# Sample Path Properties of Anisotropic Gaussian Random Fields

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## Contents

# 1 Examples of anisotropic Gaussian random fields

- Fractional Brownian sheets: An $(N,1)$-fractional Brownian sheet $W_0^H = \{W_0^H(t), t \in \mathbb{R}^N\}$ with Hurst index $H = (H_1, \ldots, H_N) \in (0,1)^N$ is a mean 0 Gaussian random field in $\mathbb{R}$ with covariance function
$$\mathbb{E}\Big[W_0^H(s)W_0^H(t)\Big] = \prod_{j=1}^{N} \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big)$$
for all $s, t \in \mathbb{R}^N$.

- Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with mean 0, stationary increments can be represented by
$$X(t) = \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\, W(d\lambda),$$
where $W(\cdot)$ is a centered complex-valued Gaussian random measure with control measure $\Delta$. The measure $\Delta$ and its density (if it exists) $f(\lambda)$ are called the spectral measure and spectral density of $X$, respectively. For example,
$$f(\lambda) = \frac{1}{\Big(\sum_{j=1}^{N} |\lambda_j|^{2H_j}\Big)^{\beta}}, \qquad \forall \lambda \in \mathbb{R}^N,$$
where $H_j \in (0,1)$ for $j = 1, \ldots, N$ and $\beta$ are certain constants; see Bonami and Estrade (2003).

- Funaki's model for random string in $\mathbb{R}^d$ is specified by the following stochastic PDE:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \dot{W},$$

where $\dot{W}\ (x,t)$ is an $\mathbb{R}^d$-valued space-time white noise. One solution is given by

$$\begin{aligned} U(t,x) = &\int_0^\infty \int_{\mathbb{R}} \Big( G(t+r, x-z) - G(t+r, z) \Big)\, \tilde{W}(dz\, dr) \\ &+ \int_0^t \int_{\mathbb{R}} G(r, x-z) W(dz\, dr), \end{aligned} \tag{1.1}$$

where $G(r,x) = (4\pi r)^{-1/2} \exp\big( - x^2/(4r)\big)$, $\tilde{W}$ and $W$ are independent; see Funaki (1983), Mueller and Tribe (2002).

Recall that a random field $X$ is called *operator-self-similar* if there exists a linear operator $A$ on $\mathbb{R}^N$ such that for all $c > 0$,

$$\{c^{-1}\, X(c^A t),\ t \in \mathbb{R}^N\} \overset{d}{=} \{X(t),\ t \in \mathbb{R}^N\}.$$

All the above three Gaussian random fields are operator self-similar with exponent $A = (a_{ij})$, where $a_{ii} = H_i^{-1}$ and $a_{ij} = 0$ if $i \neq j$.

Moreover, their sample functions share many geometric properties.

## 2 Local nondeterminsim and sectorial LND

### 2.1 Berman's local nondeterminism

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Gaussian process with mean 0 and let $J \subset \mathbb{R}_+$ be an interval.

**Definition 2.1** [Berman (1973)] $X$ *is called locally nondeterministic on* $J$ *if for every integer* $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{2.1}$$

*where* $V_m$ *is the relative prediction error:*

$$V_m = \frac{\mathrm{Var}\big(X(t_m) - X(t_{m-1}) | X(t_1), \ldots, X(t_{m-1})\big)}{\mathrm{Var}\big(X(t_m) - X(t_{m-1})\big)}$$

*and the infimum is taken over all ordered points* $t_1 < t_2 < \cdots < t_m$ *in* $J$ *with* $t_m - t_1 \leq \varepsilon$.

Berman (1973) showed (2.1) is equivalent to: for every integer $m \geq 2$, there exist positive constants $c_m$ and $\varepsilon$ such that

$$\begin{aligned} &\mathrm{Var}\Big(\sum_{k=1}^{m} u_k \left(X(t_k) - X(t_{k-1})\right)\Big) \\ &\qquad \geq c_m \sum_{k=1}^{m} u_k^2 \,\mathrm{Var}\big(X(t_k) - X(t_{k-1})\big) \end{aligned} \tag{2.2}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in $J$ with $t_m - t_1 < \varepsilon$ and $u_k \in \mathbb{R}$ $(k = 1, \ldots, m)$.

