Finite Sample Tail Behavior of Hodges-Lehmann Type Estimators

Yijun Zuo [1]

Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804
USA

E-mail: yijun.zuo@asu.edu

URL: http://math.la.asu.edu/~zuoyijun

Final version, to appear in *Statistics*, 2001

[1]Research partially supported by a grant (RIA) from Arizona State University and NSF Grant DMS-00-71976.

Abstract

The tail behavior of Hodges-Lehmann type location estimators is studied with respect to a tail performance measure. The result obtained here generalizes and complements the corresponding work on R-estimators of Jurečková (1981a). The connection between the tail behavior and the breakdown point discovered in He, Jurečková, Koenker and Portnoy (1990) for regression and monotone location estimators is extended to Hodges-Lehmann type location estimators, confirming the important role of the tail behavior as a measure of robustness of estimators.

AMS 1991 Classification: Primary 62G35; Secondary 62G05, 62F35.

Key words and phrases: Tail behavior, Hodges-Lehmann estimator, breakdown point, robustness.

Short Running title: Tail behavior of location estimators.

1 Introduction

Let $T_n = T_n(X_1, \ldots, X_n)$ be an estimator of an unknown parameter θ based on a random sample $X_1, \ldots, X_n$ from a random variable X with P_θ being the corresponding probability measure. How good is T_n as an estimator of θ? Various performance measures of T_n, including *unbiasedness* and *consistency*, have been proposed and studied. Asymptotic behavior of T_n has also been investigated, based on the behavior of

$$-\ln P_\theta(|T_n - \theta|) > a)/n, \quad \text{as } n \to \infty$$

for *fixed* a, see Bahadur (1967, 1971), Fu (1975), and Sievers (1978).

In practice, however, the sample size n is *finite* and *fixed.* In sharp contrast with large sample asymptotic approach to the performance of estimators, Jurečková (1981a) studied, for continuous and centrally symmetric F, the performance of location estimator T_n based on the measure

$$\frac{-\ln P_\theta(|T_n - \theta| > a)}{-\ln(1 - F(a))}, \quad \text{as } a \to \infty$$

for *fixed* n. He, Jurečková, Koenker and Portnoy (1990) extended this measure to the regression context, explored the tail behavior of various regression estimators, and revealed the link between the tail behavior and the breakdown point of these estimators. Other studies of the tail behavior of location estimators include Huber (1968), Jurečková (1981b, 1885), Antoch (1984a,b), Kušnier and Mizera (1999), and Lopuhaä and Rousseeuw (1991). (It is noted that some of tail behavior studies were before the study of the finite sample breakdown point.)

In this paper, the finite sample performance measure of Jurečková (1981a) is slightly modified, allowing one to consider the tail behavior of location estimators without assuming that F is centrally symmetric. We consider the finite sample tail behavior of T_n based on the behavior of

$$B(T_n; u, a) = \frac{-\ln P_\theta(u(T_n - \theta) > a)}{-\ln P_\theta(u(X - \theta) > a)}, \qquad \text{as } a \to \infty,$$

for u with $|u| = 1$ and *fixed* n. The limit above (if exists) measures the relative rate of the tail probability of the sampling distribution of T_n approaching zero with respect to the tail probability of the underlying population distribution. Clearly, for good estimator T_n one would expect the ratio $B(T_n; u, a)$ to be high for any u as $a \to \infty$. How high can $B(T_n; u, a)$ be as $a \to \infty$? We answer this question for Hodges-Lehmann type location estimators (defined in Section 2) in this paper. Specifically, we examine the behavior of

$$\underline{B}(T_n; a) = \min_{|u|=1}\{B(T_n; u, a)\} \text{ and } \overline{B}(T_n; a) = \max_{|u|=1}\{B(T_n; u, a)\}, \text{ as } a \to \infty.$$

A general tail performance result is obtained in this paper for a generalized version of Hodges-Lehmann type location estimators, generalizing and complementing the corresponding work on R-estimators of Jurečková (1981a). The connection between the tail behavior and breakdown point for regression estimators revealed in He, Jurečková, Koenker and Portnoy (1990) is extended to the Hodges-Lehmann type location estimators in the paper, confirming the important role of the tail behavior as a quantitative assessment of robustness of location estimators.

