Properties of Local Nondeterminism of Self-Similar Random Fields and Their Applications

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1 Berman's local nondeterminism

Let $X = \{X(t),\, t \in \mathbb{R}_+\}$ be a separable Gaussian process with mean 0 and let $J \subset \mathbb{R}_+$ be an interval. Assume that $\mathbb{E}[X(t)^2] > 0$ for all $t \in J$ and there exists $\delta > 0$ such that

$$\sigma^2(s,t) = \mathbb{E}\big[(X(s) - X(t))^2\big] > 0$$

for all $s,\ t \in J$ with $0 < |s - t| < \delta$.

Definition 1.1 [Berman (1973)] X *is called locally nondeterministic on* J *if for every integer* $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{1}$$

where V_m *is the relative prediction error:*

$$V_m = \frac{\mathrm{Var}\big(X(t_m) - X(t_{m-1}) | X(t_1), \ldots, X(t_{m-1})\big)}{\mathrm{Var}\big(X(t_m) - X(t_{m-1})\big)}$$

and the infimum is taken over all ordered points $t_1 < t_2 < \cdots < t_m$ *in* J *with* $t_m - t_1 \leq \varepsilon$.

Cuzick (1978) defined local ϕ-nondeterminism by replacing the variance $\sigma^2(t_m, t_{m-1})$ by $\phi(t_m - t_{m-1})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\phi(0) = 0$ and $\phi(r) > 0$ if $r > 0$.

- Berman (1973) showed that (1) is equivalent to: for every integer $m \geq 2$, there exist positive constants c_m and ε such that

$$\mathrm{Var}\Big(\sum_{k=1}^{m} u_k \left(X(t_k) - X(t_{k-1})\right)\Big) \geq c_m \sum_{k=1}^{m} u_k^2 \sigma^2(t_{k-1}, t_k) \tag{2}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in J with $t_m - t_1 < \varepsilon$ and $u_k \in \mathbb{R}$ $(k = 1, \ldots, m)$.

- Pitt (1978) used (2) to define local nondeterminism of a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ by introducing a partial order among $t_1, \ldots, t_m \in \mathbb{R}^N$.
- Another definition is given by Cuzick (1982).

Applications of LND:

(1) joint continuity and Hölder conditions of the local times of a large class of Gaussian processes; see Berman (1972, 1973, 1978, 1988), Pitt (1978), Geman and Horowitz (1980), etc.

(2) existence and regularity of intersection local times; Geman and Horowitz and Rosen (1984), Rosen (1984), Mountford (1989), and Berman (1991).

(3) Hausdorff dimension of the multiple times of Gaussian random fields; Cuzick (1982), Xiao (1995).

(4) Hausdorff dimensions of the image and level sets of Gaussian processes; Kahane (1985), Monrad and Pitt (1987).

(5) moments of the zero crossing number of a stationary Gaussian process; see Cuzick (1978).

However, it is known that the local nondeterminism is not enough for establishing fine regularity properties such as the law of the iterated logarithm and the modulus of continuity for the local times of Gaussian processes.

2 Strong local ϕ-nondeterminism (SLϕND)

Definition 2.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $0 < \mathbb{E}[X(t)^2] < \infty$ for $t \in J$, where $J \subseteq \mathbb{R}^N$ is a hyper-rectangle. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function with $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Then X is said to be* strongly locally ϕ-nondeterministic *(SLϕND) on J if there exist positive constants c_0 and r_0 such that for all $t \in J$ and all $0 < r \le \min\{|t|, r_0\}$,*

$$\mathrm{Var}\big(X(t)|X(s) : s \in J,\ r \le |s-t| \le r_0\big) \ge c_0\, \phi(r). \tag{3}$$

For Gaussian processes (i.e. $N = 1$), this definition was essentially given by Cuzick and DuPreez (1982). For Gaussian random fields, Definition 2.1 is more general than the definition of *strong local α-nondeterministism* of Monrad and Pitt (1987).

