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MULTIVARIATE MONOTONE LOCATION ESTIMATORS

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SUMMARY. A notion of monotonicity of multivariate location estimators is introduced. The relationships between this notion and other existing notions are discussed. Finite sample breakdown point properties of multivariate monotone location estimators are investigated. It turns out that the breakdown points of typical multivariate monotone location estimators are independent of the configuration of underlying observations and play an important role in the characterization of these estimators. Tail behavior of multivariate monotone location estimators is explored with respect to a finite sample "tail behavior" measure. Relationships among monotonicity, equivariance and high breakdown point of multivariate location estimators are discussed.

1. **Introduction**

Given a random sample $X^n = \{X_1, \ldots, X_n\}$ in $\mathbf{R}^d$ $(d \geq 1)$, consider a multivariate location estimator $T_n(X^n) = T_n(X_1, \ldots, X_n)$. Desirable properties of $T_n(\cdot)$ include *permutation invariance* and *translation equivariance*, that is,

$$T_n(X_1, \ldots, X_n) = T_n(X_{i_1}, \ldots, X_{i_n}), \quad \text{and} \quad T_n(X^n + b) = T_n(X^n) + b, \quad \forall\, b \in \mathbf{R}^d,$$

where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$ and $X^n + b = \{X_1 + b, \ldots, X_n + b\}$. Roughly speaking, the two properties mean that T_n does not depend on the labeling of observations and underlying coordinate systems. In the following discussion, T_n is always assumed to be permutation invariant.

Another desirable property of multivariate location estimators is high *breakdown point.* Breakdown point is a measure of robustness of an estimator. Hampel (1968, 1971) introduced the notion of breakdown point, generalizing an idea of Hodges (1967). Donoho (1982) and Donoho and Huber (1983) introduced finite sample versions of this notion. The *finite sample addition breakdown point* (ABP) of an estimator $T_n(\cdot)$ at X^n is defined as

$$\mathrm{ABP}(T_n, X^n) = \min\{\frac{m}{m+n} : \sup_{Y^m} \|T(X^n \cup Y^m) - T(X^n)\| = \infty\},$$

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where Y^m denotes a dataset of size m with arbitrary values and $X^n \cup Y^m$ denotes the contaminated dataset by adjoining Y^m to X^n. The *finite sample replacement breakdown point* (RBP) of an estimator $T_n(\cdot)$ at X^n is defined as

$$\mathrm{RBP}(T_n, X^n) = \min\{\frac{m}{n} : \sup_{X^n_m} \|T(X^n_m) - T(X^n)\| = \infty\},$$

where X^n_m denotes the contaminated dataset resulting from replacing m points of X^n with arbitrary values. In other words, the addition breakdown point ABP is the minimum addition fraction under which the estimator T_n can take on arbitrary large values, and the replacement breakdown point RBP is the minimum replacement fraction under which the estimator T_n can take on arbitrary large values. Recent studies related to breakdown point include Rousseeuw (1994), Tyler (1994), Singh (1998), Zhang and Li (1998), Chang *et al.* (1999), He and Fung (1999), He, Simpson and Wang (1999), Zuo (1999a), and Ghosh and Sengupta (1999).

The ABP and RBP have offered appealing, yet tractable, assessments of global robustness of estimators, complementing the assessment of local robustness captured by inference function approach (see Hampel, *et al.* (1986) and Rousseeuw and Leroy (1987)). A main reason for the attractiveness and popularity of the ABP and RBP as quantitative robustness assessments perhaps is because they are "distribution-free". As measures of global robustness, a natural question for the ABP and RBP is – are they also "sample-free"? From the definitions, it appears that the ABP and RBP of estimators depend not only on the sample size n but also on the configuration of a given data set X^n. Surprisingly, however, the ABP and RBP of many estimators are independent of the configuration of X^n and depend only slightly on the sample size n. This, of course, is desirable. But there do exist estimators whose ABP and RBP are "sample-dependent". So the question now is – what type of estimators will always have sample-free breakdown points? In $\mathbf{R}^1$, one answer to this question was given in He, *et al.* (1990) for translation equivariant location estimators.

In $\mathbf{R}^1$, it is intuitively appealing and desirable that the value of $T_n(X_1, \ldots, X_n)$ should not be decreasing when an observation in $\{X_1, \ldots, X_n\}$ is increasing. A location estimator $T_n(\cdot)$ in $\mathbf{R}^1$ with such property is said to be *monotone* (see He, *et al.* (1990) and Bassett (1991)). Equivalently, $T_n(\cdot)$ in $\mathbf{R}^1$ is monotone if

$$T_n(x_1, \ldots, x_n) \le T_n(y_1, \ldots, y_n), \quad \text{whenever } x_i \le y_i, \text{ for all } i\ .$$

He, *et al.* (1990) proved that monotonicity is a sufficient condition for the RBP of translation equivariant location estimators in $\mathbf{R}^1$ being universal. Rousseeuw (1994) discussed monotonicity of location estimators in $\mathbf{R}^1$ as an unconventional feature of positive breakdown estimators. Bassett (1991) studied monotone location estimators in $\mathbf{R}^1$ with high breakdown point and affine equivariance properties. (A location estimator $T_n(\cdot)$ in $\mathbf{R}^d$ $(d \ge 1)$ is said to be *affine equivariant* if $T_n(AX^n + b) = AT_n(X^n) + b$, for any nonsingular $d \times d$ matrix A and vector $b \in \mathbf{R}^d$. If A is orthogonal, T_n is said to be *rigid-body equivariant.*)

Order principle in $\mathbf{R}^1$ is clear and unambiguous, one could readily define the notion of monotonicity of univariate location estimators as above. To extend the univariate notion of monotonicity to higher dimensions, however, is much more complicated. One reason for this perhaps is because of the lack of any obvious means of order in higher dimensions (for more discussion on the ordering of multivariate data, see Barnett (1976)). Recently, Ghosh and Sengupta (1999) have introduced some appealing multivariate notions of monotonicity via contractions.