## 2.2 Strong local $\phi$-nondeterminism (SL$\phi$ND)

**Definition 2.2** *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be a real-valued, centered Gaussian random field and let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function with* $\phi(0) = 0$ *and* $\phi(r) > 0$ *for* $r > 0$*. Then* $X$ *is said to be* strongly locally $\phi$-nondeterministic *(SL$\phi$ND) on a rectangle* $J \subset \mathbb{R}^N$ *if there exist positive constants* $c_{2,1}$ *and* $r_0$ *such that for all* $t \in J$ *and all* $0 < r \le \min\{|t|,\ r_0\}$*,*

$$\mathrm{Var}\big(X(t)|X(s) : s \in J,\ r \le |s-t| \le r_0\big) \ge c_{2,1}\, \phi(r). \quad (2.3)$$

- If $N = 1$, this definition was essentially given by Cuzick and DuPreez (1982) to prove joint continuity of local times of certain Gaussian processes.
- When $N \ge 1$, Definition 2.2 is more general than the *strong local $\alpha$-nondeterministism* of Monrad and Pitt (1987).

**Applications of SL$\phi$ND**:

(1) sharp Hölder conditions for the local times; Xiao (1997a).

(2) tail probability of the local times of certain fractional Brownian motion; Kasahara et al. (1999).

(3) small ball probability estimates; Monrad and Rootzén (1995) and Shao and Wang (1995).

(4) exact Hausdorff measure of the range and graph of $X$; Talagrand (1995) and Xiao (1996, 1997a, b).

Sufficient conditions for SLND of Gaussian random fields with stationary increments have been obtained in Xiao (2005).

### 2.3 Sectorial local nondeterminism

Note that in $(2.3)$ requires that $X$ has certain isotropic property.

- The Brownian sheet $W = \{W(t), t \in \mathbb{R}_+^N\}$ does not satisfy $(2.3)$.
- Khoshnevisan and Xiao (2004) proved that $W$ satisfies the *sectorial local nondeterminism*: Let $a > 0$ be fixed. Then for all integers $n \geq 1$ and all $u$, $t^1, \ldots, t^n \in [a, \infty)^N$, we have

$$\begin{aligned} &\mathrm{Var}\big(W(u) \mid W(t^1), \ldots, W(t^n)\big) \\ &\qquad \geq \frac{a^{N-1}}{2} \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|, \end{aligned} \tag{2.4}$$

  where $t_j^0 = 0$ for every $j = 1, \ldots, N$.

## 3 Sectorial LND of fractional Brownian sheets

- **Moving average representation**

$$W_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} g(t,s) W(ds), \tag{3.1}$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard real-valued Brownian sheet and

$$g(t,s) = \prod_{j=1}^{N} \Big[ \big((t_j - s_j)_+\big)^{H_j - 1/2} - \big((-s_j)_+\big)^{H_j - 1/2} \Big]$$

with $s_+ = \max\{s, 0\}$, and where $\kappa_H > 0$ is a normalization constant.

- **Harmonizable representation**

$$W_0^H(t) = K_H^{-1} \int_{\mathbb{R}^N} \psi_t(\lambda) \widehat{W}(d\lambda), \tag{3.2}$$

where $\widehat{W}$ is the Fourier transform of white noise in $\mathbb{R}^N$ and

$$\psi_t(\lambda) = \prod_{j=1}^{N} \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}},$$

where $K_H > 0$ is a constant.

Wu and Xiao (2005) have extended the result of Khoshnevisan and Xiao (2004) to fractional Brownian sheets by using their harmonizable representation.