Applications of SLϕND:

(1) Sharp Hölder conditions for the local times; Cuzick and DuPreez (1982), Xiao (1997a).

(2) tail probability of the local times; Kasahara et al. (1999).

(3) small ball probability estimates; Monrad and Rootzén (1995) and Shao and Wang (1995).

(4) exact Hausdorff measure of the range and graph of X; Talagrand (1995) and Xiao (1996, 1997a, b).

3 Sufficient conditions for SLϕND

3.1 Stationary processes

For a stationary Gaussian process $X = \{X(t), t \in \mathbb{R}\}$ in $\mathbb{R}$ with spectral measure F, Cuzick and DuPreez (1982) have proved that if the absolutely continuous part of $dF(\lambda)$ has the property that

$$\frac{dF(\lambda/r)}{\phi(r)} \geq h(\lambda)d\lambda \qquad \forall 0 < r \leq r_0 \tag{4}$$

and

$$\int_0^\infty \frac{\log h(\lambda)}{1+\lambda^2}\, d\lambda > -\infty, \tag{5}$$

then X is strongly locally ϕ-nondeterministic.

Their proof uses the ideas from Cuzick (1977) and relies on the special properties of stationary Gaussian processes.

3.2 Gaussian random fields with stationary increments

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with stationary increments and $X(0) = 0$. According to Yaglom (1957), its covariance function $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda) + \langle s, Qt\rangle, \tag{6}$$

where $\langle x, y\rangle$ is the ordinary scalar product in $\mathbb{R}^N$, Q is an $N \times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\, \Delta(d\lambda) < \infty. \tag{7}$$

The measure Δ is called the *spectral measure* of X. We will always assume $Q = 0$. Hence X has the following stochastic integration representation:

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1) W(d\lambda) \tag{8}$$

and

$$\sigma^2(h) = \mathbb{E}\big[\big(X(t+h) - X(t)\big)^2\big] = 2 \int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\ \Delta(d\lambda). \tag{9}$$

Question: When is X strong locally ϕ-nondeterministic?

Two Known Results:

(1). A (standard) fractional Brownian motion B_H of index H $(0 < H < 1)$ is a centered, real-valued Gaussian random field with covariance function

$$\mathbb{E}\Big(B_H(t)B_H(s)\Big) = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big).$$

The spectral measure of B_H has a density given by

$$f_H(\lambda) = c(H, N)\, \frac{1}{|\lambda|^{2H+N}},$$

where $c(H,N) > 0$ is a normalizing constant. Pitt (1978) has showed that a fractional Brownian motion $B_H = \{B_H(t), t \in \mathbb{R}^N\}$ is SLϕND with $\phi(r) = r^{2H}$.

(2). Marcus (1969) and Berman (1978) showed that if $N = 1$ and $\sigma^2(h)$ is concave on $(0, \delta)$ for some $\delta > 0$, then X is *one-sided* strongly locally ϕ-nondeterministic for $\phi(r) = \sigma^2(r)$.

Theorem 3.1 [Xiao (2005)] *Let f be the density function of the absolutely continuous part of the spectral measure Δ of X. Assume that there exist two functions $\phi(r) : \mathbb{R}_+ \to \mathbb{R}_+$ and $q(\lambda) : \mathbb{R}^N \to \mathbb{R}_+$ satisfying the following conditions: $\phi(0) = 0$, $\phi(r) > 0$ for $r > 0$,*

$$\frac{f(\lambda/r)}{\phi(r)} \geq \frac{r^N}{q(\lambda)}, \qquad \forall\, r \in (0, 1] \text{ and } \lambda \in \mathbb{R}^N \tag{10}$$

and there exist positive constants c_1 and η such that

$$q(\lambda) \leq c_1\, |\lambda|^{\eta} \quad \forall\, \lambda \in \mathbb{R}^N \text{ with } |\lambda| \text{ lagrge.} \tag{11}$$

