Some Quantitative Relationships Between Two Types of Finite Sample Breakdown Point

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Abstract

The two versions of finite-sample breakdown points, the addition breakdown point and the replacement breakdown point, have been employed in the derivation of the breakdown points for various estimators. However, for a given estimator, frequently only the version that is convenient in the derivation is selected. The connections between the two versions have not yet been formally explored in the literature. In this paper, quantitative relationships between the two versions of the breakdown point are formally established. The result obtained allows one to obtain the ABP directly from the RBP of estimators and *vice versa* and thus to avoid some unnecessary, sometimes tedious, derivations of breakdown points.

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Short Running title: Relations between two types of breakdown point.

1 Introduction

The notion of breakdown point first appeared in Hodges (1967), and later was generalized by Hampel (1968, 1971). The finite sample versions of the breakdown point, including the *addition breakdown point* (ABP) and *replacement breakdown point* (RBP), were introduced in Donoho and Huber (1983), and have become most prevalent quantitative assessments of global robustness of estimators, complementing the assessment of local robustness of estimators captured by the influence function approach (see Hampel, et al. (1986) and Rousseeuw and Leroy (1987)). Indeed, one primary goal of much recent research in robustness has been the construction of high breakdown methods exemplified in the linear model by Rousseeuw (1984), and location and scatter models by Lopuhaä and Rousseeuw (1991), Donoho and Gasko (1992) and Tyler (1994). Recent studies related the breakdown point include Davies (1987), Maronna and Yohai (1991), Stromberg and Ruppert (1992) and Stromberg (1993), Müller (1995), Chen (1995), Zhang and Li (1998), Singh (1998), He and Fung (1999), He, Simpson and Wang (1999), Ghosh and Sengupta (1999), and Chang, et al (1999).

Let $X^n = \{X_1, \ldots, X_n\}$ be a sample of size n in $\mathbf{R}^d$. The finite sample addition breakdown point of an estimator T at X^n is defined as

$$\mathrm{ABP}(T, X^n) = \min\{\frac{m}{m+n} : \sup_{Y^m} \|T(X^n \cup Y^m) - T(X^n)\| = \infty\},$$

where Y^m denotes a dataset of size m with arbitrary values and $X^n \cup Y^m$ denotes the contaminated sample by adjoining Y^m to X^n. The finite sample replacement breakdown point of an estimator T at X^n is defined as

$$\mathrm{RBP}(T, X^n) = \min\{\frac{m}{n} : \sup_{X^n_m} \|T(X^n_m) - T(X^n)\| = \infty\},$$

where X^n_m denotes the corrupted sample from X^n by replacing m points of X^n with arbitrary values. In other words, the ABP and RBP of an estimator are respectively the minimum addition fraction and replacement fraction which could drive the estimator beyond any bound. For an estimator C of a covariance matrix, the above definitions of the breakdown point are applied to $T = (\log \lambda_1(C), \ldots, \log \lambda_d(C))'$ with $\lambda_1(C) \geq \ldots \geq \lambda_d(C)$ being the ordered eigenvalues of C; see Donoho and Huber (1983).

In assessing the global robustness of estimators, some studies, including Donoho (1982), Donoho and Huber (1983), Huber (1984), Donoho and Gasko (1992), Chen (1995), and He, Simpson and Wang (1999), focused on the ABP of estimators. Others, however, have concentrated attention on the RBP of estimators; see Davies (1987), Lopuhaä and Rousseeuw

(1991), Tyler (1994), and Singh (1998). Natural questions raised here are: Are there any quantitative relationships between the ABP and RBP of estimators? Do we have to separately investigate the two types of the breakdown point for the same estimator each time? Section 2 is devoted to the discussion of these questions, where quantitative relationships between the ABP and RBP of estimators are formally formulated. Examples of obtaining ABP directly from RBP and *vice versa* based on these relationships are given in Section 3. Some proofs of the results in the paper are presented in the Appendix.

