EXAM III—-STT861 Theory of Statistics and Probability I

Name (Print):_______________________ A#_______________________

If you sign below, your score will be posted on the course webpage based on the last four digits of the A#

Signature: _______________________ (**No credit will be given unless you show all of your work**.)

1. Let X have the distribution $f(x) = P(X = x) = 3/4^{x+1}$, $x = 0, 1, 2, \cdots$.
   (a) Find the moment generating function of X. (10%)
   (b) Find $E(X)$ and $\mathrm{Var}(X)$. (10%)

2. Let $(X, Y, Z) \sim \mathrm{Multinomial}(n, p_x, p_y, p_z)$.
   (a) Find the conditional distribution (pdf) of Y given $X = x$. (10%)
   (b) Find $E(Y|X)$ (10%)
   (c) Find $\mathrm{Cov}(X, Y)$. (10%)

3. Let $X_i \sim \mathrm{BIN}(1, p_i)$ be independent , $i = 1, \cdots, n$ and $Z_1, \cdots, Z_n$ be i.i.d $N(0, 1)$.
   (a) Show that $\sum_{i=1}^n (X_i - p_i)/n \xrightarrow{p} 0$ as $n \to \infty$. (10%)
   (b) Find the limiting distribution of $\sum_{i=1}^n (Z_i + \frac{1}{n})/\sqrt{n}$ as $n \to \infty$. Justify your answer. (10%)

4. Let X_i be i.i.d from $X \sim N(0, \theta)$ with $\theta > 0$ unknown.
   (a) Find the MLE $\hat{\theta}$ of θ and the MLE of $\tau(\theta)$ such that $P(X \leq \tau(\theta)) = 0.95$. (10%)
   (b) Find a pivotal quantity and an exact 95% confidence interval for θ. (10%)

5. Let X_i be i.i.d from $X \sim \mathrm{EXP}(\lambda_1)$, $i = 1, \cdots, m$, and Y_j be i.i.d from $Y \sim \mathrm{EXP}(\lambda_2)$, $j = 1, \cdots, n$.
   (a) Find the MLE of $1/\lambda_1$ and a 90% confidence interval for λ_1. (10%)
   (b) Find a pivotal quantity and an exact 90% confidence interval for λ_1/λ_2. (10%)

6. Let X_i be i.i.d from X with pdf $f(x, \eta) = e^{-(x-\eta)} I(x \geq \eta)$.
   (a) Find the MME $\tilde{\eta}$ of η. (10%)
   (b) Find the MLE $\hat{\eta}$ of η. (10%)
   (c) Find the exact distribution of $\hat{\eta}$ and $\hat{\eta} - \eta$. (10%)
   (d) Find an exact $100(1 - \alpha)\%$ confidence interval for η. (10%)
   (e) Find an approximate $100(1 - \alpha)\%$ confidence interval for η. (10%)

7. Let X_i, $i = 1, \cdots, n$, be i.i.d from $X \sim \mathrm{POI}(\lambda)$ with $\lambda > 0$ unknown.
   (a) Find the MLE of $\tau(\lambda) = P(X = 0)$. (10%)
   (b) Is the MLE of $\tau(\lambda)$ in (a) unbiased? Justify your answer. (10%)
   (c) Show that $\tilde{\tau}(\lambda) = (\frac{n-1}{n})^{\sum_{i=1}^n X_i}$ is unbiased for $P(X = 0)$. (10%)
   (d) For $\tilde{\tau}(\lambda)$ in (c), find $\mathrm{Var}(\tilde{\tau}(\lambda))$ and compare it with its asymptotic variance $(nI(\tau(\lambda)))^{-1}$. (10%)