**Theorem 3.1** [Wu and Xiao (2005)]

*For $\varepsilon > 0$, there exists a positive constant $c_{3,1}$, depending on $\varepsilon, H$ and $N$ only, such that for all $n \geq 1$, and all $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\begin{aligned}\mathrm{Var}\Big( W_0^H(u) \,\Big|\, W_0^H(t^1), \ldots, W_0^H(t^n)\Big) & \\ \geq c_{3,1} \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j}, & \end{aligned} \tag{3.3}$$

*where $t_j^0 = 0$ for every $j = 1, \ldots, N$.*

**Proof** Let $\ell \in \{1, \ldots, N\}$ be fixed, denote

$$r_\ell \equiv \min_{0 \leq k \leq n} \left|u_\ell - t_\ell^k\right|.$$

First, we prove there exists a constant $c_\ell > 0$ such that

$$\mathrm{Var}\left(W_0^H(u) \,\middle|\, W_0^H(t^1), \ldots, W_0^H(t^n)\right) \geq c_\ell \, r_\ell^{2H_\ell}. \tag{3.4}$$

This is equivalent to proving that for all $n \geq 1$ and $a_k \in \mathbb{R}$ $(1 \leq k \leq n)$,

$$\mathbb{E}\left(W_0^H(u) - \sum_{k=1}^{n} a_k W_0^H(t^k)\right)^2 \geq c_\ell \, r_\ell^{2H_\ell}. \tag{3.5}$$

By the harmonizable representation of $W_0^H$, the left hand side of (3.5), up to a constant, can be written as

$$\begin{aligned}&\mathbb{E}\Big(W_0^H(u)-\sum_{k=1}^n a_k W_0^H(t^k)\Big)^2\\&=\int_{\mathbb{R}^N}\Big|\prod_{j=1}^N(e^{iu_j\lambda_j}-1)-\sum_{k=1}^n a_k\prod_{j=1}^N(e^{it_j^k\lambda_j}-1)\Big|^2 f_H(\lambda)d\lambda,\end{aligned}\tag{3.6}$$

where

$$f_H(\lambda)=\prod_{j=1}^N|\lambda_j|^{-2H_j-1}.$$

Hence, it suffices to show

$$\begin{aligned}\int_{\mathbb{R}^N}\Big|\prod_{j=1}^N(e^{iu_j\lambda_j}-1)-\sum_{k=1}^n a_k\prod_{j=1}^N(e^{it_j^k\lambda_j}-1)\Big|^2 f_H(\lambda)\,d\lambda\\ \geq c_\ell\, r_\ell^{2H_\ell}.\end{aligned}\tag{3.7}$$

To finish the proof, we make use of Fourier analysis and the asymptotic properties of $f(\lambda)$ at infinity. □

## 4 Hausdorff dimensions of the range, graph and level sets

Let $W_1^H, \ldots, W_d^H$ be $d$ independent copies of $W_0^H$. The $(N, d)$-*fractional Brownian sheet* $W^H = \{W^H(t) : \ t \in \mathbb{R}^N\}$ is defined by

$$W^H(t) = \big(W_1^H(t), \ldots, W_d^H(t)\big), \quad t \in \mathbb{R}^N. \qquad (4.1)$$

We study the fractal properties of

- the range $W^H\big([0,1]^N\big) = \big\{W^H(t) : t \in [0,1]^N\big\}$
- the graph $\mathrm{Gr}W^H\big([0,1]^N\big) = \big\{(t, W^H(t)) : t \in [0,1]^N\big\}$
- the level set $\Gamma_x = \big\{t \in (0,\infty)^N : W^H(t) = x\big\}, \quad x \in \mathbb{R}^d$.

**Theorem 4.1** [Ayache and Xiao (2005)]

*Suppose* $0 < H_1 \le \ldots \le H_N < 1$. *With probability 1,*

$$\dim_{\mathrm{H}} W^H\big([0,1]^N\big) = \min\bigg\{d; \quad \sum_{j=1}^{N} \frac{1}{H_j}\bigg\},$$

$$\dim_{\mathrm{H}} \mathrm{Gr}W^H\big([0,1]^N\big)$$
$$= \min_{1 \le k \le N} \bigg\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1 - H_k)d; \ \sum_{j=1}^{N} \frac{1}{H_j} \bigg\},$$

*where* $\sum_{j=1}^{0} \frac{1}{H_j} := 0$.