In this paper, a notion of monotonicity of multivariate location estimators is introduced and its relations with other existing notions are discussed. Special breakdown point properties of multivariate monotone location estimators are investigated. Finite sample tail behavior of multivariate monotone location estimators are explored with respect to a performance criterion. Inherent relationships among monotonicity, breakdown point and equivariance properties of location estimators are studied.

The rest of this paper is organized as follows. In Section 2, a notion of monotonicity of multivariate location estimators is introduced. It is shown that the new notion is a generalization of existing one-dimensional notions. The relationships between the new notion and the corresponding notions of Ghosh and Sengupta (1999) are discussed. Examples of multivariate monotone location estimators are presented. In Section 3, breakdown point properties of multivariate monotone location estimators are investigated. It is shown that the monotonicity introduced in this paper is a sufficient condition for both the RBP and the ABP of multivariate translation equivariant location estimators being independent of the configuration of given data sets (Theorem 3.1) and the breakdown points of monotone location estimators play an important role in the characterization of these estimators. In Section 4, a performance measure—*finite sample tail behavior*—of multivariate location estimators is introduced and the tail behavior of multivariate monotone location estimators is studied. A link between the finite sample breakdown points and the finite sample tail behavior of multivariate monotone location estimators is established. Section 5 is devoted to the investigation of inherent relationships among monotonicity, high breakdown point and equivariance properties of location estimators.

2. A Notion of Monotyonicity of Multivariate Location Estimators

In $\mathbf{R}^d$ $(d \geq 1)$, as a point in $X^n = \{X_1, \dots, X_n\}$ moves so that its projection on a unit vector is increasing, it seems reasonable that the projection of a location estimator $T_n(\cdot)$ on the vector should not be decreasing. We now introduce a notion of monotonicity of multivariate location estimators.

Definition 2.1. A location estimator $T_n(\cdot)$ in $\mathbf{R}^d$ $(d \geq 1)$ is said to be *monotone* if for any data set $\{x_1, \dots, x_n\}$ and unit vector u in $\mathbf{R}^d$

$$u'T(x_1, \dots, x_n) \leq u'T(x_1^*, \dots, x_n^*), \quad \text{whenever } u'x_i \leq u'x_i^*, \text{ for all } i,$$

where $v'y = \sum v_i y_i$ for any vectors $v = (v_1, \dots, v_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbf{R}^d$.

It is not difficult to see that the definition in $\mathbf{R}^1$ is equivalent to that of He, *et al.* (1990), and Bassett (1991). Mann and Whitney (1947) introduced a notion of stochastically smaller for population distributions. A random variable X in $\mathbf{R}^1$ with c.d.f. $F(x)$ is *stochastically smaller* than another random variable Y in $\mathbf{R}^1$ with c.d.f. $G(x)$, denoted by $X \le_{st} Y$, if $F(x) \ge G(x)$ for every $x \in \mathbf{R}^1$. For a functional $\psi(\cdot)$ to be a *location measure* in $\mathbf{R}^1$, Bickel and Lehmann (1975) required that ψ be "stochastic order preserving", that is, $\psi(X) \le \psi(Y)$ if $X \le_{st} Y$. Consider the empirical version of stochastic order preserving, that is, replace X and Y by random samples X^n and Y^n and distributions F and G by empirical distributions F_n and G_n in the above. It is shown below that in $\mathbf{R}^1$ our notion of monotonicity is also equivalent to the "stochastic order preserving" of Bickel and Lehmann (1975) for empirical distributions. Thus our notion of monotonicity of multivariate location estimators is a generalization of previous one-dimensional notions.

THEOREM 2.1. *In $\mathbf{R}^1$, Definition 2.1 is equivalent to* (i) *the monotonicity definitions of He et al. (1990) and Bassett (1991), and to* (ii) *the stochastic order preserving for empirical distributions.*

PROOF. Part (i) is straightforward. To prove part (ii), let $F_n(\cdot)$ and $F_n^*(\cdot)$ be the respective empirical distributions for $\{X_1, \ldots, X_n\}$ and $\{X_1^*, \ldots, X_n^*\}$. Then it is readily to see that $X_i \le X_i^*$ for $1 \le i \le n$ if and only if $F_n(x) \ge F_n^*(x)$ for every $x \in \mathbf{R}^1$. Part (ii) follows. □

REMARK 2.1. (i) Clearly, other notions of monotonicity in $\mathbf{R}^d$ $(d \ge 1)$ which generalize the univariate notions exist. For example, both Ghosh and Sengupta (1999) (directional monotonic) and Zuo (1999b) defined $T_n(\cdot)$ in $\mathbf{R}^d$ $(d \ge 1)$ to be *monotone* if for any set of points $\{x_1, \ldots, x_n\}$ and unit vector u in $\mathbf{R}^d$

$$u'T_n(x_1, \ldots, x_n) \le u'T_n(x_1^*, \ldots, x_n^*), \quad \text{whenever } x_i^* = x_i + c_i u,\ c_i \ge 0, \text{ for all } i.$$

Note, however, that the monotonicity in Definition 2.1 implies this sort of monotonicity. (ii) Ghosh and Sengupta (1999) also introduced some other notions of multivariate monotonicity based on contractions, which are quite appealing in higher dimensions. On the other hand, they are not the direct generalizations of existing univariate notions. □

We now present some examples of monotone location estimators in the sense of Definition 2.1 both in $\mathbf{R}^1$ and in $\mathbf{R}^d$ $(d > 1)$.

EXAMPLE 2.1. In $\mathbf{R}^1$, define (see Serfling, 1984)

$$T_n(X^n) = \sum_{i=1}^{n_{(m)}} c_{n,i} W_{n,i},$$

where X^n is a random sample from a distribution F, $c_{n,i}, 1 \le i \le n_{(m)}$, are arbitrary constants, $W_{n,1}, \ldots, W_{n,n_{(m)}}$ are *ordered values* of $h(X_{i_1}, \ldots, X_{i_m})$ taken over the $n_{(m)} = n(n-1)\cdots(n-m+1)$ m-tuples $(i_1, \cdots, i_m)$ of distinct elements from $\{1, \cdots, n\}$, and $h(x_1, \cdots, x_m)$ is a given "kernel" function. Then $T_n(\cdot)$ is a monotone location estimator for suitable kernel functions. Some examples of such monotone location estimators are given in the following.

