Exact Hausdorff Measure of the Graph of Brownian Motion on the Sierpiński Gasket

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Abstract

Let $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G. We prove that there exist two positive constants c and C such that for every $x \in G$, $\mathbb{P}^x$-a.s. for all $t \in [0, \infty)$, we have $ct \leq \varphi\text{-}m(\mathrm{Gr}(X[0,t])) \leq Ct$, where $\mathrm{Gr}X([0,t]) = \{(s, X(s)) : \ 0 \leq s \leq t\}$ is the graph set of X,

$$\varphi(s) = s^{1+\log 3/\log 2 - \log 3/\log 5} (\log\log 1/s)^{\log 3/\log 5}, \ s \in (0, 1/8]$$

and φ-m denotes Hausdorff φ-measure.

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1 Introduction

In recent years, there has been an increased interest in studying diffusions on fractals, (see Barlow [1], Kigami [16] and references therein). Initial interest in the properties of diffusion processes and random walks on fractals came from mathematical physicists working in the theory of disordered media. Goldstein [12] and Kusuoka [19] first constructed Brownian motion X on the Sierpiński gasket G, a connected fractal subset of $\mathbb{R}^2$. Barlow and Perkins [2] have systematically investigated the properties of X. They showed that the process X, like the standard Brownian motion, is a strong Markov process with a continuous symmetric transition density $p(t,x,y)$ with respect to the normalized Hausdorff measure on G. Barlow and Perkins [2] also studied the existence and joint continuity of local time of X and proved a result for the modulus of continuity in the space variable for the local time process.

Since $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ is a special fractional diffusion introduced in Barlow [1], by Lemma 3.27 of [1], $\dim_{\mathrm{H}}\{X(t) : 0 \leq t \leq 1\} = \dim_{\mathrm{H}} G = \log 3/\log 2$, where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Furthermore, since X is point recurrent, it follows that $0 < s^{\log 3/\log 2}\text{-}m\{X(t) : \ 0 \leq t \leq 1\} < \infty$, $\mathbb{P}^x$-a.s. for any $x \in G$. This suggests that X has a local time process. Indeed, Barlow and Perkins [2] proved that there is a version, $\{L_t^x, \ t \geq 0, x \in G\}$, of the local time, which is jointly continuous in (t,x) and satisfies the following occupation density formula: for any measurable function g on G,

$$\int_0^t g(X(s))ds = \int_G g(x) L_t^x \mu(dx),$$

where μ is a multiple of the Hausdorff $s^{\log 3/\log 2}$-measure on G. They further established the following modulus of continuity for L_t^x in the space variable x: for every $N > 0$ there exists $\delta_N(\omega) > 0$ such that

$$|L_t^x - L_t^y| < C_1 |x-y|^{\frac{1}{2}(d_w - d_f)} \log \frac{1}{|x-y|} \qquad \text{if } t \leq N \ \text{ and } |x-y| \leq \delta_N(\omega),$$

where $C_1 > 0$ is an absolute constant, d_f and d_w are the Hausdorff dimension and the walk dimension of G, respectively. See Theorem 1.11 in [2]. Based on these results, Zhou [38] evaluated the Hausdorff measure of the level sets of X.

In this paper, we consider the Hausdorff measure of the graph of X, $\mathrm{Gr}(X[0,t]) = \{(s, X(s)) : \ 0 \leq s \leq t\}$. This is a random subset of $\mathbb{R}_+ \times G$. Our main result is the following theorem.

Theorem 1.1 *Let $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G, $\varphi(s) = s^{1+\log 3/\log 2 - \log 3/\log 5}(\log\log \frac{1}{s})^{\log 3/\log 5}$, $s \in (0, \frac{1}{8}]$. Then there exist two* *positive and finite constants c and C such that for every $x \in G$, $\mathbb{P}^x$-a.s. for all $t \in (0, \infty)$, we have*

$$ct \le \varphi\text{-}m\,(\mathrm{Gr}X([0,t])) \le Ct,$$

where φ-m denotes Hausdorff φ-measure.

Many authors have investigated the problem of determining the exact Hausdorff measure functions for random sets associated with Lévy processes and Gaussian processes. While the argument for obtaining lower bounds using a density theorem of Rogers and Taylor [28] has become quite standard, the proofs of the upper bounds are often more difficult. In the following, we list some related references. The Hausdorff measure of the range and graph of Lévy stable processes were evaluated by Taylor [31], Jain and Pruitt [14] and Pruitt and Taylor [27], respectively. Their arguments for proving the upper bounds rely heavily on some special properties of Lévy processes such as stationary and independent increments, and results on hitting probabilities. To be more precise, in order to construct an economic covering for the range or graph of X, Taylor [31] (see also [32]) classified the points in the *state space* into "good" points and "bad" points, according to the sojourn times of the process near these points. Results on hitting probabilities are used to estimate the number of dyadic cubes that contain bad points. See Taylor [33] for an extensive summary of related results and techniques. Corresponding problems for fractional Brownian motion (FBM) were studied, using general Gaussian techniques, by Goldman [11], Talagrand [30] and Xiao [35, 36, 37]. In constructing economic coverings of the trajectories of fractional Brownian motion, Talagrand [30] divided the points in the *parameter space* into "good" times and "bad" times according to the local asymptotic behavior of FBM at these times. The advantage of Talagrand's approach is that no results on hitting probabilities of fractional Brownian motion, which are difficult to establish due to the fact that FBM is not Markovian, are needed. On the other hand, Talagrand's argument does not apply to processes with discontinuous sample paths such as Lévy processes.

Since Brownian motion X on the Sierpiński gasket G has continuous sample paths, we have found that Talagrand's approach is more convenient to use than that for Lévy processes. In particular, for evaluating the Hausdorff measure of the sample paths of X, there is no need to study hitting probabilities of X first. Furthermore, the Markov property of X makes it possible for us to adapt an argument of Taylor [31] to prove the key estimate (i.e. Lemma 3.3) in a simpler way than that in Talagrand [30].

From Theorem 1.1, we can get the following corollary immediately.

Corollary 1.2 *Let $X = \{X(t), t \ge 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G. Then for any $t > 0$, we have*

$$\dim_{\mathrm{H}}(\mathrm{Gr}X([0,t])) = 1 + \frac{\log 3}{\log 2} - \frac{\log 3}{\log 5}.$$

Remark 1.1 In light of Theorem 1.1, it is of some interest to ask the following question: Is there a positive and finite constant C such that for every $x \in G$,

$$\varphi\text{-}m\,(\mathrm{Gr}X([0,t])) = Ct \quad \text{for all } t \in (0,\infty),\ \mathbb{P}^x\text{-a.s.?}$$

We have not been able answer this question. For the Hausdorff measure of the range of Brownian motion, P. Lévy [20] suggested that the problem can be reduced to Kolmorogorov's zero-one law. Taylor and Wendel [34] gave a nice and simple argument for any Lévy process X, using the observation that $\varphi\text{-}m\,(X([0,t]))$ is again a Lévy process with continuous and increasing sample paths. Takashima [29] considered the problem for the Hausdorff measure of the range of an arbitrary ergodic self-similar process, but there seems to be a gap in the last part of his proof of Proposition 5.1 on page 185[1]. □

Remark 1.2 Even though in this paper, we only consider Brownian motion on the Sierpiński gasket, Theorem 1.1 can be extended to Brownian motion on the nested fractals without difficulty. □

For the sake of convenience, one usually introduces the notations:

$$d_f = \frac{\log 3}{\log 2},\ d_w = \frac{\log 5}{\log 2},\ d_s = \frac{2\log 3}{\log 5},\ \gamma = \frac{1}{d_w}.$$

Following physics literature, we call d_s and d_w the spectral dimension of G and the dimension of the walk, respectively.