In Section 2, a generalized version of Hodges-Lehmann type location estimators is introduced, and a general result on the tail performance of these estimators is obtained. The connection between tail performance bounds and the breakdown point of these estimators is discovered. Applying the obtained general tail performance result to estimators such as the Hodges-Lehmann estimator, we obtain in Section 3 tail performance bounds that improve previous ones in the literature.

2 Main Results

In the univariate setting, location estimators can be derived from some test statistics which are usually distribution-free. This technique was first proposed by Hodges and Lehmann (1963). We call the estimators obtained via this technique *Hodges-Lehmann type estimators* (also called *R-estimators* in the literature). Hodges-Lehmann type estimators have strong robustness and high efficiency properties (see Randles and Wolfe (1979) and Huber (1981) for further discussions). In the following, a generalized version of Hodges-Lehmann type estimators is introduced and the finite sample tail behavior of these estimators is studied.

Let $X^n = \{X_1, \ldots, X_n\}$ be a random sample from a continuous distribution $F(x-\theta)$ and $F(\cdot)$ be a distribution with median 0. Let $V(X_1, \ldots, X_n)$ be a test statistic for testing $H_0 : \theta = 0$ against $H_a : \theta > 0$ (or $\neq 0$). H_0 is rejected if V is large. The test statistic V satisfies the following two conditions:

C1: $V(x_1 + h, \ldots, x_n + h)$ is nondecreasing in h for each $(x_1, \ldots, x_n)$.

C2: $V(X_1, \ldots, X_n)$ has a distribution with median ξ under H_0.

The Hodges-Lehmann type estimator $\hat{\theta}_n$ for θ based on $V(X_1, \ldots, X_n)$ is given by

$$\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n) = (\theta_n^* + \theta_n^{**})/2,$$

where

$$\begin{aligned} \theta_n^* = \theta_n^*(X_1, \ldots, X_n) &= \sup\{\theta : V(X_1 - \theta, \ldots, X_n - \theta) > \xi\}, \\ \theta_n^{**} = \theta_n^{**}(X_1, \ldots, X_n) &= \inf\{\theta : V(X_1 - \theta, \ldots, X_n - \theta) < \xi\}. \end{aligned}$$

Note that in the above definition of Hodges-Lehmann type estimators, F has been extended from originally (and traditionally) central symmetry (Hodges and Lehmann (1963)) to *angular symmetry* about the origin, and V has also been extended from centrally symmetric about ξ under H_0 to angularly or even *halfspace symmetric* about ξ under H_0 (for definitions and characterizations of angular or halfspace symmetry, see Zuo and Serfling (2000)). The ranges of F and V in this paper are much wider than those in the literature.

It is not difficult to see that the Hodges-Lehmann type estimator $\hat{\theta}_n$ is *translation equivariant*. (A location estimator T_n is translation equivariant, if $T_n(x^n + b) = T_n(x^n) + b$, for any b and set $x^n = \{x_1, \ldots, x_n\}$ in $\mathbf{R}^1$, where $x^n + b = \{x_1 + b, \ldots, x_n + b\}$.) For translation equivariant T_n and $X \sim F(x - \theta)$, $P_\theta(u(T_n - \theta) > a) = P_0(uT_n > a)$ and

$P_\theta(u(X-\theta) > a) = P_0(uX > a)$ for $|u| = 1$, where P_0 is the probability measure corresponding to $X - \theta$. We thus assume in our following discussion that $\theta = 0$. For simplicity P_0 is written as P. Now we establish a general tail performance result for $\hat{\theta}_n$ based on the behavior of the test statistic V.

Theorem 2.1 *Let $\hat{\theta}_n$ be a Hodges-Lehmann type estimator for θ based on $V(x_1, \ldots, x_n)$ satisfying* C1 *and* C2. *Let there be some K_n and L_n such that*

$$V(x_1,\ldots,x_n) \geq \xi \Rightarrow \sum_i \mathbf{I}(x_i \geq 0) \geq K_n, \qquad V(x_1,\ldots,x_n) \leq \xi \Rightarrow \sum_i \mathbf{I}(x_i \leq 0) \geq K_n,$$
$$\sum_i \mathbf{I}(x_i > 0) \geq L_n \Rightarrow V(x_1,\ldots,x_n) > \xi, \qquad \sum_i \mathbf{I}(x_i < 0) \geq L_n \Rightarrow V(x_1,\ldots,x_n) < \xi,$$

where $\mathbf{I}(\cdot)$ is the indicator function. Then

$$K_n \leq \liminf_{a\to\infty} \underline{B}(\hat{\theta}_n; a) \leq \limsup_{a\to\infty} \overline{B}(\hat{\theta}_n; a) \leq L_n.$$

