stochastic process taking values in $\mathbf{R}^d$. Associated to it, there are many random sets, such as the image, graph, level sets, set of multiple points and so on. On one hand, these random sets contain rich information about the fine structure of the sample paths of the process X. On the other hand, they provide examples of sets with special properties needed in other branches of mathematics such as harmonic analysis, which are usually difficult to construct otherwise.

These random sets are usually "thin" in the sense that they have Lebesgue measure 0 and they are not smooth or surface-like. The most commonly used tools in analyzing the geometric properties of such sets are Hausdorff measure and Hausdorff dimension. For any $E \subseteq \mathbf{R}^N$, we shall denote its Hausdorff dimension by $\dim E$ and its ϕ-Hausdorff measure by ϕ-$m(E)$. If $0 < \phi\text{-}m(E) < \infty$, then ϕ is said to be the correct Hausdorff measure function of E. Packing dimension and packing measure were introduced by Tricot (1982), Taylor and Tricot (1985). We denote the packing dimension and ϕ-packing measure by $\mathrm{Dim} E$ and ϕ-$p(E)$ respectively, and we refer to Falconer (1990) and Mattila (1995) for definitions and properties of Hausdorff measure, packing measure and corresponding dimensions.

The study of the geometric properties of the random sets associated to Brownian motion started in early 1950's by Lévy and Taylor. Let $B(t)$ ($t \in \mathbf{R}_+$) be Brownian motion in $\mathbf{R}^d (d \geq 2)$. Taylor (1953) proved that $\dim B([0,1]) = 2$. Lévy (1953) showed that if $d \geq 3$ and $\phi(s) = s^2 \log\log 1/s$, then ϕ-$m(B([0,1])) < \infty$. Ciesielski and Taylor (1962) proved that ϕ-$m(B([0,1])) > 0$. Therefore $\phi(s) = s^2 \log\log 1/s$ is the correct Hausdorff measure function for the image of Brownian motion in $\mathbf{R}^d$ with

$d \geq 3$. For planar Brownian motion, the Hausdorff measure problem is more difficult due to the recurrence of the process. It was proved by Ray (1963) and Taylor (1964) that the correct Hausdorff measure function is $\phi(s) = s^2 \log 1/s \log\log\log 1/s$.

The packing measure of the image set $B([0,1])$ of Brownian motion in $\mathbf{R}^d$ was evaluated by Taylor and Tricot (1985) for $d \geq 3$, and by LeGall and Taylor (1987) for $d = 2$. The corresponding problems for the graph and level sets of Brownian motion were also studied by Rezakhanlou and Taylor (1988) and Taylor (1986a).

There have been a lot of efforts to extend the results about Brownian motion to more general stochastic processes, especially to Lévy processes. See Taylor (1986b). The crucial ingredient in the study of Lévy processes is the strong Markov property. For non-Markovian processes, many of the "classical" techniques collapse and it was even difficult to obtain the Hausdorff dimension or packing dimension of the random sets associated to these processes (see Adler (1981) and Kahane (1985)).

One very natural and useful non-Markovian process is fractional Brownian motion (FBM), a Gaussian random field from $\mathbf{R}^N$ to $\mathbf{R}^d$ considered in particular by Kahane (1985). When $\alpha = 1/2$, Goldman (1988) studied the exact Hausdorff measure of the image set. The problem for the general case $0 < \alpha < 1$ was solved by Talagrand (1995). Recently, Xiao (1995b, c, 1996a, b) has studied the Hausdorff measure of the graph and level sets, and the packing measure of the image of FBM.

The purpose of this paper is to summarize recent developments in the study of fractal measures of the random sets associated to Gaussian random fields. The chosen results are related to the author's knowledge and interests, so they are by no means exhaustive. Section 2 specifies the Gaussian random fields to be considered. Section 3 surveys results on the Hausdorff measure of the image and graph of Gaussian random fields. Results on the packing dimension and packing measure of the image of fractional Brownian motion are summarized in Section 4. In Section 5, we turn to the local times and the Hausdorff measure of the level sets. Section 6 summarizes the results about the multiple points of Gaussian random fields.