We shall use $C_1, C_2, \ldots, C_{31}$ to denote unspecified positive constants.

2 Preliminaries

In this section we recall briefly the definition of Hausdorff measure and some basic facts about Brownian motion on the Sierpiński gasket. For more information we refer to Barlow [1], Barlow and Perkins [2], Falconer [9] and Mattila [25].

Let Φ be the class of functions $\phi:\ (0,\delta) \to (0,1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \le K, \quad \text{for} \quad 0 < s < \frac{1}{2}\delta.$$

For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subset \mathbb{R}^d$ is defined by

$$\phi\text{-}m(E) = \liminf_{\epsilon \to 0} \Big\{ \sum_i \phi(2r_i) :\ E \subset \bigcup_{i=1} B(x_i, r_i),\ r_i < \epsilon \Big\},$$

[1] This was observed by Yueyun Hu of Université de Paris VI in 1998.

where $B(x,r)$ denotes the open or closed ball of radius r centered at $x \in \mathbb{R}^d$. The Hausdorff dimension of E is defined by $\dim_{\mathrm{H}} E = \inf\{\alpha > 0 : \ s^{\alpha}\text{-}m(E) = 0\}$.

For any Borel measure ν on $\mathbb{R}^d$ and $\phi \in \Phi$, the upper ϕ-density of ν at $x \in \mathbb{R}^d$ is defined by

$$\bar{D}_{\nu}^{\phi}(x) = \limsup_{r \to 0} \frac{\nu\left(B(x,r)\right)}{\phi(2r)}.$$

Lemma 2.1 below, which gives a way to obtain a lower bound for ϕ-$m(E)$, is derived from a density theorem for Hausdorff measure due to Rogers and Taylor([28]).

Lemma 2.1 *For a given $\phi \in \Phi$, there exists a positive constant K such that*

$$\phi\text{-}m(E) \geq K\nu(E) \inf_{x \in E} \{\bar{D}_{\nu}^{\phi}(x)\}^{-1}.$$

In the following, we shall give the definition of the Sierpiński gasket G. Let $e_0 = (0,0)$, $e_1 = (1,0)$ $e_2 = (1/2, \sqrt{3}/2)$, $F_0 = \{e_0, e_1, e_2\}$, J_0 be the closed convex equilateral triangle with vertices e_0, e_1, e_2. Define inductively

$$F_{n+1} = F_n \cup (2^n e_1 + F_n) \cup (2^n e_2 + F_n), \quad n = 0, 1, 2, \cdots.$$

Now let $G_0' = \bigcup_{n=0} F_n$, G_0'' be the reflection of G_0' in the y-axis, $G_0 = G_0' \bigcup G_0''$. For any $n \geq 1$, let $G_n = 2^{-n} G_0$, and denote $G_{\infty} = \bigcup_{n=0}^{\infty} G_n$. Then $G = cl(G_{\infty})$ is called the (unbounded) Sierpiński gasket, where $cl(A)$ is the closure of set A. The Hausdorff dimension of G is equal to d_f, (see [13]). Let $G^{(0)}$ be the graph with vertices G_0 and with an edge between x and y in G_0 if and only if $|x - y| = 1$ and the line segment joining x and y is contained in G. For any n, let $G^{(n)} = 2^{-n} G^{(0)}$.

Definition 2.2 *$A \subset \mathbb{R}^2$ is called an n-order triangle if A is a translation of $2^{-n} J_0$ and whose vertices are three neighboring points in $G^{(n)}$.*

Let μ be the Borel measure whose support is G so that $\mu(A) = 3^{-n}$ for any n-order triangle A. Then μ is a multiple of the Hausdorff s^{d_f}-measure on G; cf. Lemma 1.1 in [2] or Remark 3.3 in [1]. Hence μ has the following property: there exist two positive and finite constants c_1 and c_2 such that

$$c_1 \rho^{d_f} \leq \mu(G \cap B(x,\rho)) \leq c_2 \rho^{d_f} \quad \text{for all} \quad x \in G, \ \rho > 0. \tag{2.1}$$

Lemma 2.3 below is from Barlow and Perkins [2] and will play a key role in this paper.

Lemma 2.3 *Let $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G, then it has a continuous symmetric transition density $p(t,x,y)$ with respect to μ. Furthermore, there exist positive constants $c_3, \ldots, c_6$ such that*

$$c_3 t^{-d_s/2} \exp\Big\{ - c_4 \Big(\frac{|x-y|^{d_w}}{t} \Big)^{\frac{1}{d_w - 1}} \Big\} \leq p(t,x,y) \leq c_5 t^{-d_s/2} \exp\Big\{ - c_6 \Big(\frac{|x-y|^{d_w}}{t} \Big)^{\frac{1}{d_w - 1}} \Big\},$$

for any $(t,x,y) \in (0,\infty) \times G \times G$.

Applying Lemma 2.3, Barlow and Perkins [2] (see also Barlow [1]) derived various sample path properties of X. We will make use of the following two lemmas. For their proofs, see Theorem 4.3, Corollary 1.7 in [2] or Theorem 2.28 in [1].

Lemma 2.4 *There are positive constants $c_5, \dots, c_8$ such that for all $x \in G$ and all $t, \delta \in (0, \infty)$,*

$$\begin{aligned} c_5 \ \exp\{-c_6(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}\} &\le \mathbb{P}^x\{|X(t) - X(0)| \ge \delta\} \\ &\le \mathbb{P}^x\Big\{\sup_{0 \le s \le t} |X(s) - X(0)| \ge \delta\Big\} \le c_7 \ \exp\{-c_8(\delta t^{-\gamma})^{\frac{1}{1-\gamma}}\}, \end{aligned}$$

where $\gamma = 1/d_w$.

Lemma 2.5 *There are positive constants c_9 and c_{10} such that for all $x \in G$,*

$$c_9 \le \lim_{\delta \to 0} \sup_{\substack{0 \le s < t \le T \\ |t-s| < \delta}} \frac{|X(t) - X(s)|}{h(|t-s|)} \le c_{10},$$

for all $T > 0$ $\mathbb{P}^x$-a.s., where $h(t) = t^\gamma (\log 1/t)^{1-\gamma}$.

We also need the following scaling relation of $X = \{X(t), t \ge 0, \mathbb{P}^x, x \in G\}$; see Remark 2.22 in Barlow and Perkins [2] or (2.13) in Barlow [1]. Let $C([0, \infty), G)$ be the space of all continuous functions $f : [0, \infty) \mapsto G$.