Proof: (i) We need only show the leftmost and the rightmost inequalities. First consider the case that $u\hat{\theta}_n = \hat{\theta}_n$. We have

$$\begin{aligned}
P(u\hat{\theta}_n > a) &= P(\hat{\theta}_n > a) \\
&\leq P(\theta^{**} > a) \\
&\leq P(V(X_1 - a, \ldots, X_n - a) \geq \xi) \\
&\leq P(X_{(n-K_n+1)} - a \geq 0) \\
&= \sum_{s=K_n}^{n} \binom{n}{s} (P(uX > a))^s (P(uX \leq a))^{n-s} \\
&= (P(uX > a))^{K_n} \left(\sum_{s=K_n}^{n} \binom{n}{s} (P(uX > a))^{s-K_n} (P(uX \leq a))^{n-s} \right),
\end{aligned}$$

where $x_{(1)} \leq \ldots \leq x_{(n)}$ are ordered values of $x_1, \ldots, x_n$. Similarly, if $u\hat{\theta}_n = -\hat{\theta}_n$, we have

$$P(u\hat{\theta}_n > a) \leq (P(uX > a))^{K_n} \left(\sum_{s=K_n}^{n} \binom{n}{s} (P(uX > a))^{s-K_n} (P(uX \leq a))^{n-s} \right).$$

Write

$$f(u;a) = \sum_{s=K_n}^{n} \binom{n}{s} (P(uX > a))^{s-K_n} (P(uX \le a))^{n-s},$$

then

$$\frac{-\ln P(u\hat{\theta}_n > a)}{-\ln P(uX > a)} \quad \ge \quad K_n + \frac{-\ln f(u;a)}{-\ln P(uX > a)}.$$

If we can show that the second term in the right hand side of the above inequality approaches zero *uniformly* with respect to u as a approaches infinity, then the desirable leftmost inequality follows. It follows, by replacing $P(uX > a)$ and $P(uX \le a)$ with 1 in $f(u;a)$, that

$$f(u;a) \le \sum_{s=K_n}^{n} \binom{n}{s} \equiv U.$$

Dropping all terms except the first one in $f(u;a)$ and utilizing Cauchy-Schwartz inequality, we have

$$f(u;a) \ge \binom{n}{K_n} (P(uX \le a))^{n-K_n} \ge \binom{n}{K_n} (P(|X| \le a))^{n-K_n}.$$

For sufficiently large a, $P(|X| \le a) \ge 1/2$, thus

$$f(u;a) \ge \binom{n}{K_n} (1/2)^{n-K_n} \equiv L.$$

Since $P(uX > a) \le P(|X| > a) \to 0$ uniformly with respect to u as $a \to \infty$, thus $\ln P(uX > a) \to -\infty$ uniformly with respect to u as $a \to \infty$. Combining this with squeeze theorem and the fact that $\ln f(u;a)$ is uniformly bounded with respect to u ($\ln L \le \ln f(u;a) \le \ln U$) gives

$$\frac{-\ln f(u;a)}{-\ln P(uX > a)} \to 0, \quad \text{uniformly with respect to } u \text{ as } a \to \infty.$$

The desired leftmost inequality now follows.

(ii) To prove the desired rightmost inequality, we first consider the case that $u\hat{\theta}_n = \hat{\theta}_n$. Let $X_1, \ldots, X_n$ be a given random sample. Consider a positive number a such that V is continuous with respect to a at $X_1 - a, \ldots, X_n - a$ (note that this is possible since