§2. Gaussian Random Fields

Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a real-valued, centered Gaussian random field with $Y(0) = 0$. We assume that $Y(t)$ $(t \in \mathbf{R}^N)$ has stationary increments and continuous covariance function $R(t,s) = EY(t)Y(s)$ given by

$$R(t,s) = \int_{\mathbf{R}^N} (e^{i<t,\lambda>} - 1)(e^{-i<s,\lambda>} - 1)\Delta(d\lambda)\ , \tag{2.1}$$

where $< x, y >$ is the ordinary scalar product in $\mathbf{R}^N$ and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbf{R}^N \backslash \{0\}$ satisfying

$$\int_{\mathbf{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\ \Delta(d\lambda) < \infty\ . \tag{2.2}$$

Then there exists a centered complex-valued Gaussian random measure $W(d\lambda)$ such that

$$Y(t) = \int_{\mathbf{R}^N} (e^{i<t,\lambda>} - 1) W(d\lambda) \tag{2.3}$$

and for any Borel sets $A,\ B \subseteq \mathbf{R}^N$, $E\Big(W(A)\overline{W(B)}\Big) = \Delta(A \cap B)$ and $W(-A) = \overline{W(A)}$. We assume that there exist constants $\delta_0 > 0,\ 0 < c_1 \le c_2 < \infty$ and a non-decreasing, continuous function $\sigma : [0, \delta_0) \to [0, \infty)$ which is regularly varying at the origin with index α $(0 < \alpha < 1)$ such that for any $t \in \mathbf{R}^N$ and $h \in \mathbf{R}^N$ with $|h| \le \delta_0$

$$E\Big[\big(Y(t+h) - Y(t)\big)^2\Big] \le c_1 \sigma^2(|h|)\ . \tag{2.4}$$

and for all $t \in \mathbf{R}^N$ and any $0 < r \le \min\{|t|,\ \delta_0\}$

$$Var\big(Y(t)|Y(s) : r \le |s-t| \le \delta_0\big) \ge c_2 \sigma^2(r)\ . \tag{2.5}$$

If (2.4) and (2.5) hold, we say that $Y(t)$ is strongly locally σ-nondeterministic. A typical example of such Gaussian random fields is the fractional Brownian motion of index α, i.e. the centered, real-valued Gaussian random field $Y(t)$ $(t \in \mathbf{R}^N)$ with covariance function

$$E(Y(t)Y(s)) = \frac{1}{2}(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha})\ .$$

See Pitt (1978) for a proof of this fact. General conditions for strong local nondeterminism of Gaussian processes $Y(t)$ $(t \in \mathbf{R})$ were given by Marcus (1968) and Berman (1972, 1978). We refer to Xiao (1995b) for more examples of strongly locally nondeterministic Gaussian random fields, and to Monrad and Pitt (1986), Cuzick and Du Peez (1982) for some applications of strong local nondeterminism in the study of sample path properties of Gaussian random fields.

We associate with $Y(t)$ a Gaussian random field $X(t)$ $(t \in \mathbf{R}^N)$ in $\mathbf{R}^d$ by

$$X(t) = (X_1(t), \cdots, X_d(t))\ , \tag{2.6}$$

where $X_1, \cdots, X_d$ are independent copies of Y. If $Y(t)$ is the FBM in $\mathbf{R}$ of index α, then $X(t)$ is called the d-dimensional FBM of index α (see Kahane (1985)). When $N = 1, \alpha = 1/2,\ X(t)$ is the ordinary d-dimensional Brownian motion. If $\alpha = 1/2,\ d = 1$, it is the multiparameter Lévy Brownian motion. It is easy to see that X is a self-similar process of exponent α, i.e. for any $a > 0$, $X(a\ \cdot) \stackrel{\mathrm{d}}{=} a^\alpha X(\cdot)$. We refer to Kôno (1991) and Taqqu (1986) for more historical comments and a bibliographical guide on self-similar processes.