Lemma 2.6 *For any $x \in G$ and all Borel sets $A \subseteq C([0, \infty), G)$,*

$$\mathbb{P}^x\Big\{2X(\frac{\bullet}{5}) \in A\Big\} = \mathbb{P}^{2x}\left\{X(\bullet) \in A\right\}$$

Hence the Brownian motion on the Sierpiński gasket is a semi-self-similar Markov process with semi-self-similarity index $\gamma = \log 2 / \log 5$. We refer to Maejima and Sato [26] for recent results about semi-self-similar processes.

3 Some Technical Lemmas

In this section, we establish some technical lemmas which will be useful for proving Theorem 1.1.

For any $t > 0$, $0 < \rho < t$, define

$$T_t(\rho) = \int_0^{+\infty} \chi_{\{|t-s| \le \rho, \ |X(t) - X(s)| \le \rho\}} ds,$$

where χ_A is the indicator function of the set A. $T_t(\rho)$ is the sojourn time of the space-time process $\{(s, X(s)),\ s \geq 0\}$ in the closed set $[t-\rho, t+\rho] \times B(X(t), \rho)$.

Lemma 3.1 *There exists a constant $C_1 > 0$ such that for all $x \in G$ and $t > 0$,*

$$\limsup_{\rho \to 0} \frac{T_t(\rho)}{\rho^{1-\frac{d_s}{2}+d_f} (\log\log 1/\rho)^{\frac{d_s}{2}}} \leq C_1, \quad \mathbb{P}^x\text{–a.s.}$$

Proof The proof is based on the moment argument. For any positive integer k and $x \in G$,

$$\begin{aligned}
\mathbb{E}^x \left[T_t(\rho)\right]^k &= \mathbb{E}^x \Big(\int_{t-\rho}^{t+\rho} \chi_{\{|X(t)-X(s)| \leq \rho\}} ds \Big)^k \\
&\leq 2^{k-1} \Big\{ \mathbb{E}^x \Big(\int_t^{t+\rho} \chi_{\{|X(t)-X(s)| \leq \rho\}} ds \Big)^k + \mathbb{E}^x \Big(\int_{t-\rho}^{t} \chi_{\{|X(t)-X(s)| \leq \rho\}} ds \Big)^k \Big\} \\
&=: 2^{k-1} \{I_1 + I_2\}.
\end{aligned}$$

By using the Markov property and Lemma 2.3, we derive

$$\begin{aligned}
I_1 &= k! \mathbb{E}^x \Big(\int_t^{t+\rho} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{k-1}} dt_k \prod_{j=1}^k \chi_{\{|X(t_j)-X(t)| \leq \rho\}} \Big) \\
&= k! \int_t^{t+\rho} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{k-1}} dt_k \int_G \cdots \int_G p(t, x, x_1) p(t_k - t, x_1, x_2) \\
&\qquad \times p(t_{k-1} - t_k, x_2, x_3) \cdots p(t_1 - t_2, x_k, x_{k+1}) \prod_{j=1}^k \chi_{\{|x_{j+1} - x_1| \leq \rho\}} \mu(dx_1) \cdots \mu(dx_{k+1}) \\
&\leq C_2^k k!\ \rho^{d_f k} \int_t^{t+\rho} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{k-1}} \prod_{j=1}^k (t_j - t_{j+1})^{-d_s/2}\, dt_k \qquad (t_{k+1} := t), \quad (3.1)
\end{aligned}$$

where in obtaining the last inequality, we have also used inequality (2.1) and the fact that $\int_G p(t, x, x_1) \mu(dx_1) = 1$.

We also need the following identity, which can be proved by using induction on k and elementary calculation. For all $k \geq 1$ and $0 < b_j < 1$,

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} \prod_{j=1}^k (t_j - t_{j+1})^{-b_j} dt_k = \frac{\prod_{j=1}^k \Gamma(1-b_j)}{\Gamma(1+k-\sum_{j=1}^k b_j)} \quad (t_{k+1} := 0). \quad (3.2)$$

By a change of variables and the identity (3.2), we obtain from (3.1) that

$$\begin{aligned}
I_1 &\leq C_2^k\, k!\, \rho^{(1-d_s/2+d_f)k} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} \prod_{j=1}^k (t_j - t_{j+1})^{-\frac{d_s}{2}} dt_k \\
&= C_2^k\, k!\, \rho^{(1-d_s/2+d_f)k} \left(\Gamma(1-d_s/2)\right)^k \left(\Gamma((1-d_s/2)k+1)\right)^{-1} \\
&\leq C_3^k\, \rho^{(1-d_s/2+d_f)k}\ (k!)^{d_s/2}.
\end{aligned}$$

In the above, the last inequality follows from the Stirling's formula.

In the same way, we have $I_2 \le C_4^k \rho^{(1-d_s/2+d_f)k}(k!)^{d_s/2}$. Thus, for every integer $k \ge 1$,

$$\mathbb{E}^x\,(T_t(\rho))^k \le C_5^k \rho^{(1-d_s/2+d_f)k}(k!)^{d_s/2}. \tag{3.3}$$

It follows that there exist positive and finite constants C_6 and C_7 such that

$$\mathbb{E}^x\Big\{\exp\Big[C_6\Big(T_t(\rho)\rho^{-(1-\frac{d_s}{2}+d_f)}\Big)^{\frac{2}{d_s}}\Big]\Big\} \le C_7.$$

This can be derived by expanding the exponential function, and then estimating the individual terms by using Jensen's inequality and (3.3). Thus for any $a > 0$, we have

$$\mathbb{P}^x\Big\{T_t(\rho) \ge \rho^{1-d_s/2+d_f}\ a^{d_s/2}\Big\} \le C_7\exp(-C_6 a). \tag{3.4}$$

Now for any fixed $t > 0$, let $b > 0$ be a constant whose value will be determined later. It follows from (3.4) that

$$\begin{aligned}&\mathbb{P}^x\Big\{T_t(e^{-n}) \ge b^{d_s/2}e^{-n(1-d_s/2+d_f)}(\log\log e^n)^{d_s/2}\Big\}\\ &\le C_7\exp\Big\{-C_6 b\log n\Big\} = C_7 n^{-C_6 b}.\end{aligned}$$

We take $b > 0$ such that $C_6 b > 1$, so that $\sum_{n=1}^\infty n^{-C_6 b} < \infty$. The Borel-Cantelli lemma, together with a standard monotonicity argument, yields

$$\limsup_{\rho\to 0}\frac{T_t(\rho)}{\rho^{1-d_s/2+d_f}\,(\log\log 1/\rho)^{d_s/2}} \le b^{d_s/2}e^{1-d_s/2+d_f},\ \ \mathbb{P}^x - \text{a.s.}$$

This finishes the proof of Lemma 3.1 with $C_1 = b^{d_s/2}e^{1-d_s/2+d_f}$. □

The following lemma gives upper and lower bounds for the small ball probability of X, which will be essential in proving Chung-type laws of the iterated logarithm for X and the upper bound in Theorem 1.1.

Lemma 3.2 *There exist constants $C_9 > 1$, $\xi_0 > 0$ such that for every $0 < \xi < \xi_0$ and every $x \in G$, we have*

$$\exp\{-C_9\xi^{-1/\gamma}\} \le \mathbb{P}^x\Big\{\sup_{0\le t\le 1}|X(t) - X(0)| < \xi\Big\} \le \exp\{-C_9^{-1}\xi^{-1/\gamma}\}, \tag{3.5}$$

where $\gamma = \frac{\log 2}{\log 5}$.