(i) The sample mean, the sample α-trimmed mean, the sample quantiles, the sample median, the well-known *Hodges-Lehmann* location estimator, defined as

$$T_n(X^n) = \text{Med}\{(X_i + X_j)/2, i \neq j\},$$

and more generally, the generalized Hodges-Lehmann location estimator, see Serfling (1984),

$$T_n(X^n) = \text{Med}\{(X_{i_1} + \cdots + X_{i_m})/m\}, \text{ for a fixed choice of } m \geq 1,$$

where "Med" denotes the sample median.

(ii) Let kernel function $h(x) = x$, then $T_n(\cdot)$ includes the class of *L-statistics*, which are monotone location estimators for any choice of nonnegative constants $c_{n,i}, 1 \leq i \leq n$.

(iii) Let $h(x_1, \cdots, x_m)$ be some linear combinations of $x_1, \cdots, x_m$ with nonnegative coefficients and $c_{n,i} = 1/n_{(m)}$, for all i. Then $T_n(\cdot)$ includes the *U-statistics*, a class of monotone location estimators. (For a discussion of classical L- and U-*statistics*, see Serfling (1980).) □

EXAMPLE 2.2. In $\mathbf{R}^d$ $(d > 1)$, define $T_n(X^n)$ as in Example 2.1 except that $W_{n,1}, \ldots, W_{n,n_{(m)}}$ now are ordered values of $h(X_{i_1}, \ldots, X_{i_m})$ based on some multivariate ranking and ordering methods, such as methods based on statistical depth functions. For discussion on data depth and the induced ordering, see Liu (1990), Donoho and Gasko (1992), and Zuo and Serfling (2000a,b,d). Some examples of multivariate monotone location estimators are given in the following.

(i) The sample mean in $\mathbf{R}^d$ $(d > 1)$ is a monotone location estimator.

(ii) Let $h(x) = x$ and $W_{n,1}, \ldots, W_{n,n_{(m)}}$ be $X_1, \ldots, X_n$ respectively, then $T_n(\cdot)$ is a monotone location estimator for any choice of nonnegative data-independent constants $c_{n,i}$, $1 \leq i \leq n$. □

3. Breakdown Points of Multivariate Monotone Location Estimators

In this section, breakdown point properties of multivariate monotone location estimators are investigated. It is shown that the ABP and RBP of translation equivariant multivariate monotone location estimators are independent of the configuration of the underlying data set. On the other hand, it is shown that the breakdown points of multivariate monotone location estimators can be used in the characterization of these estimators.

3.1 *Sample-free Breakdown points.* An interesting fact about breakdown points of estimators, as pointed out in Donoho and Huber (1983), Hampel, *et al.* (1986) and Lopuhaä and Rousseeuw (1991), is that they, in many cases, do not depend on the configuration of X^n, although from definitions it seems that they do. On the other hand, examples of location estimators whose breakdown points depend on the configuration of an underlying data set X^n do exist, see Huber (1984) and Lopuhaä (1992) for M-estimators with some special bounded ρ, Stromberg and Ruppert (1992) for some estimators in nonlinear regression setting, and Zhang and

Li (1998) for related discussions. The breakdown points of test procedures are often depend on the initial data X^n, see Zhang (1996) and the references cited therein.

As measures of global robustness, the ABP and RBP of location estimators seems desirable to be independent of the configuration of any underlying data set, i.e., to be sample-free. A natural question is – what type of location estimators will have sample-free breakdown points? In $\mathbf{R}^1$, one answer to the question was given in He, *et al.* (1990). They showed that the RBP of one-dimensional translation equivariant *monotone* location estimators is sample-free. Now, in $\mathbf{R}^d$ $(d \geq 1)$, we show that the *monotonicity* introduced in Definition 2.1 is also a sufficient condition for *both* the RBP *and* the ABP of multivariate location estimators being independent of the configuration of X^n.

THEOREM 3.1. *Suppose that T_n is a translation equivariant monotone location estimator in $\mathbf{R}^d$ $(d \geq 1)$, then* $\mathrm{RBP}(T_n, X^n)$ *and* $\mathrm{ABP}(T_n, X^n)$ *are independent of the configuration of X^n.*

PROOF. We prove the above statement only for RBP. The proof for ABP could be obtained based on a similar argument.

Assume $\mathrm{RBP}(T_n, X^n) = m/n$ for a given data set $X^n = \{X_1, \ldots, X_n\}$. We show that the replacement breakdown point for any data set $Y^n = \{Y_1, \ldots, Y_n\}$ of size n is also m/n. For any $M > 0$, there exist m contaminating points $Z_1, \ldots, Z_m$ such that for the contaminated data $X_m^n = \{X_{i_1}, \ldots, X_{i_{n-m}}, Z_1, \ldots, Z_m\}$, resulting from replacing m points in X^n by $Z_1, \ldots, Z_m$,

$$\|T_n(X_m^n)\| > M + B,$$

where $B = \max_{i,j} \|X_i - Y_j\|$, $i, j = 1, \ldots, n$. It is not difficult to see that there exists a vector b with $\|b\| \leq B$ such that for $Y^n + b = \{Y_1 + b, \ldots, Y^n + b\} \equiv \{Y_1^*, \ldots, Y_n^*\}$

$$u_0' Y_{i_j}^* \geq u_0' X_{i_j}, \quad j = 1, \ldots, n-m,$$

where $Y_{i_1}^*, \ldots, Y_{i_{n-m}}^*$ are $n-m$ points from $\{Y_1^*, \ldots, Y_n^*\}$, and u_0 is a unit vector such that $u_0' T_n(X_m^n) = \|T_n(X_m^n)\|$. Let $Y_m^n = \{Y_{i_1}^*, \ldots, Y_{i_{n-m}}^*, Z_1, \ldots, Z_m\}$. Then by the monotonicity of T_n, we get

$$\begin{aligned}\|T_n(Y_m^n)\| &\geq u_0' T_n(Y_m^n)\\ &\geq u_0' T_n(X_m^n)\\ &> M + B.\end{aligned}$$

It then follows that

$$\begin{aligned}\|T_n(Y_{i_1}, \ldots, Y_{i_{n-m}}, Z_1 - b, \ldots, Z_m - b)\| &\geq \|T_n(Y_m^n)\| - \|b\|\\ &> M + B - \|b\|\\ &\geq M.\end{aligned}$$

Hence $\mathrm{RBP}(T_n, Y^n) \leq m/n$.