We also define more general Gaussian random fields as follows. Let $X_i = \{X_i(t),\ t \in \mathbf{R}^N\}$ $(i = 1, 2, \cdots, d)$ be d independent FBMs in $\mathbf{R}$ of index α_i with $0 = \alpha_0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d < 1$. Define a Gaussian random field $Z(t)$ in $\mathbf{R}^d$ with FBM components by

$$Z(t) = (X_1(t), \cdots, X_d(t))\ . \tag{2.7}$$

Then $Z = \{Z(t),\ t \in \mathbf{R}^N\}$ has stationary increments. If α_i $(i = 1, \cdots, d)$ are not all equal, then Z is not self-similar. Sample path properties of Z have been studied by Cuzick (1978, 1982), Pitt (1978) and the author (1995a, 1995c).

Another important example of Gaussian random fields is the Brownian sheet or N-parameter Wiener process $W(t)$ $(t \in \mathbf{R}^N_+)$, see Orey and Pruitt (1973). Since $W(t)$ $(t \in \mathbf{R}^N_+)$ is not locally nondeterministic, some of the techniques used for locally nondeterministic Gaussian random fields do not apply to $W(t)$.

§3. Hausdorff Measures of the Image and Graph

As we mentioned in the introduction, the exact Hausdorff measure problem for Brownian motion were solved by Lévy (1953), Ciesielski and Taylor (1962), Ray (1963) and Taylor (1964). The Hausdorff measure of the graph of Brownian motion and Lévy stable processes were calculated by Jain and Pruitt (1968) in the transient case, by Pruitt and Taylor (1968) in the recurrent cases. In this section, we consider analogous problems for Gaussian random fields.

Let $X(t)$ $(t \in \mathbf{R}^N)$ be the Gaussian random field defined by (2.6) with $Y(t)$ satisfying (2.4) and (2.5). It is well known (see Adler (1981), Chapter 8) that with probability 1

$$\dim X([0,1]^N) = \min\Big(d, \frac{N}{\alpha}\Big)\ .$$

$$\dim Gr X([0,1]^N) = \begin{cases} \dfrac{N}{\alpha} & \text{if } N \le \alpha d \\ N + (1-\alpha)d & \text{if } N > \alpha d. \end{cases}$$

If $N > \alpha d$, then by a result of Pitt (1978), $X([0,1]^N)$ a. s. has interior points and hence has positive d-dimensional Lebesgue measure. Similar result can be proved for $X(E)$, where $E \subseteq \mathbf{R}^N$ and $\dim E > \alpha d$, see Kahane (1985). The exact Hausdorff measure of $X([0,1]^N)$ of FBM in the transient case of $N < \alpha d$ was considered first by Goldman (1988) for $\alpha = 1/2$, then by Talagrand (1995) for the general case of $0 < \alpha < 1$. Xiao (1995b) generalized their results to strongly locally nondeterministic Gaussian random fields defined by (2.6). The results can be summarized as follows: If $N < \alpha d$, then with probability 1, $0 < \phi_1\text{-}m(X([0,1]^N)) < \infty$, where $\phi_1(s) = \psi(s)^N \log\log 1/s$ and where $\psi(s)$ is the inverse function of σ.

The exact Hausdorff measure of the graph set $GrX([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\}$ for fractional Brownian motion is calculated by Xiao (1995c). The following more general theorem is proved in Xiao (1996b). Let $X(t)(t \in \mathbf{R}^N)$ be the Gaussian random field defined by (2.6). If $N < \alpha d$, then almost surely $K_1 \le \phi_1\text{-}m(GrX([0,1]^N)) \le K_2$; if $N > \alpha d$, then almost surely $K_3 \le \phi_2\text{-}m(GrX([0,1]^N)) \le K_4$, where $K_1,\ K_2,\ K_3,\ K_4$ are positive constants depending on N, d and $\sigma(s)$ only and where

$$\phi_2(r) = \frac{r^{N+d}}{(\sigma(r/(\log\log 1/r)^{1/N}))^d}\ .$$

In the case of $N = \alpha d$, the problem of finding $\phi\text{-}m(X([0,1]^N))$ for FBM is more difficult. Talagrand (1996) proved that $s^d \log 1/s \log\log\log 1/s\ \text{-}m(X([0,1]^N)) < \infty$ almost surely. This is also true for the Hausdorff measure of the graph set of $X(t)$. But in both cases, the lower bound problems remain open.