Proof We divide the proof into two parts.

(1) Lower Bound: By Lemma 2.3, we see that $p(1,z,z) \geq c_3$ for every $z \in G$ and, furthermore, there exist positive constants C_{10} and δ_1 such that for all x, y, $z \in G$ with $|x-z| \leq \delta_1$ and $|y-z| \leq \delta_1$, we have $p(1,x,y) \geq C_{10}$. Therefore, it follows from (2.1) that for any x, $z \in G$ such that $|x-z| \leq \delta_1$,

$$\mathbb{P}^x\{|X(1)-z| < \delta_1\} = \int_{\{y \in G: |y-z|<\delta_1\}} p(1,x,y)\mu(dy) \geq C_{10}c_1\delta_1^{d_f} =: C_{11}.$$

On the other hand, Lemma 2.4 implies that there exists a constant $r > 0$ such that for any $x \in G$,

$$\mathbb{P}^x\Big\{\sup_{0\leq t\leq 1}|X(t)-X(0)| \geq r-\delta_1\Big\} < \frac{1}{2}C_{11}.$$

Hence for any x, $z \in G$, if $|x-z| \leq \delta_1$, we have

$$\begin{aligned}
&\mathbb{P}^x\Big\{\sup_{0\leq t\leq 1}|X(t)-z| < r,\ |X(1)-z| < \delta_1\Big\} \\
&\geq \mathbb{P}^x\{|X(1)-z| < \delta_1\} - P^x\Big\{\sup_{0\leq t\leq 1}|X(t)-z| \geq r\Big\} \\
&\geq \mathbb{P}^x\{|X(1)-z| < \delta_1\} - \mathbb{P}^x\Big\{\sup_{0\leq t\leq 1}|X(t)-x| \geq r-\delta_1\Big\} > \frac{1}{2}C_{11}.
\end{aligned} \tag{3.6}$$

For any $s > 0$, let $\mathcal{J}_s = \sigma(X(t) : 0 \leq t \leq s)$ be the σ field generated by the random variables $X(t)$, $0 \leq t \leq s$. We observe that for all $k \geq 1$ and $z \in G$,

$$\begin{aligned}
&\mathbb{P}^z\Big\{\sup_{0\leq t\leq k}|X(t)-X(0)| < r\Big\} \\
&\geq \mathbb{P}^z\Big\{\sup_{0\leq t\leq 1}|X(t)-z| < r, \ldots, \sup_{k-1\leq t\leq k}|X(t)-z| < r, \\
&\qquad\qquad |X(1)-z| < \delta, \ldots, |X(k)-z| < \delta\Big\} \\
&= \mathbb{E}^z\Big[\mathbb{P}^z\Big\{\sup_{0\leq t\leq 1}|X(t)-z| < r, \ldots, \sup_{k-1\leq t\leq k}|X(t)-z| < r, \\
&\qquad\qquad |X(1)-z| < \delta_1, \ldots, |X(k)-z| < \delta_1 | \mathcal{J}_{k-1}\Big\}\Big] \\
&= \mathbb{E}^z\Big[\prod_{j=1}^{k-1}\mathbb{1}\Big\{\sup_{j-1\leq t\leq j}|X(t)-z| < r,\ |X(j)-z| < \delta_1\Big\} \\
&\qquad\qquad \times\mathbb{P}^{X(k-1)}\Big\{\sup_{0\leq t\leq 1}|X(t)-z| < r, |X(1)-z| < \delta_1\Big\}\Big],
\end{aligned}$$

where $\mathbb{1}(A)$ denotes the indicator random variable of event A and the last equality follows from the Markov property. Applying (3.6) to the conditional probability above and repeating this procedure, we have

$$
\begin{aligned}
&\mathbb{P}^z\Big\{ \sup_{0\le t\le k} |X(t)-X(0)| < r\Big\} \\
&\ge \frac{1}{2}C_{11}\ \mathbb{P}^z\Big\{ \sup_{0\le t\le 1} |X(t)-z| < r, \cdots, \sup_{k-2\le t\le k-1} |X(t)-z| < r, \\
&\qquad\qquad |X(1)-z| < \delta_1, \cdots, |X(k-1)-z| < \delta_1\Big\} \\
&\ge \cdots \ge \Big(\frac{1}{2}C_{11}\Big)^k = \exp(-\log(C_{11}/2)\,k). \qquad (3.7)
\end{aligned}
$$

Let $\xi_1 = r/2$. For any $\xi \in (0, \xi_1)$, choose $n = n(\xi) \ge 1$ such that $r/2^{n+1} < \xi \le r/2^n$. By Lemma 2.6 and (3.7), we see that for every $x \in G$,

$$
\begin{aligned}
\mathbb{P}^x\Big\{ \sup_{0\le t\le 1} |X(t)-X(0)| < \xi\Big\} &\ge \mathbb{P}^x\Big\{ \sup_{0\le t\le 1} |X(t)-X(0)| < \frac{r}{2^{n+1}}\Big\} \\
&= \mathbb{P}^{2^{n+1}x}\Big\{ \sup_{0\le t\le 5^{n+1}} |X(t)-2^{n+1}x| < r\Big\} \\
&\ge \exp(-\log(C_{11}/2)\,5^{n+1}) \ge \exp(-C_{12}\xi^{-1/\gamma}). \qquad (3.8)
\end{aligned}
$$

In the above, note that $2^{n+1}x \in G$.

(2) Upper Bound:

By Lemma 2.3, we can choose $0 < \delta_2 < 1$ such that for all $x \in G$,

$$
\eta := \mathbb{P}^x\Big\{|X(1)-X(0)| \le 2\delta_2\Big\} < 1. \qquad (3.9)
$$

Let $\xi_0 = \min\{\xi_1, \delta_2\}$. Then for any $\xi \in (0, \xi_0)$, there exists an $n = n(\xi) \ge 1$ such that $\frac{\delta_2}{2^{n+1}} < \xi \le \frac{\delta_2}{2^n}$. By Lemma 2.6 and the Markov property, we have

$$
\begin{aligned}
\mathbb{P}^x\Big\{ \sup_{0\le t\le 1} |X(t)-X(0)| < \xi\Big\} &\le \mathbb{P}^x\Big\{ \sup_{0\le t\le 1} |X(t)-X(0)| < \frac{\delta_2}{2^n}\Big\} \\
&= \mathbb{P}^{2^n x}\Big\{ \sup_{0\le t\le 5^n} |X(t)-2^n x| < \delta_2\Big\} \\
&\le \mathbb{P}^{2^n x}\Big\{|X(j)-X(j-1)| \le 2\delta_2,\ j = 1, \ldots, 5^n\Big\} \\
&\le \eta^{5^n} \le \exp(-C_{13}\,\xi^{-1/\gamma}). \qquad (3.10)
\end{aligned}
$$

Take $C_9 = \max\{C_{12}, C_{13}^{-1}\}$, we see that (3.5) follows from (3.8) and (3.10). This completes the proof of Lemma 3.2. □

In recent years, there has been a lot of interest in studying the small ball probabilities of stochastic processes. More precisely, let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a stochastic process with values in $\mathbb{R}^d$, and let

$$
\|Y\| = \sup_{t\in[0,1]^N} |Y(t)|,
$$

the small ball problem is to investigate the asymptotic behavior of $\log \mathbb{P}\{\|Y\| \le \epsilon\}$ when ϵ goes to 0. In contrast to the study of large deviations, small ball probabilities can only be determined for a few special examples of Y such as Lévy stable processes and certain Gaussian processes. We refer to the survey papers of Li and Shao [22] and Lifshits [24] for more details about small ball probabilities and their applications.