On the other hand, there exists some $M_1 > B$ such that for any $Z_1^*, \ldots, Z_{m-1}^*$,

$$\|T_n(X_{i_1}, \ldots, X_{i_{n-m+1}}, Z_1^*, \ldots, Z_{m-1}^*)\| < M_1 - B,$$

where $\{X_{i_1},\ldots,X_{i_{n-m+1}},Z_1^*,\ldots,Z_{m-1}^*\}$ is a contaminated data set resulting from replacing $m-1$ points in X^n by $Z_1^*,\ldots,Z_{m-1}^*$, and $X_{i_1},\ldots,X_{i_{n-m+1}}$ are $n-m+1$ points from $\{X_1,\ldots,X_n\}$. Let $Y_{m-1}^n=\{Y_{i_1},\ldots,Y_{i_{n-m+1}},Z_1,\ldots,Z_{m-1}\}$ be a contaminated data set resulting from replacing $m-1$ points in Y^n by arbitrary $m-1$ points $Z_1,\ldots,Z_{m-1}$, and $Y_{i_1},\ldots,Y_{i_{n-m+1}}$ are $n-m+1$ points from $\{Y_1,\ldots,Y_n\}$. It is not difficult to see that there exists a vector $b\in\mathbf{R}^d$ with $\|b\|\le B$ such that

$$u_0'(X_{i_j}+b)\ge u_0'Y_{i_j},\quad j=1,\ldots,n-m+1,$$

where u_0 is a unit vector such that $u_0'T_n(Y_{m-1}^n)=\|T_n(Y_{m-1}^n)\|$. Let $X_{m-1}^n=\{X_{i_1}+b,\ldots,X_{i_{n-m+1}}+b,Z_1,\ldots,Z_{m-1}\}$. It follows from monotonicity that

$$\begin{aligned}\|T_n(Y_{m-1}^n)\| &\le u_0'T_n(X_{m-1}^n)\\ &\le \|T_n(X_{m-1}^n)\|\\ &\le \|T_n(X_{i_1},\ldots,X_{i_{n-m+1}},Z_1-b,\ldots,Z_{m-1}-b)\|+\|b\|\\ &< M_1-B+\|b\|\\ &< M_1.\end{aligned}$$

Hence $\mathrm{RBP}(T_n,Y^n)\ge m/n$. The desired result now follows. □

REMARK 3.1. (i) Monotonicity is a sufficient condition for the breakdown points of multivariate location estimators being independent of the configuration of a given data set X^n. From the proof of the theorem above, however, it is easy to see that this condition can be replaced by the following *weaker* condition.

> For any data sets X^n and Y^n, and contaminating points $Z_1,\ldots,Z_m$ in $\mathbf{R}^d$ $(d\ge1,\,m\le(n+1)/2)$, there is a vector b with the same direction as that of $T_n(X_1,\ldots,X_{n-m},Z_1,\ldots,Z_m)$ and $\|b\|\le\max_{1\le i,\ j\le(n-m)}\|X_i-Y_j\|$ such that
>
> $$b'T_n(X_1,\ldots,X_{n-m},Z_1,\ldots,Z_m)\le b'T_n(Y_1+b,\ldots,Y_{n-m}+b,Z_1,\ldots,Z_m).$$

It is not difficult to see that monotone location estimator T_n always satisfies this condition. (ii) Theorem 3.1 is a multivariate generalization of the corresponding univariate result of He, *et al.* (1990). □

3.2 *Breakdown Point and Characterization.* Breakdown points of monotone location estimators are closely related to the magnitudes of these estimators and can be employed in the characterization of these estimators as shown in the following. For a unit vector $u\in\mathbf{R}^d$, let $x_{(u,i)}$ be a point in $x^n=\{x_1,\ldots,x_n\}$ such that the rank of $u'x_{(u,i)}$ in $u'x^n=\{u'x_1,\ldots,u'x_n\}$ is i, and R_i^u be the rank of $u'x_i$ among $u'x_1,\ldots,u'x_n$.

THEOREM 3.2. *Let T_n be a monotone location estimator in $\mathbf{R}^d$ $(d\ge1)$.*

(i) *If T_n is translation equivariant, then $\mathrm{RBP}(T_n,X^n)\ge m/n$ if and only if $u'X_{(u,t^*)}-M\le u'T_n(X^n)\le u'X_{(u,s^*)}+M$, for any given X^n and unit vector u, some constant $M>0$, and $t^*\le m$ and $s*\ge n-m+1$.*

(ii) *If T_n is translation and scale equivariant, then the above result hold for $M=0$.*

PROOF. (i) *Necessity* Suppose that $\mathrm{RBP}(T_n, X^n) \geq m/n$ for a given X^n in $\mathbf{R}^d$. By the translation equivariance and monotonicity of T_n, we have

$$\begin{aligned} u'T_n(X_1, \ldots, X_n) &= u'X_{(u,m)} + u'T_n(X_1^*, \ldots, X_n^*) \\ &\geq u'X_{(u,m)} + u'T_n(X_1^{**}, \ldots, X_n^{**}), \end{aligned}$$

where $X_i^* = X_i - X_{(u,m)}$, $X_i^{**} = X_i^* \mathbf{I}(R_i^u \leq m)$ for all i, and $\mathbf{I}(\cdot)$ is an indicator function. Since the RBP of T_n is sample-free (Theorem 3.1) and is $\geq m/n$, thus for a contaminated data set $\{Y_1, \ldots, Y_n\}$ resulting from replacing $(m-1)$ 0 vectors in the data set $\{0, \ldots, 0\}$ by $(m-1)$ arbitrary vectors, there exists a positive number M such that $\|T_n(Y_1, \ldots, Y_n)\| < M$. Hence we have