The Hausdorff dimension of the image and graph of $Z(t)$ defined by (2.7) were considered by Cuzick (1978) and Xiao (1995a). The exact Hausdorff measure of $Z([0,1]^N)$ and $GrZ([0,1]^N)$ have been obtained by Xiao (1995c).

The Hausdorff measure of the image and graph of the Brownian sheet in $\mathbf{R}^d$ were calculated by Ehm (1981).

§4. The Packing Dimension and Packing Measure of the Image of FBM

Many authors have applied the packing measure and packing dimension to study sample path properties of processes with stationary independent increments (see Taylor (1986a, b), LeGall and Taylor (1987), Fristedt and Taylor (1994), and the references therein). In this section, we summarize recent results about the packing dimension and packing measure of the image of fractional Brownian motion.

Kahane (1985) proved that for every Borel set $E \subset \mathbf{R}^N$, almost surely

$$\dim X(E) = \min\left(d,\ \frac{1}{\alpha}\dim E\right).$$

It is natural to ask whether a similar equality holds for packing dimension. Since $\dim \le \mathrm{Dim}$, it is easy to see that for every Borel set $E \subseteq \mathbf{R}^N$ with $\dim E = \mathrm{Dim} E$,

$$\mathrm{Dim} X(E) = \min(d,\ \frac{1}{\alpha}\mathrm{Dim} E)\quad a.\ s. \tag{4.1}$$

For those sets E with $\dim E \ne \mathrm{Dim} E$, (4.1) is not trivial. The first progress in this direction was made by Perkins and Taylor (1987), who showed that if $X(t)$ $(t \in \mathbf{R}_+)$ is Brownian motion in $\mathbf{R}^d$ with $d \ge 2$, then with probability 1,

$$\mathrm{Dim} X(E) = 2\mathrm{Dim} E \quad \text{for every Borel set} \quad E \subseteq \mathbf{R}_+. \tag{4.2}$$

This result is stronger than (4.1) since the exceptional null set does not depend on E. Xiao (1993) proved that if $N \le \alpha d$, then (4.2) holds for fractional Brownian motion, and if $N > \alpha d$, (4.2) fails. In this case, Talagrand and Xiao (1996) has shown that (4.1) does not hold in general by constructing a Cantor-type set E such that $\mathrm{Dim} X(E)$ is strictly smaller than the right hand side of (4.1), and they also obtained the best lower bound for $\mathrm{Dim} X(E)$: with probability 1,

$$\mathrm{Dim} X(E) \ge \max\left\{\min\{d,\ \frac{1}{\alpha}\dim E\},\ \frac{\mathrm{Dim} E \cdot d}{\alpha d + \mathrm{Dim} E \cdot (N - \alpha d)}\right\}. \tag{4.3}$$

Example shows that the lower bound in (4.3) can not be improved by knowing $0 < \dim E < \mathrm{Dim} E$. Xiao (1995d) proved that almost surely

$$\mathrm{Dim} X(E) = \frac{1}{\alpha}\mathrm{Dim}_{\alpha d} E\ , \tag{4.4}$$

where $\mathrm{Dim}_s E$ is the packing dimension profile of E introduced by Falconer and Howroyd (1995). Similar result holds for the Brownian sheet.

Another problem of interest is the exact packing measure of the image and graph of fractional Brownian motion. We remark that while the exact Hausdorff measure of the image and graph of $X(t)$ depend on the properties of the large tail distribution of the sojourn measure of $X(t)$, which can be obtained through the moment argument (see Goldman (1988) and Xiao (1995b)), the packing measure of the image set depends on the small tail (or small ball) probability of the sojourn measure, for which the moment method does not give much information. For this reason, the packing measure problems may be more difficult than the corresponding Hausdorff measure problems.