Then for every hypercube $I = [-T, T]^N$, there exists a constant $0 < c_2 < \infty$ such that for all $t \in I \backslash \{0\}$ and all $0 < r \leq \min\{1, |t|\}$,

$$\mathrm{Var}\Big(X(t)\big|X(s) :\ s \in I,\ |s - t| \geq r\Big) \geq c_2\, \phi(r). \tag{12}$$

Proof It is sufficient to show that there exists a constant c_2 such that for every $t \in I\backslash\{0\}$ and $0 < r \leq \min\{1, |t|\}$, the inequality

$$\mathbb{E}\bigg(X(t) - \sum_{k=1}^{n} a_k X(s_k)\bigg)^2 \geq c_2\, \phi(r) \tag{13}$$

holds for all integers $n \geq 1$, all $a_k \in \mathbb{R}$ and $s_k \in [-T,\, T]^N$ satisfying $|s_k - t| \geq r,\ (k = 1, 2, \ldots, n)$.

This relies on two observations:

- SLND is determined by the behavior of the spectral measure Δ at infinity;
- use the Fourier analytic argument of Kahane (1985) and Pitt (1978).

To connect $\phi(|h|)$ with the function $\sigma^2(h)$, we assume that the spectral measure Δ is absolutely continuous and its density function $f(\lambda)$ satisfies the following condition [when $N = 1$, this is due to Berman (1988)]:

$$0 < \underline{\alpha} = \frac{1}{2} \liminf_{\lambda \to \infty} \frac{\beta_N \, |\lambda|^N f(\lambda)}{\Delta\{\xi : |\xi| \geq |\lambda|\}} \leq \frac{1}{2} \limsup_{\lambda \to \infty} \frac{\beta_N \, |\lambda|^N f(\lambda)}{\Delta\{\xi : |\xi| \geq |\lambda|\}} = \overline{\alpha} < 1, \tag{14}$$

where $\beta_1 = 2$ and for $N \geq 2$, $\beta_N = \mu(S^{N-1})$ is the area of S^{N-1}.

Define $\phi(r) = \Delta\{\xi : |\xi| \geq r^{-1}\}$ and $\phi(0) = 0$. Then the function ϕ is non-decreasing and continuous on $[0, \infty)$.

Theorem 3.2 [Xiao (2005)] *Under the assumption (14), we have*

$$0 < \liminf_{h \to 0} \frac{\sigma^2(h)}{\phi(|h|)} \leq \limsup_{h \to 0} \frac{\sigma^2(h)}{\phi(|h|)} < \infty. \tag{15}$$

Moreover, for every $T > 0$, X is strongly locally ϕ-nondeterministic on the hypercube $[-T, T]^N$.

Example 3.1 [Benassi, Jaffard and Roux (1997), Bonami and Estrade (2003)] Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a mean zero Gaussian random field with stationary increments and spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{s^2(\lambda/|\lambda|)}{|\lambda|^{2H+N}},$$

where $H \in (0, 1)$ and s is an even square integrable function on S^{N-1}.

Example 3.2 [Anh et al. (1999, 2000, 2001)] Suppose $X = \{X(t),\, t \in \mathbb{R}^N\}$ has the spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{c(\gamma, \beta, N)}{|\lambda|^{2\gamma}(1 + |\lambda|^2)^\beta},$$

where γ and β are constants satisfying

$$\beta + \gamma > \frac{N}{2}, \quad 0 < \gamma < 1 + \frac{N}{2}$$

and $c(\gamma, \beta, N) > 0$ is a normalizing constant. It is called the *fractional Riesz-Bessel motion with indices β and γ.* Anh et al. (2000, 2001) have shown that these Gaussian random fields can be used for modelling simultaneously long range dependence and intermittency.

It is easy to check that

- Condition (14) is satisfied with $\underline{\alpha} = \overline{\alpha} = \gamma + \beta - \frac{N}{2}$.
- since the spectral density $f_{\gamma,\beta}(x)$ is regularly varying at infinity of order $2(\beta+\gamma) > N$, a result of Pitman (1968) implies that, if $\gamma+\beta-\frac{N}{2} < 1$, then $\sigma(h)$ is regularly varying at 0 of order $\gamma + \beta - N/2$:

$$\sigma(h) \sim c_{\gamma,\beta}\, |h|^{\gamma+\beta-N/2} \qquad \text{as } \ h \to 0.$$

- Theorem 3.2 implies that X is SLND with respect to $\sigma^2(h)$.