$$u'T_n(X_1, \ldots, X_n) \geq u'X_{(u,m)} - M.$$

Similarly, let $m^* = n - m + 1$, then $u'T_n(X_1, \ldots, X_n) \leq u'X_{(u,m^*)} + M$. Hence for any unit vector u and X^n, there is a constant $M > 0$ such that

$$u'X_{(u,t^*)} - M \leq u'T_n(X_1, \ldots, X_n) \leq u'X_{(u,s^*)} + M.$$

To prove *sufficiency*, we show that $(m-1)$ contaminating points are not sufficient to break down T_n. Assume, w.l.o.g., that $Z^n = \{Z_1, \ldots, Z_{m-1}, X_m, \ldots, X_n\}$ is a contaminated data set resulting from replacing $(m-1)$ points in X^n by arbitrary points. Since

$$u'X_{(u,m)} - M \leq u'T_n(X^n) \leq u'X_{(u,n-m+1)} + M,$$

for any given X^n and unit vector $u \in \mathbf{R}^d$, thus

$$u'X_{(u,1)} - M \leq u'T_n(Z^n) \leq u'X_{(u,n)} + M,$$

for any unit vector $u \in \mathbf{R}^d$. Hence, $\|T_n(Z^n)\|$ is bounded. Sufficiency now follows.

(ii) Clearly, this follows if we can show that the $T_n(X_1^{**}, \ldots, X_n^{**})$ in (i) now equals 0. Since there are at least $n - m + 1$ 0's in $\{X_1^{**}, \ldots, X_n^{**}\}$, and since RBP $\geq m/n$, thus

$$\sup_s \|sT_n(x_1^{**}, \ldots, x_n^{**})\| = \sup_s \|T_n(sx_1^{**}, \ldots, sx_n^{**})\| < N,$$

for some constant $N > 0$. Therefore $T_n\,(x_1^{**}, \ldots, x_n^{**}) = 0$. □

REMARK 3.2. The above result reveals interconnections between the RBP of monotone location estimators and the magnitudes of these estimators in $\mathbf{R}^d$ $(d \geq 1)$. The result is generalizations of the corresponding results of He, *et al.* (1990) and Bassett (1991) in $\mathbf{R}^1$. □

Although, the breakdown points of typical multivariate monotone location estimators are sample-free as well as distribution-free, there still exist intimate connections between the breakdown points and sampling distribution of multivariate monotone location estimators and the underlying population distribution. In fact, it is shown in the next that the breakdown points of typical multivariate monotone location estimators are closely linked with a tail performance measure of these estimators.

4. Tail Behavior of Multivariate Monotone Location Estimators

Let $T_n = T_n(X_1, \ldots, X_n)$ be an estimator of an unknown parameter θ based on a random sample $X_1, \ldots, X_n$ from a random vector X in $\mathbf{R}^d$ ($d \geq 1$) with $F(x-\theta)$ and P_θ being the corresponding distribution function and probability measure. How good is T_n as an estimator of θ? Various performance measures of T_n, including *unbiasedness*, *consistency*, and *mean squared error*, have been proposed and studied. Asymptotic behavior of T_n in $\mathbf{R}^1$ has also been investigated, based on the behavior of $-\ln P_\theta(|T_n - \theta|) > a)/n$, for a *fixed* a as $n \to \infty$, see Bahadur (1967, 1971), Fu (1975), and Sievers (1978).

In practice, however, the sample size n is *finite* and *fixed*. In sharp contrast with asymptotic approach to the performance of estimators, Jurečková (1981) studied, for centrally symmetric F in $\mathbf{R}^1$, the performance of location estimator T_n based on the behavior of

$$\frac{-\ln P_\theta(|T_n - \theta| > a)}{-\ln(1 - F(a))}, \quad \text{as } a \to \infty$$

for *fixed* n. He, *et al.* (1990) studied the tail behavior of regression estimators based on a similar measure. Kušnier and Mizera (1999) considered the tail behavior of one-dimension translation and scale equivariant location estimators for some special absolutely continuous distribution F in $\mathbf{R}^d$.

Generalizing the univariate tail performance measure of location estimators in Jurečková (1981) to higher dimensions, we now introduce a measure of performance of multivariate location estimator T_n in $\mathbf{R}^d$ ($d \geq 1$) as follows (also see Zuo (1999b)). For a *fixed* n, a unit vector u, clearly one would like the "tail" (halfspace) probability $P_\theta(u'(T_n - \theta) > a)$ to be as small as possible for a fixed $a > 0$ and to approach zero as fast as possible when $a \to \infty$. On the other hand, the behavior of this tail probability depends intrinsically on the behavior of the tail probability $P_\theta(u'(X-\theta) > a)$ of an underlying distribution. Thus one could introduce a finite sample tail performance measure of T_n in $\mathbf{R}^d$ based on the behavior of

$$B(T_n; u, a) = \frac{-\ln P_\theta(u'(T_n - \theta) > a)}{-\ln P_\theta(u'(X - \theta) > a)}, \qquad \text{as } a \to \infty,$$

for a unit vector u and a *fixed* n. The limit above (if exists) measures the relative rate of the tail probability of the sampling distribution of T_n approaching zero w.r.t. the tail probability of the underlying population distribution. Clearly, for good estimator T_n, one would expect the quantity $B(T_n; u, a)$ to be high for any unit vector u as $a \to \infty$. How high could $B(T_n; u, a)$ be as $a \to \infty$? For translation equivariant monotone location estimators, it is bounded below and above by 1 and n. That is, the tails of the sampling distribution of translation equivariant monotone location estimator T_n approach zero at least as fast as the tails of the underlying distribution and could be at most n times as fast.