$V(x_1 - h, \ldots, x_n - h)$ has at most countably many discontinuous points with respect to h). Now if $V(X_1 - a, \ldots, X_n - a) > \xi$, then $\theta_n^{**} > a$. We thus have

$$\begin{aligned}
P(u\hat{\theta}_n > a) &= P(\hat{\theta}_n > a) \\
&\geq P(V(X_1 - a, \ldots, X_n - a) > \xi) \\
&\geq P(X_{(n-L_n+1)} - a > 0) \\
&= \sum_{s=L_n}^{n} \binom{n}{s} (P(uX > a))^s (P(uX \leq a))^{n-s} \\
&= (P(uX > a))^{L_n} \left(\sum_{s=L_n}^{n} \binom{n}{s} (P(uX > a))^{s-L_n} (P(uX \leq a))^{n-s} \right).
\end{aligned}$$

Likewise, for $u\hat{\theta}_n = -\hat{\theta}_n$, we have

$$\begin{aligned}
P(u\hat{\theta}_n > a) &= P(-\hat{\theta}_n > a) \\
&\geq (P(uX > a))^{L_n} \left(\sum_{s=L_n}^{n} \binom{n}{s} (P(uX > a))^{s-L_n} (P(uX \leq a))^{n-s} \right).
\end{aligned}$$

Set

$$g(u; a) = \sum_{s=L_n}^{n} \binom{n}{s} (P(u'X > a))^{s-L_n} (P(uX \leq a))^{n-s},$$

then

$$\frac{-\ln P(u\hat{\theta}_n > a)}{-\ln P(uX > a)} \leq L_n + \frac{-\ln g(u; a)}{-\ln P(uX > a)}.$$

Now employing an argument similar to that utilized in part (i), we have

$$\frac{-\ln g(u; a)}{-\ln P(uX > a)} \to 0, \quad \text{uniformly with respect to } u \text{ as } a \to \infty.$$

The desired rightmost inequality now follows. □

Remark 2.1 Roughly speaking, the theorem says that the tails of the sampling distribution of $\hat{\theta}_n$ approach zero (in natural log scale) at least K_n times as fast as the tails of the underlying population distribution and can be L_n times as fast, as a approaches infinity. □

As shown in Jurečková (1981a), tail behavior measures the robustness of location estimators. This finding was extended to the regression estimators in He, Jurečková, Koenker and Portnoy (1990), where they also discovered the connection between the finite sample tail behavior and the finite sample breakdown point. The latter notion, introduced in Donoho (1982) and Donoho and Huber (1983), has emerged as one standard quantitative assessment of global robustness of estimators. The finite sample replacement breakdown point (RBP) of an estimator $T_n(\cdot)$ at X^n is defined to be the minimum number of contaminating points m such that after replacing original m points in X^n by m contaminating points, one can drive the estimator beyond any bound.

Now we extend the finding in He, Jurečková, Koenker and Portnoy (1990) for regression estimators to Hodges-Lehmann type estimators, and show that the tail performance bounds of Hodges-Lehmann type estimators are also intimately connected to their breakdown point.

Theorem 2.2 *Let $\hat{\theta}_n$, K_n, and L_n be the Hodges-Lehmann type estimator and the numbers in Theorem 2.1. Then $K_n \leq RBP(\hat{\theta}_n) \leq L_n$.*

Proof: We first show that $K_n \leq \text{RBP}(\hat{\theta}_n)$. Suppose, on the contrary, that some $K_n - 1$ contaminating points $X_1^*, \ldots, X_{K_n-1}^*$ can break down $\hat{\theta}_n(X^n)$, where $X^n = \{X_1, \ldots, X_n\}$. Assume, w.l.o.g., that $\hat{\theta}_n(X_1^*, \ldots, X_{K_n-1}^*, X_{K_n}, \ldots, X_n) \geq M = \max_i\{|X_i|\} + 1$ after replacing the original first $K_n - 1$ points in X^n by some $K_n - 1$ contaminating points. Then, $\theta_n^{**}(X_1^*, \ldots, X_{K_n-1}^*, X_{K_n}, \ldots, X_n) \geq M$. Hence $V(X_1^* - \theta, \ldots, X_{K_n-1}^* - \theta, X_{K_n} - \theta, \ldots, X_n - \theta) \geq \xi$ for any $\theta < M$. Let $Z^n = \{Z_1, \ldots, Z_n\} = \{X_1^* - \theta, \ldots, X_{K_n-1}^* - \theta, X_{K_n} - \theta, \ldots, X_n - \theta\}$. By the definition of K_n in Theorem 2.1, $\sum_i \mathbf{I}(Z_i \geq 0) \geq K_n$ for any $\theta < M$. But this is impossible since if we set $\theta = \max\{X_{K_n}, \ldots, X_n\} + 1/2$ then $\theta < M$ and $\sum_i \mathbf{I}(Z_i \geq 0) < K_n$. This implies that $K_n \leq \text{RBP}(\hat{\theta}_n)$.