The following defines a class of Gaussian random fields for which (14) does not hold, but Theorem 3.1 is applicable.

Example 3.3 For any given constants $0 < \alpha_1 < \alpha_2 < 1$ and any increasing sequence $\{b_n\}$ of real numbers such that $b_0 = 0$ and $b_n \to \infty$, define the function f on $\mathbb{R}^N$ by

$$f(\lambda) = \begin{cases} |\lambda|^{-(2\alpha_1+N)} & \text{if } |\lambda| \in (b_{2k}, b_{2k+1}], \\ |\lambda|^{-(2\alpha_2+N)} & \text{if } |\lambda| \in (b_{2k+1}, b_{2k+2}]. \end{cases}$$

Using the functions f as spectral densities, we obtain a quite large class of Gaussian random fields with stationary increments that are different from the fractional Brownian motion. If X is such a random field, then it can be shown that there exist positive constants c_3 and c_4 such that

$$c_3^{-1} \, |h|^{2\alpha_2} \le \sigma^2(h) \le c_4 \, |h|^{2\alpha_1} \qquad \forall \, |h| \le 1$$

and

$$c_4^{-1} \, r^{2\alpha_2} \le \phi(r) \le c_4 \, r^{2\alpha_1} \qquad \forall \, 0 < r \le 1.$$

We can choose $\{b_n\}$ such that the following properties hold:

(i) $\phi(r) \asymp r^{2\alpha_2}$ for $r \in (0, 1)$.

(ii) $\sigma^2(h) \asymp |h|^{2\alpha_2}$ for all $h \in \mathbb{R}^N$ with $|h| \le 1$.

(iii) Condition (14) is not satisfied.

(iv) the corresponding Gaussian random field X is SLϕND on all hypercubes $I = [-T, T]^N$.

We have shown that many of the sample path properties of X, such as the small ball probabilities, Hausdorff dimension and LIL for the local times, are completely determined by the index α_2.

4 Anisotropic random fields: Sectorial LND

4.1 Examples of anisotropic Gaussian random fields

- Fractional Brownian sheets: An $(N,1)$-fractional Brownian sheet $W_0^H = \{W_0^H(t), t \in \mathbb{R}^N\}$ with Hurst index $H = (H_1, \ldots, H_N) \in (0,1)^N$ is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbb{E}\big[W_0^H(s)W_0^H(t)\big] = \prod_{j=1}^{N} \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big), \quad s,t \in \mathbb{R}^N.$$

- Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with mean 0, stationary increments and spectral density function

$$f(\lambda) = \frac{1}{\sum_{j=1}^N |\lambda_j|^{2\beta_j + N}}, \qquad \forall \lambda \in \mathbb{R}^N,$$

where $\beta_j \in (0,1)$ for $j = 1, \ldots, N$; see Bonami and Estrade (2003).

- Funaki's model for random string in $\mathbb{R}^d$ is specified by the following stochastic PDE:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \dot{W},$$

where $\dot{W}\,(x,t)$ is an $\mathbb{R}^d$ valued space-time white noise. One solution is given by

$$\begin{aligned} U(t,x) = \int_0^\infty \int_{\mathbb{R}} \Big(G(t+r, x-z) - G(t+r, z)\Big)\, \tilde{W}(dz\, dr) \\ + \int_0^t \int_{\mathbb{R}} G(r, x-z) W(dz\, dr), \end{aligned} \tag{16}$$

where $G(r,x) = (4\pi r)^{-1/2} \exp\big(-x^2/(4r)\big)$, $\tilde{W}$ and W are independent; see Funaki (1983), Mueller and Tribe (2002).