Since for translation equivariant T_n, $P_\theta(u'(T_n - \theta) > a) = P_0(u'T_n > a)$ and $P_\theta(u'(X-\theta) > a) = P_0(u'X > a)$, where P_0 is the probability measure corresponding to $X - \theta$, we assume in the following that $\theta = 0$. (Location parameter θ could be the point of symmetry of a centrally, angularly or even halfspace symmetric distribution F. For definitions and discussions of these symmetric distributions, see Zuo and Serfling (2000c). In our following discussion, however, symmetry is not required.) For simplicity, P_0 is written as P.

THEOREM 4.1. *Let T_n be a translation equivariant monotone location estimator in $\mathbf{R}^d$ $(d \geq 1)$ such that $T_n(0,\ldots,0) = 0$. Then*

$$1 \leq \liminf_{a\to\infty} \inf_u B(T_n; u, a) \leq \limsup_{a\to\infty} \sup_u B(T_n; u, a) \leq n,$$

where u is a unit vector in $\mathbf{R}^d$ and a is some positive number.

PROOF. The middle inequality is trivial. It is easy to see that $T_n(b,\ldots,b) = b$ for any vector $b \in \mathbf{R}^d$. Suppose that for a given unit vector u and a positive number a there is a $b \in \mathbf{R}^d$ such that $u'b = a$. By the conditions on T_n we get

$$\begin{aligned} P(u'T_n > a) &= P(u'T_n(X_1,\ldots,X_n) > a) \\ &\geq P(u'X_{(u,1)} > u'b) \\ &= P(u'X_i > a,\ i = 1,\ldots,n) \\ &= (P(u'X > a))^n. \end{aligned}$$

Then

$$\begin{aligned} \frac{-\ln P(u'T_n > a)}{-\ln P(u'X > a)} &\leq \frac{-n\ln P(u'X > a)}{-\ln P(u'X > a)} \\ &= n, \end{aligned}$$

which implies the desired rightmost inequality. Similarly,

$$\begin{aligned} P(u'T_n > a) &= P(u'T_n(X_1,\ldots,X_n) > a) \\ &\leq P(u'X_{(u,n)} > u'b) \\ &= 1 - P(u'X_{(u,n)} \leq a) \\ &= 1 - P(u'X_i \leq a,\ i = 1,\ldots,n) \\ &= 1 - (1 - P(u'X > a))^n. \end{aligned}$$

Now it is not difficult to see that

$$\lim_{a\to\infty} \frac{\ln(1 - (1 - P(u'X > a))^n)}{\ln P(u'X > a)} = 1,$$

uniformly with respective to u. The desired leftmost inequality follows. $\square$

REMARK 4.1. (i) It can be seen that the above theorem holds if $T_n(\cdot)$ is translation equivariant such that for any unit vector u

$$(1)\;\; u'X_{(u,1)} > 0 \Rightarrow u'T(X^n) > 0 \;\text{ and }\; (2)\;\; u'X_{(u,n)} < 0 \Rightarrow u'T(X^n) < 0.$$

The result obtained under (1) and (2) generalizes the corresponding one-dimensional result of Jurečková (1981) (see Zuo (1999b)). Note that monotone location estimators always satisfy (1) and (2). (ii) $T(0,\ldots,0) = 0$ holds if $T(\cdot)$ is *odd*, that is, $T_n(x_1,\ldots,x_n) = -T_n(-x_1,\ldots,-x_n)$, or more strongly if $T(\cdot)$ is *scale equivariant*, that is, $T_n(cx_1,\ldots,cx_n) = cT_n(x_1,\ldots,x_n)$ for any scalar c and for any $x_1,\ldots,x_n$ in $\mathbf{R}^d$. □

Note that the minimum contaminating number m in the definition of RBP is bounded between 1 and n, the same lower and upper bounds as those in Theorem 4.1. We thus see a connection between the tail behavior and the finite sample RBP of multivariate monotone location estimators. This connection is further explored in the remainder of this section.

It certainly would be interesting to know about exact values of the $\liminf_{a\to\infty} \inf_u B(T_n;u,a)$ and $\limsup_{a\to\infty}\sup_u B(T_n;u,a)$ in Theorem 4.1 in one interesting example. We now explore the answer for Hodges-Lehmann estimator.

EXAMPLE 4.1. Let HL_n be the well-known Hodges-Lehmann location estimator, defined as

$$T_n(X^n) = \mathrm{Med}\{(X_i + X_j)/2,\;\; i \neq j\}.$$

First we derive the RBP of the HL_n. Assume that the RBP of the estimator is m/n. Then strategically replacing m points in original data set $X^n = \{X_1,\ldots,X_n\}$ by carefully chosen m "bad" points could lead to a contaminated majority of the total $n(n-1)/2$ points $(X_i + X_j)/2, i \neq j$. Hence

$$n(n-1)/2 - (n-m)(n-m-1)/2 \geq (n-m)(n-m-1)/2.$$

It follows that $m = \lceil (2n - 1 + \sqrt{2n^2 - 2n + 1})/2 \rceil$ and $n-m+1 = \lfloor (\sqrt{2n^2 - 2n + 1} -1)/2 \rfloor + 1$, which, based on the following theorem, are the exact values of the $\liminf_{a\to\infty}\inf_u B(T_n;u,a)$ and $\limsup_{a\to\infty}\sup_u B(T_n;u,a)$ in Theorem 4.1. Here $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor and ceiling functions respectively. □

THEOREM 4.2. *Let T_n be a translation and scale equivariant monotone location estimator with RBP$= m/n$. Then*

$$m \leq \liminf_{a\to\infty}\inf_u B(T_n;u,a) \leq \limsup_{a\to\infty}\sup_u B(T_n;u,a) \leq n - m + 1.$$

PROOF. By Theorems 3.3 and 3.1, we have $u'x_{(u,m)} \leq u'T_n(x_1,\ldots,x_n) \leq u'x_{(u,n-m+1)}$ for any $x_1,\ldots,x_n$ in $\mathbf{R}^d$. Thus for any unit vector u and positive number a,

$$P(u'T_n > a) \;\;\leq\;\; P(u'X_{(u,n-m+1)} > a)$$

$$
\begin{aligned}
&= \sum_{s=m}^{n} \binom{n}{s} \left(P(u'X > a)\right)^{s} \left(P(u'X \le a)\right)^{n-s} \\
&= \left(P(u'X > a)\right)^{m} \sum_{s=m}^{n} \binom{n}{s} \left(P(u'X > a)\right)^{s-m} \left(P(u'X \le a)\right)^{n-s}.
\end{aligned}
$$