2 Quantitative relationships between ABP and RBP

Since the introduction of the ABP and RBP by Donoho and Huber in 1983, the two versions of breakdown point have often been discussed separately for estimators, and treated as two essentially independent notions. Some authors are favorable to the ABP in the discussion of the robustness property of estimators, whereas others prefer the RBP, which they believe is more simple, realistic and generally more applicable; see, for example, the remark on pp. 117-118 of Rousseeuw and Leroy (1987). Are there any quantitative connections between the ABP and the RBP of estimators? No answer has been given in the literature. For a specific location estimator (the Stahel-Donoho location estimator), Davies (1987) informally derived the RBP from the ABP of the estimator. In this section, general quantitative interrelationships between the ABP and the RBP are formally established for a large class of estimators.

A surprising fact about the breakdown points of an estimator T at X^n is that they, in many cases, depend only on the sample size n and are independent of the configuration of the X^n (though, from the definitions, it seems that they should depend on the sample X^n). On the other hand, for the breakdown points as *global* robustness assessments of estimators, this "sample-free" property is definitely desirable. It is noteworthy that *ad hoc* estimators whose breakdown points depend on the configuration of the X^n *do* exist, see Huber (1984), Lopuhaä (1992) and Zhang and Li (1998) for related discussions.

In the following, we confine attention to estimators whose breakdown points are independent of the configuration of X^n, and whose RBPs are of the form $\lfloor an+b \rfloor/n$ or $\lceil an+b \rceil/n$. In fact, this *is* the case for most of location and scatter estimators. Here $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor and ceiling functions respectively (i.e. $\lfloor x \rfloor$ is the greatest integer no greater than x and $\lceil x \rceil$ is the smallest integer no smaller than x). We have the following results (proofs are given in the Appendix).

Theorem 2.1 *Let T be an estimator of interest in $\mathbf{R}^d$ and X^n be a dataset of size n.*
(a) *The addition breakdown point of T can be expressed as*

$$\mathrm{ABP}(T, X^n) = (\lfloor \epsilon n + \beta \rfloor + m)/(n + \lfloor \epsilon n + \beta \rfloor + m),$$

if the replacement breakdown point of T can be expressed as

$$\mathrm{RBP}(T, X^n) = \lfloor (\epsilon n + (\beta + m + \epsilon))/(\epsilon + 1) \rfloor / n,$$

(b) *The replacement breakdown point of T can be expressed as*

$$\mathrm{RBP}(T, X^n) = \lfloor (\epsilon n + (\beta + m + r))/(\epsilon + 1) \rfloor / n,$$

if the addition breakdown point of T can be expressed as

$$\mathrm{ABP}(T, X^n) = (\lfloor \epsilon n + \beta \rfloor + m)/(n + \lfloor \epsilon n + \beta \rfloor + m),$$

where ϵ, β and integer m are some constants independent of n and X^n, $0 \le \beta \le \epsilon \le 1$, and $\max\{1 - (\epsilon n + \beta + m)/(\epsilon + 1),\ \max_N\{D^-(N)\}\} \le r/(\epsilon + 1) < 1 + \min_N\{D^-(N)\}$ with $N = \lfloor (\epsilon + 1)n + \beta + m \rfloor$ and $D^-(n) = \lfloor (\epsilon n + \beta + m)/(\epsilon + 1) \rfloor - (\epsilon n + \beta + m)/(\epsilon + 1)$.

Theorem 2.2 *Let T be an estimator of interest in $\mathbf{R}^d$ and X^n be a dataset of size n.*
(a) *The addition breakdown point of T can be expressed as*

$$\mathrm{ABP}(T, X^n) = (\lceil \epsilon n + \beta \rceil + m)/(n + \lceil \epsilon n + \beta \rceil + m),$$

if the replacement breakdown point of T can be expressed as

$$\mathrm{RBP}(T, X^n) = \lceil (\epsilon n + \beta + m)/(\epsilon + 1) \rceil / n,$$

(b) *The replacement breakdown point of T can be expressed as*

$$\mathrm{RBP}(T, X^n) = \lceil (\epsilon n + (\beta + m + r))/(\epsilon + 1) \rceil / n,$$

if the addition breakdown point of T can be expressed as

$$\mathrm{ABP}(T, X^n) = (\lceil \epsilon n + \beta \rceil + m)/(n + \lceil \epsilon n + \beta \rceil + m),$$

where ϵ, β and integer m are some constants independent of n and X^n, $0 \le \beta \le \epsilon \le 1$, and $\max\{-(\epsilon n + \beta + m)/(\epsilon + 1), \max_N\{D^+(N)\} - 1\} < r/(\epsilon + 1) \le \min_N\{D^+(N)\}$ with $N = \lfloor (\epsilon + 1)n + \beta + m \rfloor$ and $D^+(n) = \lceil (\epsilon n + \beta + m)/(\epsilon + 1) \rceil - (\epsilon n + \beta + m)/(\epsilon + 1)$.