Recall that a random field X is called *operator self-similar* if there exists a linear operator A on $\mathbb{R}^N$ such that for all $c > 0$,

$$\{c^{-1} X(c^A t),\ t \in \mathbb{R}^N\} \stackrel{d}{=} \{X(t),\ t \in \mathbb{R}^N\}.$$

All the above three Gaussian random fields are operator self-similar with exponent $A = (a_{ij})$, where $a_{ii} = H_i^{-1}$ and $a_{ij} = 0$ if $i \neq j$.

Moreover, their sample functions share many geometric properties.

4.2 Sectorial LND of fractional Brownian sheets

• Moving average representation

$$W_0^H(t) = \kappa_{_H}^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} g(t,s) W(ds), \tag{17}$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard real-valued Brownian sheet and

$$g(t,s) = \prod_{j=1}^N \Big[\big((t_j - s_j)_+\big)^{H_j - 1/2} - \big((-s_j)_+\big)^{H_j - 1/2} \Big]$$

with $s_+ = \max\{s, 0\}$, and where $\kappa_{_H}$ is the normalizing constant.

• Harmonizable representation

$$W_0^H(t) = K_{_H}^{-1} \int_{\mathbb{R}^N} \psi_t(\lambda) \widehat{W}(d\lambda), \tag{18}$$

where $\widehat{W}$ is the Fourier transform of white noise in $\mathbb{R}^N$ and

$$\psi_t(\lambda) = \prod_{j=1}^N \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}},$$

where $K_{_H}$ is the normalizing constant; see Herbin (2004).

Fractional Brownian sheets are not LND. Khoshnevisan and Xiao (2004) have proved that Brownian sheet has the property of *sectorial local non-determinism* (in short, SeLND). Wu and Xiao (2005) have extended it to fractional Brownian sheets by using their harmonizable representation.

Theorem 4.1 (Sectorial LND) *For all positive real numbers ε, there exists a positive constant c_5 depending on ε, H and N only, such that for all positive integers $n \geq 1$, and all $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\mathrm{Var}\left(W_0^H(u) \,\middle|\, W_0^H(t^1), \ldots, W_0^H(t^n)\right) \geq c_5 \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j}, \quad (19)$$

where $t^0 = 0$.

4.3 Geometry of fractional Brownian sheets

Let $W_1^H, \ldots, W_d^H$ be d independent copies of W_0^H. Then **the** (N, d)-**fractional Brownian sheet** with Hurst index $H = (H_1, \ldots, H_N)$ is the Gaussian random field $W^H = \{W^H(t) : \ t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$W^H(t) = \left(W_1^H(t), \ldots, W_d^H(t)\right), \quad t \in \mathbb{R}^N. \quad (20)$$

We study the fractal properties of

- the range $W^H\big([0,1]^N\big) = \big\{W^H(t) : \ t \in [0,1]^N\big\}$
- the graph $\mathrm{Gr}W^H\big([0,1]^N\big) = \big\{(t, W^H(t)) : \ t \in [0,1]^N\big\}$
- the level set $L_x = \big\{t \in (0,\infty)^N : W^H(t) = x\big\}, \quad x \in \mathbb{R}^d$.

For convenience, we assume

$$0 < H_1 \leq \ldots \leq H_N < 1. \tag{21}$$

Theorem 4.2 [Ayache and Xiao (2004)] *Let* $W^H = \{W^H(t),\ t \in \mathbb{R}^N_+\}$ *be an* (N, d)*-fractional Brownian sheet with Hurst index* $H = (H_1, \ldots, H_N)$ *satisfying (21). Then with probability 1,*

$$\dim_{\rm H} W^H\big([0,1]^N\big) = \min\left\{d; \quad \sum_{j=1}^N \frac{1}{H_j}\right\}$$

and

$$\dim_{\rm H} {\rm Gr} W^H\big([0,1]^N\big) = \min_{1\leq k\leq N}\left\{\sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1-H_k)d;\ \sum_{j=1}^N \frac{1}{H_j}\right\},$$

where $\sum_{j=1}^0 \frac{1}{H_j} := 0$.