Now we show that $\text{RBP}(\hat{\theta}_n) \leq L_n$. Let $X_i^* = y$ for $1 \leq i \leq L_n$ and let $y \to \infty$, then for any θ, $\sum_i \mathbf{I}(Z_i > 0) \geq L_n$ for $Z^n = \{X_1^* - \theta, \ldots, X_{L_n}^* - \theta, X_{L_n+1} - \theta, \ldots, X_n - \theta\}$ as $y \to \infty$. By the condition in Theorem 2.1, $V(Z^n) > \xi$ for any finite θ as $y \to \infty$. Hence $\theta_n^*(Z^n) \to \infty$, implying $\mathrm{RBP}(\hat{\theta}_n) \leq L_n$. $\square$

Remark 2.2 (i) The RBP of a Hodges-Lehmann type estimator $\hat{\theta}_n$ has the same lower and upper bounds of its tail behavior, implying that the tail behavior is a measure of robustness of Hodges-Lehmann type estimators. (ii) It can be shown that $n - L_n + 1 \leq \mathrm{RBP}(\hat{\theta}_n) \leq n - K_n + 1$. Furthermore, for many V and $\hat{\theta}_n$, $\mathrm{RBP}(\hat{\theta}_n) = K_n = n - L_n + 1$. $\square$

Remark 2.3 For a class of *monotonic* and translation equivariant location estimators and for continuous and centrally symmetric F satisfying some tail condition, He, Jurečková, Koenker and Portnoy (1990) established a connection between the breakdown point and the tail behavior of the estimators. Results in this section do not follow from He, Jurečková, Koenker and Portnoy (1990). $\square$

3 Examples

Now we apply the general result in Theorem 2.1 to specific Hodges-Lehmann type estimators. Clearly, the key point is to find the K_n and L_n.

Example 3.1 Let $V(X_1, \ldots, X_n) = \sqrt{n}\,\overline{X}_n/S$ be the one-sample t-statistic for testing $H_0 : \theta = 0$ vs. $H_a : \theta > 0$. H_0 is rejected when $V(X_1, \ldots, X_n)$ is large. It is readily seen that C1 and C2 are satisfied with $\xi = 0$ and symmetric F. Since $V(x_1 - \theta, \ldots, x_n - \theta)$ is continuous and strictly decreasing in θ, the Hodges-Lehmann type estimator $\hat{\theta}_n$ for θ is $\hat{\theta}_n = \theta_n^* = \theta_n^{**}$, and $V(X_1 - \hat{\theta}_n, \ldots, X_n - \hat{\theta}_n) = 0$. Thus $\hat{\theta}_n = \overline{X}_n$. It is not difficult to see that the K_n and L_n in Theorem 2.1 now are 1 and n respectively. Thus we have the following result. $\square$

Corollary 3.1 *Let $\overline{X}_n$ be the sample mean. Then*

$$1 \leq \liminf_{a \to \infty} \underline{B}(\overline{X}_n; a) \leq \limsup_{a \to \infty} \overline{B}(\overline{X}_n; a) \leq n.$$

Example 3.2 Let $U_i = U_i(X_1, \ldots, X_n)$, $i = 1, \ldots, k$, be *odd* and a translation equivariant statistic. (A statistic T_n is said to be odd if $T_n(-x_1, \ldots, -x_n) = -T_n(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n$). Set $V(X_1, \ldots, X_n) = \sum_{i=1}^k \mathbf{I}(U_i > 0)$. Reject the test of $H_0 : \theta = 0$ (vs. H_a: $\theta > 0$) for large value of V. Clearly, $V(x_1 + h, \ldots, x_n + h)$ is nondecreasing in h and it is not difficult to show that $V(X_1, \ldots, X_n)$ is centrally symmetric about $\xi = k/2$ when H_0 is true. Thus C1 and C2 are satisfied. It is also not difficult to show that the Hodges-Lehmann type estimator $\hat{\theta}_n$ for θ now is