Since Lemma 3.2 only gives upper and lower bounds for the small ball probability of X, it is natural to ask the following

Question 3.1: For every $x \in G$, does the limit

$$\lim_{\xi \to 0} \xi^{1/\gamma} \log \mathbb{P}^x \Big\{ \sup_{0 \le t \le 1} |X(t) - X(0)| < \xi \Big\} \ \text{ exist?} \tag{3.11}$$

Bertoin ([6], p. 220) gives a nice proof for the existence of the limit (3.11) for Lévy stable processes. However, that proof uses special properties of Lévy stable processes such as the self-similarity and hitting probability estimates. For Brownian motion on the Sierpiński gasket, the same argument and Lemma 2.6 only yield that the limit

$$\lim_{n \to \infty} 2^{-n/\gamma} \log \sup_{|x| < 2^{-n}} \mathbb{P}^x \Big\{ \sup_{0 \le t \le 1} |X(t) - X(0)| < 2^{-n} \Big\} \ \text{ exists.}$$

Other possible techniques for proving the existence of limit in (3.11) include the methods using Laplace transforms (Kac [15]), subadditivity (see, for example, De Acosta [8] and Kuelbs and Li [17]), correlation inequalities for Gaussian processes (cf. Li [21] and Li and Shao [22], Section 6) and the renewal theorem (cf. Li, Peres and Xiao [23]). But the last three arguments require the considered processes to have the unimodality property and, hence, do not seem to work for Brownian motion X on the Sierpiński gasket.

We should mention that the large deviations and short time asymptotic behavior of Brownian motions on the Sierpiński gasket or the nested fractals have been studied by Kumagai [18], Fukushima et al [10] and Ben Arous and Kumagai [5]. In particular, Kumagai [18] have applied the renewal theorem in studying the short time behavior of the transition density $p_t(x, y)$ of X as $t \to 0$. However, it is not clear to us whether these arguments are related to the above small ball problem.

Denote $\psi_1(t) = t^{\gamma} (\log \log 1/t)^{-\gamma}$ and for any $k \ge 1$, let $t_k = 2^{-k^2}$.

Lemma 3.3 *There exists a constant $C_{14} > 0$ such that for every $\tau \ge 0$ and every $x \in G$,*

$$\mathbb{P}^x \Big(\bigcap_{k=n}^{2n} \Big\{ \sup_{\tau \le t \le \tau + t_k} |X(t) - X(\tau)| \ge b \psi_1(t_k) \Big\} \Big) \le \exp(-C_{14} n^{1/3})$$

for all n large enough, where $b > 0$ is a constant satisfying $C_9 (b/4)^{-1/\gamma} = 1/3$.

Proof For any $n \le k \le 2n$, define the following events

$$D_k = \Big\{\omega \in \Omega : \sup_{\tau \le t \le \tau + t_k} |X(t, \emptyset) - X(\tau, \emptyset)| \ge b\psi_1(t_k)\Big\},$$
$$D'_k = \Big\{\omega \in \Omega : \sup_{\tau + t_{k+1} \le t \le \tau + t_k} |X(t, \emptyset) - X(\tau, \emptyset)| \ge \frac{b}{2}\psi_1(t_k)\Big\},$$
$$D''_k = \Big\{\omega \in \Omega : \sup_{\tau \le t \le \tau + t_{k+1}} |X(t, \emptyset) - X(\tau, \emptyset)| \ge \frac{b}{2}\psi_1(t_k)\Big\}.$$

Then we have

$$D_k \subset D'_k \cup D''_k, \qquad \bigcap_{k=n}^{2n} D_k \subseteq \Big(\bigcap_{k=n}^{2n} D'_k\Big) \bigcup \Big(\bigcup_{k=n}^{2n} D''_k\Big).$$

Thus for every $x \in G$,

$$\mathbb{P}^x\Big\{\bigcap_{k=n}^{2n} D_k\Big\} \le \mathbb{P}^x\Big\{\bigcap_{k=n}^{2n} D'_k\Big\} + \mathbb{P}^x\Big\{\bigcup_{k=n}^{2n} D''_k\Big\} =: I_3 + I_4.$$

For any $n \le k \le 2n$, choose $m(k)$ such that $5^{-m(k)-1} < t_k \le 5^{-m(k)}$. As follows from the Markov property and an argument similar to that in the proof of Lemma 3.2, we have

$$\begin{aligned}
I_3 &\le \prod_{k=n}^{2n} \sup_{y \in G} \mathbb{P}^y\Big\{\sup_{0 \le t \le t_k - t_{k+1}} |X(t) - y| \ge \frac{b}{2}\psi_1(t_k)\Big\} \\
&\le \prod_{k=n}^{2n} \sup_{y \in G} \mathbb{P}^y\Big\{\sup_{0 \le t \le 5^{-m(k)}} |X(t) - y| \ge \frac{b}{2}\Big(\frac{t_k}{\log\log 1/t_k}\Big)^\gamma\Big\} \\
&= \prod_{k=n}^{2n} \sup_{y \in G} \mathbb{P}^y\Big\{\sup_{0 \le t \le 1} |2^{m(k)} X(\frac{t}{5^{m(k)}}) - 2^{m(k)} y| \ge \frac{b}{2} 2^{m(k)}\Big(\frac{t_k}{\log\log 1/t_k}\Big)^\gamma\Big\} \\
&\le \prod_{k=n}^{2n} \sup_{y \in G} \mathbb{P}^{2^{m(k)} y}\Big\{\sup_{0 \le t \le 1} |X(t) - 2^{m(k)} y| \ge \frac{b}{4}(\log\log 1/t_k)^{-\gamma}\Big\}.
\end{aligned}$$

Setting

$$1 - p_k = \sup_{y \in G} \mathbb{P}^{2^{m(k)} y}\Big\{\sup_{0 \le t \le 1} |X(t) - 2^{m(k)} y| \ge \frac{b}{4}\Big(\frac{1}{\log\log 1/t_k}\Big)^\gamma\Big\},$$

we see that

$$I_3 \le \prod_{k=n}^{2n} (1 - p_k) \le \exp\Big(-\sum_{k=n}^{2n} p_k\Big).$$

By Lemma 3.2 and the fact that $C_9(b/4)^{-1/\gamma} = 1/3$ we have

$$\begin{aligned} p_k &= \inf_{y\in G} \mathbb{P}^{2^{m(k)}y}\Big\{ \sup_{0\le t\le 1} |X(t) - 2^{m(k)}y| < \frac{b}{4}(\log\log 1/t_k)^{-\gamma}\Big\} \\ &\geq \exp\Big(- C_9(\frac{b}{4})^{-\frac{1}{\gamma}} \log\log 1/t_k\Big) \\ &= (\log 2)^{-\frac{1}{3}} k^{-\frac{2}{3}}. \end{aligned}$$