**Theorem 4.2** [Ayache and Xiao (2005)]

*(i). If* $\sum_{j=1}^{N} \frac{1}{H_j} < d$ *then for every* $x \in \mathbb{R}^d$, $\Gamma_x = \emptyset$ *a.s.*

*(ii). If* $\sum_{j=1}^{N} \frac{1}{H_j} > d$, *then for any* $x \in \mathbb{R}^d$ *and* $0 < \varepsilon < 1$, *with positive probability*

$$\dim_{\rm H}\big(\Gamma_x \cap [\varepsilon, 1]^N\big) = \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \quad \textit{if} \ \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j}.$$

The proofs of Theorems 4.1 and 4.2 only rely on the two-point distribution of $(W^H(s), W^H(t))$.

- Conjecture: If $\sum_{j=1}^{N} \frac{1}{H_j} = d$ then for every $x \in \mathbb{R}^d$, $\Gamma_x = \emptyset$ a.s.
- Theorem 4.2 suggests: If $\sum_{j=1}^{N} \frac{1}{H_j} > d$, then $W^H$ has a local time.
- Regularity of local times relies on joint distribution of

$$(W^H(t^1), \ldots, W^H(t^n))$$

for all $n \ge 2$.

## 5 Hausdorff dimension of $W^H(E)$

**Question:** What is $\dim_{\mathrm{H}} W^H(E)$ for an arbitrary Borel set $E \subset (0,\infty)^N$ almost surely?

**Partial Answer 1.** If $E = (0,1]^N$ or having positive Lebesgue measure, then

$$\dim_{\mathrm{H}} W^H(E) = \min\left\{ d; \sum_{j=1}^{N} \frac{1}{H_j} \right\}, \quad \text{a.s.}$$

**Partial Answer 2.** If $E = E_1 \times E_2$ with $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{P}} E_1$, then

$$\dim_{\mathrm{H}} W^H(E) = \min\left\{ d; \frac{\dim_{\mathrm{H}} E_1}{H_1} + \frac{\dim_{\mathrm{H}} E_2}{H_2} \right\} \quad \text{a.s.}$$

**Observation:** Knowing $\dim_{\mathrm{H}} E$ alone is not enough to determine $\dim_{\mathrm{H}} W^H(E)$.

**A Counterexample:** Let $E = E_1 \times E_2$, $F = E_2 \times E_1$ with $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{P}} E_1$, then

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} F$$

but in general

$$\dim_{\mathrm{H}} W^H(E) \neq \dim_{\mathrm{H}} W^H(F).$$

## Hausdorff dimension contour

Let $\mathcal{M}_c^+(E)$ denote the family of finite Borel measures on $E$ with compact support in $E$. For $\mu \in \mathcal{M}_c^+(E)$, we define

$$\Lambda_\mu = \Bigg\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N : \\ \limsup_{r \to 0^+} \frac{\mu\left(R(t,r)\right)}{r^{\langle \lambda, H^{-1} \rangle}} = 0 \text{ for } \mu\text{-a.e. } t \in \mathbb{R}^N \Bigg\},$$

where $H^{-1} = \left( {H_1}^{-1}, \ldots, {H_N}^{-1} \right)$ and

$$R(t,r) = \prod_{j=1}^{N} \left[ t_j - r^{1/H_j},\ t_j + r^{1/H_j} \right].$$

**Key property**: For any $a \in (0,\infty)^N$, $\sup_{\lambda \in \Lambda_\mu} \langle a, \lambda \rangle$ is achieved on the boundary of $\Lambda_\mu$.

We call the boundary of $\Lambda_\mu$, denoted by $\partial \Lambda_\mu$, the *Hausdorff dimension contour* of $\mu$.