Theorem 4.3 [Ayache and Xiao (2004)] *Let* $W^H = \{W^H(t),\ t \in \mathbb{R}^N_+\}$ *be an* (N, d)*-fractional Brownian sheet with Hurst index* $H = (H_1, \ldots, H_N)$ *satisfying (21). If* $\sum_{j=1}^N 1/H_j < d$ *then for every* $x \in \mathbb{R}^d$, $L_x = \emptyset$ *a.s. If* $\sum_{j=1}^N 1/H_j > d$*, then for any* $x \in \mathbb{R}^d$ *and* $0 < \varepsilon < 1$*, with positive probability*

$$\begin{aligned}\dim_{\rm H}\big(L_x \cap [\varepsilon, 1]^N\big) &= \min\left\{\sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d,\ 1 \leq k \leq N\right\} \\ &= \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d, \quad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^k \frac{1}{H_j}.\end{aligned}$$

Theorem 4.3 suggests that if $\sum_{j=1}^N 1/H_j > d$, then W^H has a local time. The following result on the joint continuity of the local times was proved by Xiao and Zhang (2002) under the extra condition $H_j d < 1$ for $j = 1, \ldots, N$.

The present form is due to Ayache, Wu and Xiao (2005). The key is to apply SeLND.

Theorem 4.4 [Ayache, Wu and Xiao (2005)] *If $d < \sum_{j=1}^{N} 1/H_j$, then for all closed intervals $T \subset (0,\infty)^N$, W^H has a jointly continuous local time on T.*

We also prove the following sharp Hölder conditions for the local times.

Theorem 4.5 [Ayache, Wu and Xiao (2005)] *Let $W^H = \{W^H(t),\ t \in \mathbb{R}_+^N\}$ be a fractional Brownian sheet in $\mathbb{R}^d$ with Hurst index $H = (H_1, \ldots, H_N)$. We assume that for some integer $\tau \in \{1, \ldots, N\}$,*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{22}$$

Let L be its jointly continuous local time, and for an interval $I \subset (0,\infty)^N$, write $L^*(I) = \sup_{x\in\mathbb{R}^d} L(x, I)$. Then, there are finite constants c_6 and c_7 such that for every $t \in (0,\infty)^N$ and every interval $T \subset (0,\infty)^N$

$$\limsup_{r\to 0} \frac{L^*([t - \langle r\rangle, t + \langle r\rangle])}{r^{\beta_\tau}(\log\log r^{-1})^{N-\beta_\tau}} \le c_6, \text{ a.s.} \tag{23}$$

and

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*([t - \langle r\rangle, t + \langle r\rangle])}{r^{\beta_\tau}(\log r^{-1})^{N-\beta_\tau}} \le c_7, \quad \text{a.s.}, \tag{24}$$

where $\beta_\tau = N - \tau - H_\tau d + \sum_{\ell=1}^{\tau} H_\tau/H_\ell$ is the Hausdorff dimension of the level set $\Gamma_x = \{t \in (0,\infty)^N : B^H(t) = x\}$.

5 Self-similar stable random fields

The class of symmetric α-stable (SαS) self-similar processes and random fields is very large; see Samorodnitsky and Taqqu (1994). Of special interest are the following

- linear fractional stable motions
- harmonizable fractional stable motion
- the stable sheet
- fractional stable sheets

The first two are stable analogues of fractional Brownian motion, and the later two are stable analogues of the Brownian sheet and fractional Brownian sheets. See Taqqu and Wolpert (1983), Maejima (1983), Cambanis and Maejima (1989), Ehm (1981), Xiao (2005).

The notion of local nondeterminism has been extended to SαS random fields by Nolan (1988, 1989), which is useful in studying the local times and self-intersection local times of certain self-similar stable processes with stationary increments. See Kôno and Shieh (1993), Shieh (1993) and Xiao (1995).