If $X(t)$ is Brownian motion in $\mathbf{R}^d$ with $d \geq 3$, Taylor and Tricot (1985) showed that with probability 1, $s^{1/2}/\log\log 1/s$ -$p(X([0,t])) = Kt$ for any $t > 0$. By using a method which is quite different from that of Taylor and Tricot, Xiao (1996a) proved that if $N = 1$ and $1 < \alpha d$, then there exists a constant $K > 0$ such that ϕ_3-$p(X([0,t])) = K\ t$ almost surely, where $\phi_3(s) = s^{\frac{1}{\alpha}}/(\log\log 1/s)^{\frac{1}{2\alpha}}$. In the critical case of $1 = \alpha d$, the problem of finding the packing measure of $X([0,1])$ is much deeper. Except in the special case of planar Brownian motion, for which LeGall and Taylor (1987) proved that ϕ-$p(X([0,1]))$ is either zero or infinite, this problem is still open.

Rezakhanlou and Taylor (1988) calculated the packing measure of the graph set of Brownian motion in $\mathbf{R}^d$. By modifying the proof of Thereom 5.1 in Xiao (1996a), it can be shown that in the transient case, $\phi_3(s)$ is also the correct packing measure function for the graph of FBM. If $N \geq \alpha d$, the problem remains open.

No results about the packing measure of the image and graph of the Brownian sheet have been proved.

§5. Local Times and the Hausdorff Measure of the Level Sets

The study of the local times of Gaussian processes was initiated by Berman (1969, 1972). In order to unify and generalize his earlier work on local times of Gaussian processes, Berman (1973) introduced the concept of "local nondeterminism" (LND). Pitt (1978) generalized LND to Gaussian random fields and proved Hölder conditions in the set variable for the local times. For an excellent survey on the study of local times before early 1980s, we refer to Geman and Horowitz (1980).

It is known (cf. Pitt (1978), Kahane (1985)) that for any rectangle $I \subseteq \mathbf{R}^N$, if

$$\int_I \int_I \frac{dt\ ds}{\sigma(|t-s|)^d} < \infty\ , \tag{5.1}$$

then almost surely the local time $L(x, I)$ of $X(t)$ $(t \in I)$ exists and is square integrable. For many Gaussian random fields including fractional Brownian motion, the condition (5.1) is also necessary for the existence of local times. The joint continuity as well as Hölder conditions in both space variable and (time) set variable of the local times of locally nondeterministic Gaussian processes and fields have been studied by Berman (1969, 1972, 1973), Pitt (1978), Davis (1976), Kôno (1977), Cuzick (1982), Geman and Horowitz (1980), Geman, Horowitz and Rosen (1984), Csörgő, Lin and Shao (1995) and recently by the author (1996b).

Let $l(x,t)$ be the local time of a standard Brownian motion $B(t)(t \geq 0)$ in $\mathbf{R}$. Kesten (1965) proved the law of iterated logarithm

$$\limsup_{h \to 0} \sup_x \frac{l(x,h)}{(2h \log\log 1/h)^{1/2}} = 1 \quad a.s. \tag{5.2}$$

Perkins (1981) proved the following global result

$$\limsup_{h \to 0} \sup_{0 \leq t \leq 1-h} \sup_x \frac{l(x,t+h) - l(x,h)}{(2h \log 1/h)^{1/2}} = 1 \quad a.s. \tag{5.3}$$

Xiao (1996b) generalized these results partially to the local time of strongly locally nondeterministic Gaussian random fields.

Let $X(t)(t \in \mathbf{R}^N)$ be the Gaussian random field defined by (2.7) and $N > \alpha d$. For any $B \in \mathcal{B}(\mathbf{R}^N)$ define $L^*(B) = \sup_x L(x,B)$. Xiao (1996b) proved that there exists a $K > 0$ such that for any $\tau \in \mathbf{R}^N$ almost surely

$$\limsup_{r \to 0} \frac{L^*(B(\tau,r))}{\phi_4(r)} \leq K \ , \tag{5.4}$$

and for any rectangle $T \subseteq \mathbf{R}^N$, there exists a constant $K > 0$ such that almost surely

$$\limsup_{r \to 0} \sup_{t \in T} \frac{L^*(B(t,r))}{\phi_5(r)} \leq K \ , \tag{5.5}$$

where $B(\tau,r)$ is the (open) ball centered at τ with radius r and

$$\phi_4(r) = \frac{r^N}{\sigma(r(\log\log 1/r)^{-1/N})^d}, \qquad \phi_5(r) = \frac{r^N}{\sigma(r(\log 1/r)^{-1/N})^d} \ .$$