Thus for all n large enough,

$$I_3 \leq \exp(-C_{15} n^{1/3}).$$

On the other hand, it follows from the Markov property and Lemma 2.4 that for every $x \in G$,

$$\begin{aligned} \mathbb{P}^x(D_k'') &= \mathbb{P}^x\Big\{ \sup_{\tau\le t\le \tau+t_{k+1}} |X(t) - X(\tau)| \geq \frac{b}{2}\psi_1(t_k)\Big\} \\ &\leq \sup_{y\in G} \mathbb{P}^y\Big\{ \sup_{0\le t\le t_{k+1}} |X(t) - X(0)| \geq \frac{b}{2}\psi_1(t_k)\Big\} \\ &\leq C_{16} \exp\Big(- C_{17}(\frac{b}{2}\psi_1(t_k) t_{k+1}^{-\gamma})^{\frac{1}{1-\gamma}}\Big) \\ &\leq C_{16} \exp(-C_{18} k) \qquad (\text{ for } \ n \ \text{ large enough}). \end{aligned} \tag{3.12}$$

Therefore for all n large enough,

$$I_4 \leq \sum_{k=n}^{2n} \mathbb{P}^x\{D_k''\} \leq \exp(-C_{19} n). \tag{3.13}$$

Combining (3.12) and (3.13), we have that for all $x \in G$ and $\tau \geq 0$,

$$\begin{aligned} \mathbb{P}^x\Big(\bigcap_{k=n}^{2n} \Big\{ \sup_{\tau\le t\le \tau+t_k} |X(t) - X(\tau)| \geq b\psi_1(t_k)\Big\}\Big) &\leq \exp(-C_{15} n^{1/3}) + \exp(-C_{19} n) \\ &\leq \exp(-C_{14} n^{1/3}) \end{aligned}$$

for all n large enough. The proof of Lemma 3.3 is finished. □

4 Chung's Law of the Iterated Logarithm

Let $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G. Our main result of this section is Theorem 4.1, which characterizes the local and uniform oscillations of X. The inequalities (4.1) and (4.2) extend Chung's law of the iterated logarithm and the Csörgő-Révész modulus of non-differentiability for the ordinary Brownian motion, respectively. See Csörgő and Révész [7].

Theorem 4.1 *Let $X = \{X(t), t \geq 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G. Then there exists a positive and finite constant C_{20} such that for any fixed $\tau \geq 0$ and for every $x \in G$,*

$$C_{20}^{-1} \leq \liminf_{r \to 0} \sup_{t \in [\tau, \tau + r]} \frac{|X(t) - X(\tau)|}{(r / \log\log r^{-1})^\gamma} \leq C_{20} \qquad \mathbb{P}^x\text{-}a.s. \tag{4.1}$$

and for any fixed $T > 0$, there exists a positive and finite constant C_{21} such that for every $x \in G$,

$$C_{21}^{-1} \leq \liminf_{r \to 0} \inf_{\tau \in [0,T]} \sup_{t \in [\tau, \tau + r]} \frac{|X(t) - X(\tau)|}{(r / \log r^{-1})^\gamma} \leq C_{21} \qquad \mathbb{P}^x\text{-}a.s. \tag{4.2}$$

Remark 4.1 The Blumenthal zero-one law and (4.1) imply that for every $x \in G$, there exists a constant $C_{22}(x) \in [C_{21}^{-1}, C_{21}]$ such that

$$\liminf_{r \to 0} \sup_{t \in [0,r]} \frac{|X(t) - X(0)|}{(r / \log\log r^{-1})^\gamma} = C_{22}(x) \qquad \mathbb{P}^x\text{-a.s.}$$

We conjecture that $C_{22}(x)$ is independent of $x \in G$. A similar conjecture for the limsup type law of the logarithm of X was made by Barlow and Perkens [2, p.580], and, to our knowledge, it is still open.

We mention that the situation for the asymptotics of X near ∞ is known. By applying a 0-1 law of Barlow and Bass [3], Fukushima et al [10] have proved that there is a constant $0 < C_{23} < \infty$ such that for every $x \in G$,

$$\liminf_{r \to \infty} \sup_{0 \leq t \leq r} \frac{|X(t)|}{(r / \log\log r)^\gamma} = C_{23} \qquad \mathbb{P}^x\text{-a.s.} \tag{4.3}$$

See also Bass and Kumagai [4] for more general results of this type. However, it does not seem possible to adapt their arguments to the current problem. □

Remark 4.2 It would also be interesting to know whether there is a constant $0 < C_{24} < \infty$ such that for every $x \in G$,

$$\liminf_{r \to 0} \inf_{\tau \in [0,T]} \sup_{t \in [\tau, \tau + r]} \frac{|X(t) - X(\tau)|}{(r / \log r^{-1})^\gamma} = C_{24} \qquad \mathbb{P}^x\text{-a.s.} \tag{4.4}$$

This question is closely related to Question 3.1, but perhaps it is simpler. □

Proof of Theorem 4.1 The inequalities in (4.1) follow from Lemmas 3.2, 3.3 and the Borel-Cantelli lemma in a standard way. Hence we omit the details.

The proof of (4.2) is also quite standard, cf. Csörgő and Révész ([7], Section 1.6). We give it here for convenience of the reader. Without loss of generality, we assume $T = 1$. Let $K > 0$ be a constant such that $M = 1 - C_9 K^{-\gamma} > 0$. Denote $\psi_2(r) = (r/\log r^{-1})^\gamma$. For each $n \geq 1$, we divide $[0, 1]$ into 5^n subintervals of length 5^{-n}. Consider the events

$$A_n = \Big\{ \sup_{t \in [0, 5^{-n}]} |X(t + k5^{-n}) - X(k5^{-n})| \geq K\psi_2(5^{-n}) \ \text{ for all } k = 0, 1, \dots, 5^n - 1 \Big\}.$$

Applying the Markov property 5^n times, Lemma 2.6 and Lemma 3.2, we deduce that

$$\begin{aligned}
\mathbb{P}^x\{A_n\} &\leq \Big[\sup_{y \in G} \mathbb{P}^y \Big\{ \sup_{t \in [0, 5^{-n}]} |X(t) - X(0)| \geq K\psi_2(5^{-n}) \Big\} \Big]^{5^n} \\
&= \Big[\sup_{x \in G} \mathbb{P}^{2^n x} \Big\{ \sup_{t \in [0,1]} |X(t) - 2^n x| \geq K(\log 5^n)^{-\gamma} \Big\} \Big]^{5^n} \\
&\leq \big(1 - \exp\{-C_9 K^{-\gamma} \log 5^n\}\big)^{5^n} \\
&= \Big(1 - 5^{-C_9 K^{-\gamma} n}\Big)^{5^n} \\
&\leq \exp(-5^{Mn}).
\end{aligned}$$

The Borel-Cantelli lemma implies that $\mathbb{P}\{A_n \text{ i.o. }\} = 0$. Hence

$$\liminf_{n \to \infty} \inf_{\tau \in [0,1]} \sup_{t \in [\tau, \tau + 5^{-n}]} \frac{|X(t) - X(\tau)|}{\psi_2(5^{-n})} \leq K \quad \mathbb{P}^x\text{-a.s.}$$

and the right-hand inequality in (4.2) follows.