- no covariance to measure dependence of $X(t^1), \ldots, X(t^n)$.
- Nolan (1989) relies on the L^α-representations of symmetric α-stable random fields [Hardin (1982) or Samorodnitsky and Taqqu (1994)] and the approximation properties of normed or quasi-normed linear spaces.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α)-stable field. Then there exists a family of functions $\{\kappa(t, \cdot), t \in \mathbb{R}^N\}$ in the space $L^\alpha(E, \mathcal{B}, \mu; \mathbb{R}^d)$ such that

$$\mathbb{E} \exp\Big(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) = \exp\Big(-\Big\|\sum_{j=1}^n \langle u^j, \kappa(t^j)\rangle\Big\|_\alpha^\alpha\Big). \tag{25}$$

Nolan (1989) defines LND of X in terms of the family $\{\kappa(t, \cdot), t \in \mathbb{R}^N\}$.

Definition 5.1 *An (N, d, α)-stable field X is called locally nondeterministic on an interval $T \subset \mathbb{R}^N$ if its representation $\{\kappa(t, \cdot), t \in \mathbb{R}^N\}$ satisfies the following conditions:*

(a) $\|\kappa_j(t)\|_\alpha > 0$ *for all* $t \in T$ *and* $j = 1, \ldots, d$.

(b) $\|\kappa_j(s) - \kappa_j(t)\|_\alpha > 0$ *for all* $s, t \in T$ *with* $|s - t|$ *sufficiently small and* $j = 1, \ldots, d$.

(c) For all integers $n \geq 1$, arbitrary $t^1, \ldots, t^n \in T$ and all $j = 1, \ldots, d$, define M_j^n to be the subspace of $L^\alpha(E, \mathcal{B}, \mu)$ spanned by $\{\kappa_l(t^k) : 1 \leq l \leq d, 1 \leq k \leq n$ and $(l, k) \neq (j, n)\}$. Then for all $j = 1, \ldots, d$,

$$\inf_{t^1 \in T} \frac{\|\kappa_j(t^1) - M_j^1\|_\alpha}{\|\kappa_j(t^1)\|_\alpha} > 0 \tag{26}$$

and

$$\liminf \frac{\|\kappa_j(t^n) - M_j^n\|_\alpha}{\|\kappa_j(t^n) - \kappa_j(t^{n-1})\|_\alpha} > 0, \tag{27}$$

where the liminf is taken over all "ordered" $t^1, \ldots, t^n \in T$ with $|t^n - t^1| \to 0$, and $\|\kappa_j(t^n) - M_j^n\|_\alpha$ denotes the "L^α-distance" between $\kappa_j(t^n)$ and M_j^n.

Similar to Gaussian random fields, there are several different senses of strong local nondeterminism for (N, d, α)-stable fields.

For simplicity, we start by considering only isotropic $(N, 1, \alpha)$-stable fields with stationary increments. We define the strong local ϕ-nondeterminism for such SαS random fields as follows.

Definition 5.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $(N, 1, \alpha)$-stable field with stationary increments and $X(0) = 0$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$ and let $T \subset \mathbb{R}^N$. Then X is said to be* strongly locally ϕ-nondeterministic *(SLϕND) on T if, in addition to (a) and (b) in Definition 5.1, there exists a constant $c_8 > 0$ such that for all integers $n \geq 1$, all $t, s^1, \ldots, s^n \in T$ sufficiently close,*

$$\|\kappa(t) - M^n\|_\alpha^\alpha \geq c_8\, \phi\big(\min_{0 \leq j \leq n} |t - s^j|\big), \tag{28}$$

where M^n denotes the subspace of $L^\alpha(E, \mathcal{B}, \mu)$ spanned by $\{\kappa(s^1), \ldots, \kappa(s^n)\}$ and $s^0 = 0$.

We have proven that

- harmonizable fractional $(N, 1, \alpha)$-stable fields with $\alpha \in [1, 2]$ and Hurst index $H \in (0, 1)$
- linear fractional stable motions with $\alpha \in (0, 2]$ and Hurst index $H \in (0, 1)$

are strongly locally ϕ-nondeterministic with $\phi(r) = r^H$.