If $X(t)$ $(t \in \mathbf{R}^N)$ is the d-dimensional fractional Brownian motion of index α $(0 < \alpha < 1)$ with $N > \alpha d$. Then it follows from (5.4) and (5.5) that for any $\tau \in \mathbf{R}^N$ almost surely

$$\limsup_{r \to 0} \frac{L^*(B(\tau,r))}{r^{N-\alpha d}(\log\log 1/r)^{\alpha d/N}} \leq K \ , \tag{5.6}$$

and for any rectangle $T \subseteq \mathbf{R}^N$, almost surely

$$\limsup_{r \to 0} \sup_{t \in T} \frac{L^*(B(t,r))}{r^{N-\alpha d}(\log 1/r)^{\alpha d/N}} \leq K \ . \tag{5.7}$$

By (5.2) and (5.3), we see that (5.6) and (5.7) are the best possible. In the case of real-valued Gaussian processes (i.e. $N = d = 1$), results similar to (5.4) for $L(x, B(\tau, r))$ with $x \in \mathbf{R}$ fixed instead of $L^*(B(t, r))$ were obtained by Kôno (1977), Cuzick (1982) and recently by Csörgő, Lin and Shao (1995). Since the Gaussian processes considered by Csörgő, Lin and Shao (1995) are strongly locally nondeterministic, inequality (5.5) confirms a conjecture made by Csörgő, Lin and Shao (1995) for Gaussian processes.

Another problem considerd in Xiao (1996b) is the exact Hausdorff measure of the level sets of $X(t)$ $(t \in \mathbf{R}^N)$. The exact Hausdorff measure function for the level set of Lévy stable process was obtained by Taylor and Wendel (1966). In the case of Brownian motion in $\mathbf{R}$, Perkins (1981) proved that almost surely for any $x \in \mathbf{R},\ t \in \mathbf{R}_+$,

$$(r \log\log 1/r)^{1/2}\text{-}m\left(X^{-1}(x) \cap [0, t]\right) = \frac{1}{\sqrt{2}}\ L(x, [0, t])\ . \tag{5.8}$$

Barlow, Perkins and Taylor (1986) extended this result to some Lévy processes.

The exact Hausdorff measure of the level sets of certain stationary Gaussian processes was considered by Davies (1976, 1977), in which she adapted partially the method of Taylor and Wendel (1966) and it was essential to assume the stationarity and $N = 1$. The following theorem is proved in Xiao (1996b).

Let $X(t)(t \in \mathbf{R}^N)$ be the Gaussian random field defined by (2.6) with $Y(t)(t \in \mathbf{R}^N)$ further satisfying $E(Y(t+h) - Y(t))^2 = \sigma^2(|h|)$ and $N > \alpha d$. Let T be a closed cube in $\mathbf{R}^N \backslash \{0\}$ Then there exists a finite constant $K_5 > 0$ such that for every fixed $x \in \mathbf{R}^d$, with probability 1

$$K_5 L(x, T) \le \phi_1\text{-}m\left(X^{-1}(x) \cap T\right) < \infty\ . \tag{5.9}$$

The proof of (5.9) is very much different from those of the previous work on this subject. In fact the proof of the lower bound relies much less on the specific properties of the process than those of Taylor and Wendel (1966) and Davies (1976, 1977), hence can be applied to other random fields such as the Brownian sheet. The proof of the upper bound is based upon the approach of Talagrand (1996). We believe that there exist some finite constant $K_6 > 0$ such that $\phi_1\text{-}m(X^{-1}(x) \cap T) \le K_6 L(x, T)$. It seems that this can not be proved by the method of Xiao (1996b).

Results analogous to (5.4), (5.5) and (5.9) also hold for the Gaussian random field $Z(t)$ with FBM components defined by (2.7). The proofs are similar to those of Xiao (1996b), with several careful modifications.