To prove the lower bound, we let $\beta > 1/\gamma$ be a constant and, for each integer $n \geq 1$, let $h_n = 5^{-n}(\log 5^n)^{-\beta}$. Consider the following partition of $[0, 1]$:

$$\tau_i = i h_n \qquad i = 0, 1 \dots, m_n := \lfloor 5^n (\log 5^n)^\beta \rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Let $K > 0$ be a constant whose value will be determined later. It follows from the Markov property and Lemma 3.2 that for every $x \in G$,

$$\begin{aligned}
&\mathbb{P}^x \Big\{ \min_{1 \leq i \leq m_n} \sup_{t \in [\tau_i, \tau_i + 5^{-n}]} |X(t) - X(\tau_i)| < K\psi_2(5^{-n}) \Big\} \\
&\leq m_n \sup_{y \in G} \mathbb{P}^y \Big\{ \sup_{t \in [0, 5^{-n}]} |X(t) - X(0)| < K\psi_2(5^{-n}) \Big\} \\
&\leq m_n \exp\Big(- C_9^{-1} K^{-\gamma} \log 5^n \Big). \qquad (4.5)
\end{aligned}$$

We take $K > 0$ small such that $C_9^{-1} K^{-\gamma} > 1$. Hence the last term in (4.5) is summable. The Borel-Cantelli lemma implies that there exists a constant $C_{25} > 0$ such that for every

$x \in G$,

$$\liminf_{n\to\infty} \min_{1\le i\le m_n} \sup_{t\in[\tau_i,\tau_i+5^{-n}]} \frac{|X(t)-X(\tau_i)|}{\psi_2(5^{-n})} \ge C_{25} \quad \mathbb{P}^x\text{-a.s.} \tag{4.6}$$

Now for any $\tau \in [0,1]$, let $i \ge 1$ be the integer satisfying $\tau_i \le \tau < \tau_{i+1}$. We note that

$$\sup_{t\in[\tau,\tau+5^{-n}]} |X(t)-X(\tau)| \ge \sup_{t\in[\tau_i,\tau_i+5^{-n}]} |X(t)-X(\tau_i)| - 2 \sup_{\substack{t,s\in[0,1]\\ |s-t|\le h_n}} |X(s)-X(t)|, \tag{4.7}$$

and by the choice of β and Lemma 2.5, we derive

$$\limsup_{n\to\infty} \sup_{\substack{t,s\in[0,1]\\ |s-t|\le h_n}} \frac{|X(s)-X(t)|}{\psi_2(5^{-n})} = 0 \qquad \mathbb{P}^x\text{-a.s.} \tag{4.8}$$

It is clear that the lower bound in (4.2) follows from (4.6), (4.7), (4.8) and a standard monotonicity argument. This completes the proof. $\square$

5 Hausdorff Measure of the Graph

Let $X = \{X(t), t \ge 0, \mathbb{P}^x, x \in G\}$ be the Brownian motion on the Sierpiński gasket G. For any $t > 0$, let $\mathrm{Gr}(X[0,t]) = \{(s, X(s)) : 0 \le s \le t\}$ be the graph of X on $[0,t]$ and denote

$$\varphi(s) = s^{1+\log 3/\log 2 - \log 3/\log 5} (\log\log 1/s)^{\log 3/\log 5}, \quad s \in (0, 1/8].$$

Proof of Theorem 1.1 We divide the proof into two parts.

(i) **Upper bound:** In order to prove that there exists a finite constant C such that φ-$m(\mathrm{Gr}(X[0,t])) \le Ct$ $\mathbb{P}^x$-a.s. for all $t > 0$, we need to construct, for every $\epsilon > 0$, an economical covering, say $\{E_n,\ n \ge 1\}$, of $\mathrm{Gr}(X[0,t])$ such that $\mathrm{diam}(E_n) < \epsilon$ and $\sum_{n=1}^{\infty} \varphi(\mathrm{diam} E_n) \le Ct$ $\mathbb{P}^x$-a.s., where $\mathrm{diam} E$ denotes the diameter of the set E. It will be clear that the construction of such a covering is closely related to the oscillation of the sample path of X. In particular, the idea of distinguishing between "good" and "bad" points in the state space goes back to Taylor [32]. Our approach follows that of Talagrand [30], who distinguishes between "good" times and "bad" times in the parameter space.

For any integers $n \ge 1$ and $k \ge 0$, let $C_{n,k} = [kt_n, (k+1)t_n]$, where $t_n = 2^{-n^2}$. For any $t > 0$, we set

$$\Lambda_n = \Big\{ C_{m,k} : n \le m \le 2n, k \ge 0, C_{m,k} \cap [0,t] \ne \emptyset \Big\}.$$

Then for every $C_{2n,k} \in \Lambda_n$ and every integer m, $n \le m \le 2n$, there exists a unique subinterval $C_{m,k'} \in \Lambda_n$, denoted by $C_{2n,k}(m)$, such that $C_{2n,k} \subset C_{2n,k}(m)$. We denote the left endpoint of $C_{2n,k}(m)$ by $t_{2n,k}(m)$.

For each subinterval $C_{2n,k} \in \Lambda_n$, we call $C_{2n,k}$ a "bad interval", if

$$\sup_{s \in C_{2n,k}(m)} |X(s) - X(t_{2n,k}(m))| \geq b\Big(\frac{t_m}{\log\log 1/t_m}\Big)^\gamma \quad \text{for all} \ \ m = n, \dots, 2n,$$

where $b > 0$ is the constant defined in Lemma 3.3.

If $C_{2n,k}$ is not a "bad interval", we define $m_* = m_*(k)$ by

$$m_* = \min\Big\{m : n \leq m \leq 2n, \sup_{s \in C_{2n,k}(m)} |X(s) - X(t_{2n,k}(m))| < b\Big(\frac{t_m}{\log\log 1/t_m}\Big)^\gamma\Big\}$$

and call $C_{2n,k}(m_*)$ a "good interval". It is clear that all "good intervals" and "bad intervals" in Λ_n constitute a non-overlapping covering of $[0, t]$. In the following, we construct an economic covering of $\mathrm{Gr}X([0, t])$ using these "good intervals" and "bad intervals".

If $C_{2n,k}(m_*)$ is a "good interval", then

$$\sup_{s \in C_{2n,k}(m_*)} |X(s) - X(t_{2n,k}(m_*))| < b\Big(\frac{t_{m_*}}{\log\log 1/t_{m_*}}\Big)^\gamma.$$

Let $n_1 = n_1(k) \geq 1$ be the integer such that

$$2^{-n_1-1} \leq b\Big(\frac{t_{m_*}}{\log\log 1/t_{m_*}}\Big)^\gamma < 2^{-n_1}.$$

Let

$$\Gamma_k = \Big\{A : A \ \text{ is an } \ n_1\text{-order triangle and } \ A \cap X(C_{2n,k}(m_*)) \neq \emptyset\Big\}.$$

Then

$$X(C_{2n,k}(m_*)) \subset \bigcup_{A \in \Gamma_k} A.$$

Since $\mathrm{diam}(X(C_{2n,k}(m_*))) < 2^{-n_1+1}$ and $\mathrm{diam}(A) = 2^{-n_1}$ for any n_1-order triangle A, it follows from the construction of (the unbounded) Sierpiński gasket G that $\#\Gamma_k \leq 6$, where $\#E$ is the cardinality of the set E.