Remark 2.1 (i) It can be shown that the above two theorems could be extended to the cases where m depend only on n, d and (or) some structural (geometrical) properties of X^n such as $c(X^n)$ (the maximum number of data points contained in any $d-1$ dimensional hyperplane), but not on the values of $X_i \in X^n$. (ii) The above results hold for scale parameter or scatter matrix estimators if the breakdown point definition for these estimators in Section 1 is adopted, and are independent of the type of bias resulting from the contamination. (iii) If the RBP of an estimator T is of the form, say $\lfloor an+b \rfloor/n$ (of course, $a \le 1/2$), then the RBP can be *uniquely* expressed as $\lfloor(\epsilon n + (\beta + m + \epsilon))/(\epsilon+1)\rfloor/n$ with $\epsilon = a/(1-a)$, $m = \lfloor (b-a)/(1-a) \rfloor$ and $\beta = (b-a)/(1-a) - m$. On the other hand, if the ABP of T is of the form $(\lfloor \epsilon n + \beta \rfloor + m)/(n + \lfloor \epsilon n + \beta \rfloor + m)$, then the ϵ, β and the m are *uniquely* identified. □

3 Examples

Applying the results in Section 2, we can obtain the ABP directly from the RBP of estimators and *vice versa*, and hence avoid some unnecessary, sometimes tedious, derivations. This is illustrated in the following examples.

Example 3.1 *Obtain the RBP directly from the ABP of the sample median in* $\mathbf{R}^1$. The ABP of the sample median, given in Donoho and Huber (1983), is 1/2. To utilize Theorem 2.1 to get RBP we need to identify ϵ, β and m. It is readily seen that the ϵ, β and m for this estimator are *uniquely* determined and $\epsilon = 1$, $\beta = 0$ and $m = 0$. By the theorem the RBP of the estimator is $\lfloor (n+r)/2 \rfloor/n$, with $\max\{1/2,\ \max_N\{\lfloor N/2 \rfloor - N/2\}\} \le r/2 < 1 + \min_N\{\lfloor N/2 \rfloor - N/2\}$, and $N = \lfloor 2n \rfloor = 2n$. Thus, $1 \le r < 2$. (Consequently, the RBP of the estimator can be written as $\lfloor (n+1)/2 \rfloor/n$.) One can verify that this RBP is exactly the same as the one derived from the definition. □

Example 3.2 *Obtain the ABP directly from the RBP of the* L_1*-median in* $\mathbf{R}^d (d \ge 1)$. The L_1-median (also called the spatial median in the literature) is defined to be the vector T such that

$$T = \arg\min_{x \in \mathbf{R}^d} \sum_{i=1}^{n} \|X_i - x\|,$$

for any sample $X_1, \ldots, X_n$. Lopuhaä and Rousseeuw (1991) showed that $\mathrm{RBP}(T, X^n) = \lfloor (n+1)/2 \rfloor/n$. To utilize Theorem 2.1 we need to identify ϵ, β and m. It is seen that for this estimator $\epsilon = 1$, $\beta = 0$ and $m = 0$ are uniquely identified. Now by the theorem the ABP of this estimator is 1/2. Following the lines of the proof of Theorem 2.2 of Lopuhaä and Rousseeuw (1991), one can verify that $1/2$ is exactly the ABP of the L_1-median. Here the ABP is obtained without the long derivation. $\square$