Results similar to (5.4) and (5.5) for the local time of the Brownian sheet were obtained by Ehm (1981). The Hausdorff measure of the level sets for the Brownian sheet in $\mathbf{R}$ was consider by Zhou (1995). For $d > 1$ the problem has not been solved.

For Gaussian random fields considered in this paper as well as the Brownian sheet, the problems of lower bound for the limits considered above remain open.

We remark that many of the techniques in studying the local times of Gaussian random fields can be applied to other stochastic processes, such as self-similar stable processes. For studies of these aspects, we refer to Nolan (1988), Kôno and Shieh (1993), Xiao (1996c) and references therein. It is natural to ask whether the results analogous to (5.4) and (5.5) hold for the local times of self-similar stable processes.

§6. Multiple Points of Gaussian Random Fields

Let $X(t)(t \in \mathbf{R}^N)$ be a random field with values in $\mathbf{R}^d$. A point $x \in \mathbf{R}^d$ is called a k-multiple point of X(t) if there exist k distinct $t_1, \cdots, t_k \in \mathbf{R}^N$ such that $X(t_1) = \cdots = X(t_k) = x$. If $k = 2$ (or 3), x is also called a double (or triple) point. The set of k-multiple times is denoted by $M_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^{Nk},\ X(t_1) = \cdots = X(t_k)\}$, and the set of k-multiple points is denoted by L_k. For results on the existence and the Hausdorff dimension of the multiple points of Brownian motion, we refer to Taylor (1986b). Wolpert (1978) studied the existence of multiple points of Brownian motion by the method of self-intersection local times. His approach has been extended to more general stochastic processes including FBM and the Brownian sheet (see Rosen (1984)). Similar methods have also been applied by Berman (1991), Shieh (1993), and Zhong and Xiao (1995) to study the multiple points of certain Gaussian and stable processes.

Kôno (1978) and Goldman (1981) studied the existence of k-multiple points ($k \geq 2$) for fractional Brownian motion. Similar problem for $Z(t)$ defined by (2.7) was also considered by Cuzick (1982). Their results can be summarized as

(a) If $Nk < (k-1)\sum_{i=1}^d \alpha_i$, then $X(t)$ a.s. has no k-multiple point.

(b) If $Nk > (k-1)\sum_{i=1}^d \alpha_i$, then $X(t)$ a.s. has k-multiple points.

In the critical case of $Nk = (k-1)\sum_{i=1}^d \alpha_i$, the existence problem of k-multiple points had remained open until recently Talagrand (1996) proved the following theorem: Let $X(t)$ $(t \in \mathbf{R}^N)$ be fractional Brownian motion of index α in $\mathbf{R}^d$. If $Nk = (k-1)\alpha d$, then almost surely there exist no k-multiple points.

With a little modification, Talagrand's argument also proves the similar result for Gaussian random field $Z(t)$ defined by (2.7).

When k-multiple points exist, the Hausdorff dimension of M_k were obtained by Weber (1983) for FBM and by Rosen (1984) for both FBM and the Brownian sheet respectively. Similar result for $Z(t)$ was proved earlier by Cuzick (1982).

The Hausdorff dimension of L_k was calculated in Xiao (1992) for FBM. His argument does not solve the problem of finding $\dim L_k$ for $Z(t)$ defined by (2.7), part of the reason is that there is no uniform Hausdorff dimension result for the image of $Z(t)$. The question about $\dim L_k$ for $Z(t)$ was raised by Cuzick (1982) and is still open.

For Brownian motion in $\mathbf{R}^2$ or $\mathbf{R}^3$, the exact Hausdorff measure of the set of k-multiple points was considered by LeGall (1986). For FBM, Talagrand (1996) showed that if $N < \alpha d$ and $Nk > (k-1)\alpha d$, then with probability 1, ϕ_6-$m(L_k)$ is σ-finite, where $\phi_6(s) = s^{Nk/\alpha-(k-1)d} \left(\log\log 1/s\right)^k$. The lower bound of ϕ-$m(L_k)$ is not known.

For the Brownian sheet in the critical case of $2Nk = (k-1)d$, the existence problem for k-multiple points has not been solved.

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