Observe that for every $A \in \Gamma_k$, $A = \bigcup_{j=1}^{3^{m_*^2-n_1}} A_j$, where A_j is an m_*^2-order triangle, thus $X(C_{2n,k}(m_*))$ can be covered by $6 \cdot 3^{m_*^2-n_1}$ triangles with diameter $2^{-m_*^2}$. Since

$$\mathrm{Gr}X(C_{2n,k}(m_*)) \subseteq C_{2n,k}(m_*) \times X(C_{2n,k}(m_*)),$$

we see that the graph of X on $C_{2n,k}(m_*)$ can also be covered by $6 \cdot 3^{m_*^2-n_1}$ sets of the form $C_{2n,k}(m_*) \times A_j$ and the diameters of these sets are at most $2^{-m_*^2+1}$.

Since $\gamma d_f = d_s/2$, we have

$$\begin{aligned} & 6\cdot 3^{-n_1+m_*^2}\varphi(2^{-m_*^2+1}) \\ & \le C_{26}\Big(\frac{t_{m_*}}{\log\log 1/t_{m_*}}\Big)^{\gamma d_f} t_{m_*}^{-d_f} t_{m_*}^{1+d_f-\frac{d_s}{2}} (\log\log 1/t_{m_*})^{\frac{d_s}{2}} \\ & = C_{26} t_{m_*}. \end{aligned}$$

Therefore, summing over the coverings of the graph of X on all the "good" intervals in Λ_n, we have for n large enough,

$$\begin{aligned} & \sum_{C_{2n,k}(m_*) \ \text{is "good"}} 6\cdot 3^{-n_1+m_*^2}\varphi(2^{-m_*^2+2}) \\ & \le C_{26} \sum_{C_{2n,k}(m_*) \ \text{is "good"}} t_{m_*} \le C_{26} t. \end{aligned} \tag{5.1}$$

Here we have used the fact that the good intervals are non-overlapping.

By Lemma 3.3 and the Markov property, when n is large enough, for every $x \in G$, we have

$$\mathbb{P}^x\{C_{2n,k} \ \text{ is "bad" }\} \le \exp\{-C_{14} n^{1/3}\}.$$

Thus for n large enough,

$$\mathbb{P}^x\Big\{\#\{C_{2n,k}\in\Lambda_n,\ C_{2n,k} \ \text{ is "bad" }\} \ge 2\frac{t}{t_{2n}} n^2 \ \exp(-C_{14} n^{1/4})\Big\} \le \frac{1}{n^2}.$$

By the Borel-Cantelli lemma, $\mathbb{P}^x$-a.s. there exists $n_0(\omega)$ such that for any $n \ge n_0(\omega)$, we have

$$\#\{C_{2n,k}\in\Lambda_n,\ C_{2n,k} \ \text{ is "bad"}\} < 2\frac{t}{t_{2n}}\ n^2 \ \exp(-C_{14} n^{\frac{1}{4}}). \tag{5.2}$$

On the other hand, Lemma 2.5 implies that, when n is large enough,

$$\mathrm{diam} X(C_{2n,k}) \le C_{27} t_{2n}^{\gamma}\Big(\log\frac{1}{t_{2n}}\Big)^{1-\gamma}.$$

Now let $n_2 \ge 1$ be the integer satisfying

$$2^{-n_2-1} \le C_{26} t_{2n}^{\gamma}\Big(\log\frac{1}{t_{2n}}\Big)^{1-\gamma} < 2^{-n_2}.$$

By repeating the argument above for the "good interval" case, we see that for each "bad" interval $C_{2n,k}$, $\mathrm{Gr}X(C_{2n,k}))$ can be covered by

$$6\cdot 3^{4n^2-n_2} \tag{5.3}$$

sets of the form $C_{2n,k} \times A'_j$, where A'_j's are $4n^2$-order triangles. The diameters of such sets are at most 2^{-4n^2+1}.

Summing over the coverings of the graph of X on all the "bad" intervals in Λ_n and using (5.2) and (5.3), we obtain

$$\begin{aligned} &\sum_{C_{2n,k} \text{ is "bad"}} 6 \cdot 3^{4n^2-n_2} \ \varphi(2^{-4n^2+1}) \\ &\le C_{28} \frac{t}{t_{2n}} \ n^2 \ \exp(-C_{14} n^{\frac{1}{4}}) \cdot t_{2n} \Big(\log \frac{1}{t_{2n}} \Big)^{(1-\gamma)d_f} \Big(\log\log \frac{1}{t_{2n}} \Big)^{\frac{d_s}{2}} \\ &= \epsilon_n \, t, \end{aligned} \tag{5.4}$$

where $\epsilon_n \to 0$ when $n \to \infty$. It follows from (5.1) and (5.4) that $\mathbb{P}^x$-a.s., φ-$m(\mathrm{Gr}X([0,t])) \le C_{29}t$. By the continuity of measures, we see that there exists a finite constant $C > 0$ such that for every $x \in G$, $\mathbb{P}^x$-a.s., φ-$m(\mathrm{Gr}X([0,t])) \le Ct$ for all $t > 0$.

(ii) **Lower Bound.**

For any $t > 0$, we define a random Borel measure ν on $\mathrm{Gr}X([0,t])$ by

$$\nu(B) = \lambda_1\{s \in [0,t] : (s, X(s)) \in B\} \quad \text{for any } \ B \subset \mathbb{R}_+ \times G.$$

Then $\nu(\mathbb{R}^1 \times G) = \nu(\mathrm{Gr}X([0,t])) = t$. It follows from Lemma 3.1 that there exists a constant C_{29} such that for any fixed $s \in [0,t]$, $\mathbb{P}^x$-a.s.,

$$\limsup_{\rho \to 0} \frac{\nu(B((s,X(s)),\rho))}{\rho^{1-\frac{d_s}{2}+d_f} \left(\log\log \frac{1}{\rho}\right)^{\frac{d_s}{2}}} \le C_{30}. \tag{5.5}$$

Let

$$E(\omega) = \{(s, X(s)) : s \in [0,t] \ \text{ and } \ (5.1) \text{ holds}\}.$$

Then $E(\omega) \subset \mathrm{Gr}X([0,t])$ and a Fubini argument shows that $\nu(E(\omega)) = t$ $\mathbb{P}^x$-a.s.. By Lemma 2.1, we have $\mathbb{P}^x$-a.s. $\varphi - m(\mathrm{Gr}X([0,t])) \ge C_{31}t$. It follows from the continuity of measures that there exists a constant $c > 0$ such that for every $x \in G$, $\mathbb{P}^x$-a.s., φ-$m(\mathrm{Gr}X([0,t])) \ge ct$ for all $t > 0$. The proof of Theorem 1.1 is completed. $\square$

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