Example 3.3 *Obtain the ABP directly from the RBP of the least median of squares (LMS) estimator.* The LMS estimator was defined in Rousseeuw (1984) to be the vector $\hat{\theta}$ which minimizes $\mathrm{Med}_{1\le i\le n} r_i^2$, where r_i are the residuals $y_i - x_{i1}\hat{\theta}_1 - \cdots - x_{id}\hat{\theta}_d$ in the linear model $y_i = x_{i1}\theta_1 + \cdots + x_{id}\theta_d + e_i,\ \ i = 1, \cdots, n$. Rousseeuw (1984) proved that for $p > 1$ and observations in general position (any d of them give a unique determination of θ), $\mathrm{RBP}(\hat{\theta}, X^n) = (\lfloor n/2 \rfloor - d + 2)/n$, where $X_i = \{y_i, x_{i1}, \cdots, x_{id}\}$. Set $\epsilon = 1$, $\beta = 0$ and $m = -2d+3$, then by Theorem 2.1, $\mathrm{ABP}(\hat{\theta}, X^n) = (n-2d+3)/(2n-2d+3)$. Following the proof of Rousseeuw (1984), we can derive the above ABP result. Such a derivation, however, is tedious and becomes unnecessary after we have Theorem 2.1. $\square$

Example 3.4 *Obtain the ABP directly from the RBP of the S-estimators of multivariate location and scatter in* $\mathbf{R}^d (d \ge 1)$. Rousseeuw and Yohai (1984) introduced S-estimators in the regression setting, Davies (1987) defined the S-estimators $\mu^{(n)}$ and $\Sigma^{(n)}$ of the location and scatter parameters μ and Σ in elliptical distributions. He considered the *simultaneous* RBP of $\mu^{(n)}$ and $\Sigma^{(n)}$ (the simultaneous breakdown point of estimators is the minimum fraction of contamination which makes at least one of the estimators break down), and set

$$T(X^n) = \|\mu^{(n)}\| + \sum_{i=1}^{d} \left(\sigma_j^{(n)} + \sigma_j^{(n)^{-1}} \right),$$

where $\sigma_1^{(n)}, \ldots, \sigma_d^{(n)}$ are the eigenvalues of $\Sigma^{(n)}$. He showed that, under certain conditions, the simultaneous RBP of $\mu^{(n)}$ and $\Sigma^{(n)}$ is $(\lfloor \epsilon^* n \rfloor + 1)/n$, for $n(1 - 2\epsilon^*) = d + 1$ and X^n in general position (i.e., no more than d data points of X^n lie in any $d-1$ dimensional subspace). Now, set $\epsilon = 1$, $\beta = 0$ and $m = -d$, then by Theorem 2.1, we immediately have that $\mathrm{ABP}(T, X^n) = (n-d)/(2n-d)$. Although one can establish the above result with a proof similar to that of Davies' (1987), directly applying Theorem 2.1 is much easier and more efficient.

Lopuhaä and Rousseeuw (1991) showed that, under certain conditions, the RBPs of the S-estimators $\mu^{(n)}$ and $\Sigma^{(n)}$ are $\mathrm{RBP}(\mu^{(n)}, X^n) = \mathrm{RBP}(\Sigma^{(n)}, X^n) = \lceil rn \rceil / n$, for X^n in general position and $r = (n-d)/(2n)$, complementing the result of Davies (1987). Now, set $\epsilon = 1$, $\beta = 0$ and $m = -d$, then by Theorem 2.2, we immediately have that $\mathrm{ABP}(\mu^{(n)}, X^n) = \mathrm{ABP}(\Sigma^{(n)}, X^n) = (n-d)/(2n-d)$. $\square$

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Appendix: Proofs

Lemma 3.1 *For any $x \in \mathbf{R}$, $\epsilon \geq 0$,*

$$\lfloor x \rfloor = \lfloor (\epsilon \lfloor x \rfloor + x + r)/(\epsilon + 1) \rfloor, \quad \text{and} \quad \lceil x \rceil = \lceil (\epsilon \lceil x \rceil + x - r^*)/(\epsilon + 1) \rceil,$$

where $-(x - \lfloor x \rfloor) \leq r < \epsilon + 1 - (x - \lfloor x \rfloor)$, and $-(\lceil x \rceil - x) \leq r^* < \epsilon + 1 - (\lceil x \rceil - x)$.