$$\hat{\theta}_n = \text{median}\{U_i,\ i = 1, \ldots, k\}.$$

Let $U_i = X_i$, $i = 1, \ldots, n$. Then $V(X_1, \ldots, X_n)$ is the well-known *sign statistic* (see Randles and Wolfe (1979)) and the Hodges-Lehmann type estimator $\hat{\theta}_n$ is the sample median. It is readily seen that the K_n and L_n in Theorem 2.1 now are $\lfloor (n+1)/2 \rfloor$ and $\lfloor (n+2)/2 \rfloor$ respectively, where "$\lfloor x \rfloor$" is the largest integer no larger than x. Employing Theorem 2.1, we have for univariate sample median the following result. □

Corollary 3.2 *Let Med_n be the sample median. Then*

$$\lfloor (n+1)/2 \rfloor \le \liminf_{a \to \infty} \underline{B}(Med_n; a) \le \limsup_{a \to \infty} \overline{B}(Med_n; a) \le \lfloor (n+2)/2 \rfloor.$$

Example 3.3 (continuation) Let U_i, $i = 1, \ldots, n(n+1)/2$, be the ordered $n(n+1)/2$ values of $(X_k + X_l)/2$, $k \le l = 1, \ldots, n$. Let $W^+ = \sum_{i=1}^n \mathbf{I}(X_i > 0) R^+(|X_i|)$ be the *Wilcoxon signed rank statistic* (see Wilcoxon (1945)), where $R^+(|x_i|)$ is the rank of $|x_i|$ in $|x_1|, \ldots, |x_n|$. Then it is not difficult to show that $V(X_1, \ldots, X_n) = W^+$, and

$$\hat{\theta}_n = \text{median}\{\frac{X_i + X_j}{2}, i \le j = 1, \ldots, n\},$$

which is often called the *Hodges-Lehmann estimator* (HL_n) in the literature. Let $W^- = \sum_{i=1}^n \mathbf{I}(X_i < 0) R^+(|X_i|)$ and $W = W^+ - W^-$. It is not difficult to see that W is symmetric about the origin when the null hypothesis $H_0 : \theta = 0$ is true, and $V(X_1, \ldots, X_n) \ge (\le)\ \xi \iff W \ge (\le)\ 0$. To find out the corresponding K_n and L_n in Theorem 2.1, we assume that

$W(x_1,\ldots,x_n) \geq (\leq)\ 0$ implies that there are at least K_n nonnegative (nonpositive) points. We thus conclude that $-(n-K_n)(n-K_n+1)/2+(2n-K_n+1)K_n/2 \geq 0$. Likewise, let L_n be the number in Theorem 2.1, we conclude that $L_n(1+L_n)/2-(n+L_n+1)(n-L_n)/2 > 0$. It follows, by solving these inequalities for K_n and L_n respectively, that

$$K_n \geq (2n+1-\sqrt{2n^2+2n+1})/2, \quad \text{and} \quad L_n > (\sqrt{2n^2+2n+1}-1)/2.$$

Let "$\lceil\cdot\rceil$" be the ceiling function, that is, $\lceil x\rceil$ is the smallest integer no smaller than x. We thus obtain the following result. □

Corollary 3.3 *Let HL_n be the Hodges-Lehmann estimator. Then*

$$\begin{aligned}\lceil(2n+1-\sqrt{2n^2+2n+1})/2\rceil &\leq \liminf_{a\to\infty} \underline{B}(HL_n;a) \\ &\leq \limsup_{a\to\infty} \overline{B}(HL_n;a) \leq \lfloor(\sqrt{2n^2+2n+1}-1)/2\rfloor+1.\end{aligned}$$

Remark 3.1 Corollary 3.3 extends Theorem 3.4 of Jurečková (1981a) which was established for continuous centrally symmetric F. The performance bounds obtained here are sharper than those in Jurečková (1981a). □