For a Borel set $E \subset (0,\infty)^N$, we define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_c^+(E)} \partial \Lambda_\mu.$$

**Theorem 5.1** [Wu and Xiao, 2005]

*For an arbitrary Borel set $E \subset (0,\infty)^N$,*

$$\dim_{\mathrm{H}} W^H(E) = \min\left\{ d;\ s(H,E) \right\}, \qquad \textit{a.s.}$$

*where $s(H,E) = \sup_{\lambda \in \Lambda(E)} \langle \lambda, H^{-1} \rangle$.*

**Natural Questions:**

If $s(H,E) \le d$, then $\dim_{\mathrm{H}} W^H(E) = s(H,E)$ a.s.

**Q1.** What more can be said about $W^H(E)$?

If $s(H,E) > d$, then $\dim_{\mathrm{H}} W^H(E) = d$ a.s.

**Q2.** Does $W^H(E)$ have interior points almost surely?

**Some history**:

Brownian motion: Kahane (1966), Kaufman (1975);

Fractional Brownian motion: Pitt (1978) and Kahane (1985a, 1985b);

Gaussian random fields with stationary increments: Shieh and Xiao (2004);

The Brownian sheet: Mountford (1989), Khoshnevisan and Xiao (2004), Khoshnevisan, Wu and Xiao (2005).

## 6 Salem set

◇ For $\beta > 0$, $F \subset \mathbb{R}^d$ is called an *$M_\beta$-set*, if $\exists\, \nu$ on $F$ such that

$$\widehat{\nu}(\xi) = o(|\xi|^{-\beta}) \quad \text{as } \xi \to \infty.$$

◇ The Fourier dimension of $F$:

$$\dim_{\rm F} F = \sup\big\{\gamma \ge 0 : \ F \text{ is an } \ M_{\gamma/2}\text{-set} \ \big\}.$$

We always have $\dim_{\rm F} F \le \dim_{\rm H} F$.

◇ $F$ is called a *Salem set* if

$$\dim_{\rm F} F = \dim_{\rm H} F.$$

**Answer to Q1:**

**Theorem 6.1** [Wu and Xiao, 2005]
*If $s(H,E) \le d$, then $W^H(E)$ is almost surely a Salem set with Fourier dimension $s(H,E)$.*

**Proof** Since $\dim_{\rm F} W^H(E) \le \dim_{\rm H} W^H(E) = s(H,E)$ a.s., we only need to prove $\dim_{\rm F} W^H(E) \ge s(H,E)$. This is done in the next theorem. □

Define $\mu_{W^H}(\bullet) = \mu\left\{t : W^H(t) \in \bullet\right\}$, the image measure of $\mu$ under the mapping $t \mapsto W^H(t)$.

The key of proving Theorem 6.1 is to find the asymptotic behavior of $\widehat{\mu_{W^H}}(\xi)$ as $\xi \to \infty$.

**Theorem 6.2** [Wu and Xiao, 2005]
*Let* $\tau_j : \mathbb{R}_+ \to \mathbb{R}_+$ *(*$j = 1, \ldots, N$*) be non-decreasing functions satisfying* $\tau_j(0) = 0$ *and*

$$\tau_j(2r) \le c_{6,1}\, \tau_j(r) \quad \textit{for all} \;\; r \ge 0.$$

*If* $\mu$ *is a Borel probability measure on* $[\varepsilon, T]^N$ *such that*

$$\mu\big(R(t,r)\big) \le c_{6,2} \prod_{j=1}^N \tau_j(r^{1/H_j}), \quad \forall\, t \in \mathbb{R}_+^N, \tag{6.1}$$

*where* $R(t,r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$*. Then there exists a constant* $\varrho > 0$ *such that almost surely,*

$$\limsup_{|\xi|\to\infty} \frac{\big|\widehat{\mu_{W^H}}(\xi)\big|}{\sqrt{\Big(\prod_{j=1}^N \tau_j\big(|\xi|^{-\frac{1}{H_j}}\big)\Big)\, \log^{\varrho} |\xi|}} < \infty. \tag{6.2}$$