Similarly, let $m^* = n - m + 1$, we have

$$
P(u'T_n > a) \;\ge\; \left(P(u'X > a)\right)^{m^*} \sum_{s=m^*}^{n} \binom{n}{s} \left(P(u'X > a)\right)^{s-m^*} \left(P(u'X \le a)\right)^{n-s}.
$$

The desired result now follows. □

Dropping scale equivariance of T_n in the above theorem and adding a condition to the underlying population distributions, we have

THEOREM 4.3. *Let T_n be a translation equivariant monotone location estimator with RBP= m/n and X be a random vector such that $\lim_{a\to\infty} \ln P(u'X > a + c)/\ln P(u'X > a) = 1$ for any unit vector u in $\mathbf{R}^d$ and fixed c. Then*

$$
m \le \liminf_{a\to\infty} \inf_{u} B(T_n; u, a) \le \limsup_{a\to\infty} \sup_{u} B(T_n; u, a) \le n - m + 1.
$$

PROOF. By Theorems 3.2 and 3.1, we have for any $X^n = \{X_1, \ldots, X_n\}$ that

$$
\begin{aligned}
P(u'T_n > a) &\le P(u'X_{(u,n-m+1)} > a - M) \\
&= \sum_{s=m}^{n} \binom{n}{s} \left(P(u'X > a - M)\right)^{s} \left(P(u'X \le a - M)\right)^{n-s} \\
&= \left(P(u'X > a - M)\right)^{m} \sum_{s=m}^{n} \binom{n}{s} \left(P(u'X > a - M)\right)^{s-m} \\
&\quad \left(P(u'X \le a - M)\right)^{n-s}.
\end{aligned}
$$

Similarly, let $m^* = n - m + 1$, we have

$$
\begin{aligned}
P(u'T_n > a) \;\ge\; & \left(P(u'X > a + M)\right)^{m^*} \sum_{s=m^*}^{n} \binom{n}{s} \left(P(u'X > a + M)\right)^{s-m^*} \\
& \left(P(u'X \le a + M)\right)^{n-s}.
\end{aligned}
$$

The desired result now follows. □

REMARK 4.2. (i) Theorem 4.3 holds for a wide range of distributions (including multivariate normal distributions). In $\mathbf{R}^1$, the theorem yields a result which generalizes a corresponding result of He, *et al.* (1990). (ii) The above two theorems reveal inherent connections between the finite sample tail behavior and the finite sample breakdown points of monotone location estimators. High RBP of a monotone location estimator T_n corresponds to an attractive tail performance of

T_n. (iii) These results also provide probabilistic interpretations of the finite sample RBP based on the finite sample tail behavior of monotone location estimators. $\square$

Breakdown points of translation equivariant multivariate monotone location estimators determine lower tail performance bounds of estimators. High breakdown points means appealing tail performance. The question now is - how high could the breakdown points be? Are breakdown point, monotonicity and equivariance compatible with each other or antipathetic to each other? In the next, we discuss relationships among monotonicity, equivariance and breakdown point.

5. Monotonicity, Affine Equivariance and High Brakdown Points

Monotonicity seems to be a natural and intuitively appealing property of multivariate location estimators. It is also a sufficient condition for translation equivariant location estimators having sample-free breakdown points, a desirable property of location estimators. However, combining monotonicity with affine equivaniance and high breakdown points is not trivial.

5.1 *Univariate case.* A data analyst desiring a location estimator T_n not biased toward any placement of coordinate axes and scale of measurements might require *affine equivariance* of T_n. However, monotonicity is not always accompanied with affine equivariance and high breakdown points, as pointed out in Rousseeuw (1994). In fact, Bassett (1991) proved that in $\mathbf{R}^1$ the sample median is the only location estimator which is monotone, affine equivariant and with the best possible breakdown points (ABP=1/2).

At this point one asks-how important is affine (or rigid-body) equivariance? The answer of course depends on the statistician and the data analytic situation. Affine (or rigid-body) equivariance may be less compelling, for example, when the original coordinates and measurement scale have valuable interpretations. In such case, more monotone location estimators which are translation equivariant and with the best breakdown points exist. For example, in $\mathbf{R}^1$ define

$$T_n(X^n) = \alpha X_{(\lfloor (n+1)/2 \rfloor)} + (1-\alpha) X_{(\lfloor (n+2)/2 \rfloor)},$$

where $\alpha \in (0,1)$ and $X_{(1)} \leq, \ldots, \leq X_{(n)}$ are ordered values of $X_1, \ldots, X_n$. Then it is not difficult to see that $T_n(\cdot)$ is monotone, translation equivariant and with the best breakdown points (ABP=1/2). Also there are monotone, translation equivariant M-*estimators* in $\mathbf{R}^1$ with the best breakdown points. Define an M-estimator T_n in $\mathbf{R}^1$ as any solution of the equation

$$\sum_{i=1}^{n} \psi(X_i - T) = 0,$$

with respect to T, where $\psi(\cdot)$ is an appropriate nondecreasing odd bounded function. One example of $\psi(\cdot)$ is Huber $\psi_c(\cdot)$ function, $\psi_c(x) = \min\{c, max\{-c, x\}\}$, with some finite value $c > 0$, see Huber (1964). We have

THEOREM 5.1. *Let T_n be an M-estimator in $\mathbf{R}^1$ corresponding to the nondecreasing odd function ψ such that $\psi(x) = \psi(k)$ for $x \geq k$, $k > 0$. Then T_n is monotone, translation equivariant with replacement breakdown point $\lfloor (n+1)/2 \rfloor$ and addition breakdown point $1/2$.*

PROOF. Monotonicity and translation equivariance are straightforward. We prove here only the RBP result, the ABP result could be proved similarly, or could be obtained directly from a quantitative relationship between RBP and ABP (see Zuo (1999a)).