Remark 3.2 It is readily seen that the lower tail performance bounds in Corollaries 3.1 and 3.2 are the RBP of the corresponding estimators. For HL_n, assume its RBP is m. Then replacing m points in original data set by carefully chosen m "bad" points could lead to a contaminated majority of the total $n(n+1)/2$ points $(X_i+X_j)/2, 1 \leq i \leq j \leq n$. Hence $n(n+1)/2-(n-m)(n-m+1)/2 \geq (n-m)(n-m+1)/2$. Solving this inequality for m reveals that $m = \lceil(2n+1-\sqrt{2n^2+2n+1})/2\rceil$, which is exact the lower tail performance bound in Corollary 3.3, and $n-m+1 = \lfloor(\sqrt{2n^2+2n+1}-1)/2\rfloor+1$, which is exactly the upper tail performance bound in Corollary 3.3. These findings reveal the intimate connection between the tail behavior and the breakdown point and confirm the important role of the tail behavior as a quantitative assessment of robustness of location estimators. □

Acknowledgments

The author is grateful to Robert Serfling for his helpful comments and constant encouragement. The work is partially supported by a grant (RIA) from Arizona State University and NSF Grant DMS-00-71976.

References

[1] Antoch, J. (1984a). Behavior of L-estimators of location from the point of view of large deviations. In *Robustness of Statistical Methods and Nonparametric Statistics* (D. Rasch and M. L. Tiku, eds) 4-8. Deutscher Verlag der Wissenschaften, Berlin.

[2] Antoch, J. (1984b). Behavior of estimators of location in non-regular cases: A Monte Carlo Study. *Proc. 3rd Prague Symp. on Asympt. Statis.* (P. Mandl and M. Hušková, eds) 185-196. Elsevier, Amsterdam.

[3] Bahadur, R. R. (1967). Rates of Convergence of Estimators and Test Statistics. *Ann. Math. Statist.* **38** 303-324.

[4] Bahadur, R. R. (1971). *Some Limit Theorems in Statistics.* SIAM. Philadelphia.

[5] Donoho, D. L. (1982). Breakdown properties of multivariate location estimators. Ph.D. qualifying paper, Dept. Statistics, Harvard University.

[6] Donoho, D. L., and Huber, P. J. (1983). The notion of breakdown point. In *A Festschrift foe Erich L. Lehmann* (P. J. Bickel, K. A. Doksum and J. L. Hodges, Jr., eds.) 157-184. Wadsworth, Belmont, CA.

[7] Fu, J. C. (1975). The rate of convergence of consistent point estimators. *Ann. Statist.* **3** 234-240.

[8] He, X., Jurečková, J., Koenker,R. and Portnoy, S. (1990). Tail behavior of regression estimators and their breakdown points. *Econometrica* **58** 1195–1214.

[9] Hodges, J. L. Jr., and Lehmann, E. L. (1963). Estimates od location based on rank tests. *Ann. Math. Statist.* **34** 598-611.

[10] Huber, P. J. (1968). Robust confidence limits. *Z. Wahrscheinlichkeitstheorie werw. Geb.* **10** 269-278.

[11] Huber, P. J. (1981). *Robust Statistics.* John Wiley & Sons.

[12] Jurečková, J. (1981a). Tail behavior of location estimators. *Ann. Statist.* **9** 578-585.

[13] Jurečková, J. (1981b). Tail behavior of location estimators in non-regular cases. *Comment. Math. Univ. Carolinae* **22** 365-375.

[14] Jurečková, J. (1985). Tail behavior of L-estimators and M-estimators. *Proc. 4th Pannonian Symposium* (W. Wertz, ed) Vol. 1, 205-217.

[15] Kušnier, J. and Mizera, I. (1999). Tail behavior and breakdown properties of equivariant estimators of location. To appear in *Ann. Inst. Statist. Math.*

[16] Lopuhaä, H. P., and Rousseeuw, P. J. (1991). Breakdown points of affine equivariant estimators of multivariate location and covariance matrices. *Ann. Statist.* **19** 229-248.

[17] Randles, R. H., and Wolfe, D. A. (1979). *Introduction to the Theory of Nonparametric Statistics.* Wiley, New York.

[18] Sievers, G. L. (1978). Estimation of location: A large deviation comparison. *Ann. Statist.* **6** 610-618.

[19] Wilcoxon, F. (1945). Individual comparisons by ranking methods. *Biometrics* **1** 80-83.

[20] Zuo, Y., and Serfling, R. (2000). On the performance of some robust nonparametric location measures relative to a general notion of multivariate symmetry. *J. Statist. Plann. Inference* **84** 55-79.