## 7 Interior points

**Theorem 7.1** [Wu and Xiao, 2005] *If a Borel set $E \subset (0,\infty)^N$ carries a probability measure $\mu$ such that*

$$\sup_{t\in\mathbb{R}_+^N}\int_{\mathbb{R}_+^N}\frac{\log_+^{(N+1)\gamma}\left[\left(\sum_{j=1}^N |s_j-t_j|^{2H_j}\right)^{-1}\right]}{\left(\sum_{j=1}^N |s_j-t_j|^{2H_j}\right)^{d/2}}\,\mu(ds)\le c_{7,1} \qquad (7.1)$$

*for some finite constants $c_{7,1} > 0$ and $\gamma > N$, where $\log_+ x = \max\{1, \log x\}$. Then $W^H(E)$ has interior points almost surely.*

**Proof** We prove that $W^H$ has a local time on $E$, denoted by $L(x,E)$, which is continuous in $x$. Sectorial local nondeterminism plays a useful role in the proof. □

The following sufficient condition is given in terms of the Hausdorff dimension contour of $E$.

**Corollary 7.2** *If $E \subset (0,\infty)^N$ is a Borel set with $s(H,E) > d$, then $W^H(E)$ a.s. has interior points.*

Applying Theorem 7.1 to the Brownian sheet, we get a sufficient condition which improves that of Khoshnevisan ad Xiao (2004).

## 8 Local times of fBs

Let $T \subset (0,\infty)^N$ be an interval, and let $\mu_{_T}$ be the occupation measure of $W^H$ on $T$ defined by

$$\mu_{_T}(\bullet) = \lambda_N\big\{t \in T : \, W^H(t) \in \bullet\big\}.$$

If $\mu_{_T} \ll \lambda_d$, then $W^H$ is said to have a *local time* on $T$, defined by $L(x,T) \equiv d\mu_{_T}/d\lambda_d(x)$.

**Why Important**: If a local time $L(x,\bullet)$ is jointly continuous, it can be extended to be a finite Borel measure supported on the level set

$$\Gamma_x = \big\{t \in T : \, W^H(t) = x\big\}.$$

The following result on the joint continuity of the local times was proved by Xiao and Zhang (2002) under the extra condition $H_j d < 1$ for every $j = 1, \ldots, N$. The present form is due to Ayache, Wu and Xiao (2005). The key is to apply the sectorial local nondeterminism.

**Theorem 8.1** [Ayache, Wu and Xiao (2005)]
*If $d < \sum_{j=1}^N 1/H_j$, then for all closed intervals $T \subset (0,\infty)^N$, $W^H$ has a jointly continuous local time on $T$.*

We also prove the following sharp Hölder conditions for the local times.

**Theorem 8.2** [Ayache, Wu and Xiao (2005)]

*Let $L$ be its jointly continuous local time, and for an interval $I \subset T$, write $L^*(I) = \sup_{x \in \mathbb{R}^d} L(x, I)$. If for some integer $\tau \in \{1, \ldots, N\}$,*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{8.1}$$

*Then, there are finite constants $c_{8,1}$ and $c_{8,2}$ such that for every $t \in T$,*

$$\limsup_{r \to 0} \frac{L^*([t - \langle r \rangle, t + \langle r \rangle])}{r^{\beta_\tau} (\log\log r^{-1})^{N - \beta_\tau}} \le c_{8,1}, \ \text{a.s.} \tag{8.2}$$

*and*

$$\limsup_{r \to 0} \ \sup_{t \in T} \ \frac{L^*([t - \langle r \rangle, t + \langle r \rangle])}{r^{\beta_\tau} (\log r^{-1})^{N - \beta_\tau}} \le c_{8,2}, \qquad \text{a.s.}, \tag{8.3}$$

*where*

$$\beta_\tau = N - \tau - H_\tau d + \sum_{\ell=1}^{\tau} H_\tau / H_\ell$$

*is the Hausdorff dimension of the level set $\Gamma_x = \{t \in (0, \infty)^N : W^H(t) = x\}$.*