Proof: Consider the cases $x = z$ and $x = z + \alpha$, where z is an integer and $0 < \alpha < 1$. The result follows in a straightforward fashion. □

Lemma 3.2 *Let $f(n) = \lfloor an + b \rfloor$ and $g(n) = \lfloor cn + d \rfloor$ (or $f(n) = \lceil an + b \rceil$ and $g(n) = \lceil cn + d \rceil$), $f(n) = g(n)$ on a sequence $\{n_i\}$ with $1 \leq n_1 \leq n_2 \leq \ldots n_i \leq \ldots$. Then $a = c$ and*

$$\max_{n_i}\{D^-(n)\} \leq d - b < 1 + \min_{n_i}\{D^-(n)\} \quad (or \ \max_{n_i}\{D^+(n)\} - 1 < d - b \leq \min_{n_i}\{D^+(n)\}),$$

where $D^-(n) = \lfloor an + b \rfloor - (an + b)$, and $D^+(n) = \lceil an + b \rceil - (an + b)$.

Proof: We consider only the floor function case, the proof for the ceiling function case is similar. Suppose that $a \neq c$, w.l.o.g., assume $a > c$, then there exists a $N > 0$ such that $an + b - (cn + d) = (a - c)n + (b - d) > 1$, when $n > N$. Thus, when $n_i > N$, $f(n_i) \neq g(n_i)$, contradicting the assumption that $f(n) = g(n)$ on the sequence $\{n_i\}$. Therefore $a = c$. Now let $D^-(n) = \lfloor an + b \rfloor - (an + b)$, and $D^-(n_j) = \max_{n_i}\{D^-(n)\}$ for some j, then $d - b \geq D^-(n_j)$, since otherwise $f(n_j) \geq g(n_j) + 1$, contradicting the assumption that $f(n) = g(n)$ on the sequence $\{n_i\}$. Similarly, let $D^-(n_k) = \min_{n_i}\{D^-(n)\}$ for some k, then $d - b < 1 + D^-(n_k)$, since otherwise $f(n_k) \leq g(n_k) - 1$, contradicting the assumption that $f(n) = g(n)$ on the sequence $\{n_i\}$. □

Proof of Theorem 2.1. (a) For any n, let $N = \lfloor (\epsilon + 1)n + \beta + m \rfloor$. By Lemma 3.1,

$$\lfloor \epsilon n + \beta \rfloor + m = \lfloor \frac{\epsilon N + (\beta + m + \epsilon)}{\epsilon + 1} \rfloor.$$

The replacement breakdown point of T now implies that if we add at least $\lfloor \epsilon n + \beta \rfloor + m$ points to a given data set of size n (or, if we replace at least $\lfloor \epsilon n + \beta \rfloor + m$ points in a given data set of size $\lfloor (\epsilon+1)n + \beta + m \rfloor$), the estimator T will break down. That is

$$\mathrm{ABP}(T, X^n) = \frac{\lfloor \epsilon n + \beta \rfloor + m}{n + \lfloor \epsilon n + \beta \rfloor + m}.$$

(b) Again for any n, let $N = \lfloor (\epsilon+1)n + \beta + m \rfloor$. By Lemma 3.1,

$$\lfloor \epsilon n + \beta \rfloor + m = \lfloor (\epsilon N + (\beta + m))/(\epsilon + 1) \rfloor.$$

The addition breakdown point of T now implies that for a given data set of size N , in order to corrupt the estimator T, if and only if we replace at least $\lfloor \epsilon n + \beta \rfloor + m$ points by "bad points" (or if and only if the data set contains at least $\lfloor \epsilon n + \beta \rfloor + m$ "bad points"). Therefore,

$$\mathrm{RBP}(T, X^N) = \frac{\lfloor (\epsilon N + (\beta + m))/(\epsilon + 1) \rfloor}{N},$$

for $N = \lfloor (\epsilon+1)n + \beta + m \rfloor$. Since the replacement breakdown point of T is of the form $\lfloor an + b \rfloor / n$, by Lemma 3.2 we conclude that

$$\mathrm{RBP}(T, X^n) = \frac{\lfloor (\epsilon n + (\beta + m + r))/(\epsilon + 1) \rfloor}{n},$$

for any positive integer n with $\max_N\{D^-(N)\} \le r/(\epsilon+1) < 1 + \min_N\{D^-(N)\}$. Since $\lfloor (\epsilon n + (\beta + m + r))/(\epsilon + 1) \rfloor \ge 1$, the desired result now follows. $\square$

Proof of Theorem 2.2. Similar to that of Theorem 2.1. $\square$

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