We now prove that $m = \lfloor (n+1)/2 \rfloor - 1$ contaminating points $Z_1, \ldots, Z_m$ are not sufficient to break down T_n. Let $X_m^n = \{X_{i_1}, \ldots, X_{i_{n-m}}, Z_1, \ldots, Z_m\}$ be the contaminated data set resulting from replacing m points in X^n by arbitrary $Z_1, \ldots, Z_m$. We show that $T_n(X_m^n)$ is bounded. Since, otherwise, suppose that $|T_n(X_m^n)| > \max_{1 \leq j \leq n-m}\{|X_{i_j}|\} + k$, then it is not difficult to see that

$$\sum_{i=1}^{n} \psi(X_i - T_n(X_m^n)) \neq 0.$$

Thus $m = \lfloor (n+1)/2 \rfloor - 1$ contaminating points are not sufficient to break down T_n. Combining this with the upper bound of RBP (see Lopuhaä and Rousseeuw (1991)) of translation equivariant location estimators, we obtain the desired result. □

At this point one may also ask-how high should breakdown points of an estimator be? The answer again depends on the statistician and the data analytic situation. Contaminations between 1% and 10% are sufficiently common that a blind application of procedures with breakdown points of less than 10% might be dangerous. However, as Huber (1993) pointed out, it is difficult to find a rational justification for a breakdown point higher than 25%, if this should result in a serious efficiency loss at the model, or in excessive computation costs. If estimators which can resist 30% contamination or outliers are good enough, we then have trimmed-means and Hodges-Lehmann estimators which are monotone and affine equivariant and with breakdown point about 30%.

It is noted that some well-known location estimators in $\mathbf{R}^1$ such as LMS estimators (Rousseeuw (1984)) and some M-estimators (Rousseeuw (1994)), which are affine equivariant and with the best breakdown points, are not monotone. So in $\mathbf{R}^1$ monotonicity is more restrictive than it appears.

5.2 *Multivariate Case.* Monotonicity is also very restrictive in higher dimensions. In fact, some prevalent affine (or rigid-body) equivariant multivariate medians such as the L_1-median (Haldane (1948)), the halfspace median (Donoho and Gasko (1992)) and the simplicial median (Liu (1990)) are not monotone. Let $X_1, \ldots, X_n$ be a sample from F in $\mathbf{R}^d$ ($d \geq 1$). The sample L_1*-median* (also called spatial median in the literature) is defined to be the point θ such that

$$\theta = \arg \inf_{x \in \mathbf{R}^d} \sum_{i=1}^{n} \|X_i - x\|,$$

where $\|\cdot\|$ denotes the usual Euclidean norm. The L_1-median is rigid-body equivariant. The sample *halfspace median*, denoted by H-median, is defined to be the point θ such that

$$\theta = \arg\max_{x\in\mathbf{R}^d} \inf_{\|u\|=1} \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(u'X_i \geq u'x),$$

where u is a unit vector in $\mathbf{R}^d$. The H-median is affine equivariant. The sample *simplicial median*, denoted by S-median, is defined be the point θ such that

$$\theta = \arg\max_{x\in\mathbf{R}^d} \binom{n}{d+1}^{-1} \sum_{1\leq i_1<\cdots<i_{d+1}\leq n} \mathbf{I}\{x\in S[X_{i_1},\ldots,X_{i_{d+1}}]\},\ x\in\mathbf{R}^d.$$

The S-median is affine equivariant. The ABP of L_1-median is $1/2$ (50%), the ABP of H-median could reach $1/3$ for large sample, and the ABP of S-median is at most $1/(d+2)$, see Lopuhaä and Rousseeuw (1991), Donoho and Gasko (1992), and Chen (1995), also see Zuo and Serfling (2000c).

THEOREM 5.2. *The sample H-, L_1-, and S-medians in $\mathbf{R}^d$ are not monotone.*

PROOF. Without loss of generality, assume that we have a sample of size 4 in $\mathbf{R}^2$ with $x_1 = A = (0,0)$, $x_2 = B = (2,0)$, $x_3 = C = (0,1)$ and $x_4 = D = (2,1)$. It is not very difficult to verify that the point $E = (1,1/2)$ is the sample H-, L_1-, and S-medians. Now let point x_4 move to point $x_5 = F = (2,2)$, and assume that $x_1^* = x_1, \ldots x_3^* = x_3, x_4^* = x_5$ is another observed sample. Then it is seen that the intersecting point of AF and BC is the H-, L_1-, and S-medians of this sample. Take u to be the unit vector parallel with the vector $\overrightarrow{AD}$. Then it is easy to see that $u'x_i \leq u'x_i^*,\ \ i = 1,\ldots,4$. But $u'T(x_1,\ldots,x_4) > u'T(x_1^*,\ldots,x_4^*)$. Thus the sample H-, L_1-, and S-medians are not monotone estimators. □

REMARK 5.1. Ghosh and Sengupta (1999) introduced the notions of (weak) monotonicity and directional monotonicity, and were unable to verify the monotonic status of the nonlinear L_1-median and halfspace median directly (see page 369 of Ghosh and Sengupta (1999)). Instead, they verified a weaker version of directional monotonicity for the L_1-median, and obtained some partial results for some of these highly nonlinear estimators. Now based on the proof of Theorem 5.2, we can conclude that the L_1-, the S-, and the halfspace medians are not directional monotonic. Similarly, one can show that these leading multivariate location estimators are also not monotonic or weakly monotonic in the sense of Ghosh and Sengupta (1999). □

In $\mathbf{R}^d$ $(d > 1)$, it is not clear that if there exist or if one can construct location estimators which are monotone, affine equivariant and with the best breakdown points (ABP=1/2). However, it seems clear that the notion of monotonicity of location estimators introduced in this article is more restrictive than it appears. There is some trade-off among monotonicity, high breakdown point, and affine equivariance. Adopted a different approach, Ghosh and Sengupta (1999) introduced a notion of monotonicity based on contractions, which has led to a successful construction of location estimators with monotonicity, rigid-body (affine) equivariance and high breakdown point.

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