7.4.1)

(a) $\hat{R}(X=0) = 0$

$\hat{R}(X=1) = 2$ or 3

$\hat{R}(X=2) = 6$ or 7

$\hat{R}(X=3) = 8$

(b) $\frac{3R}{8} = g(R)$ for $g(x) = \frac{3x}{8}$. So ;

$$\hat{E}(X) = g(\hat{R})$$

7.4.4) $$L(\theta|x) = \frac{1}{2^n} \cdot e^{-\sum_{i=1}^{n} |x_i - \theta|}$$

$$\Rightarrow \ell(\theta|x) = -n \log 2 - \sum_{i=1}^{n} |x_i - \theta|$$

$$\Rightarrow \ell'(\theta|x) = \sum_{i=1}^{n} \frac{(x_i - \theta)}{|x_i - \theta|}$$

So if n is odd:

$$\theta = X_{(\frac{n+1}{2})}$$

But if n is even, θ could be any number between $X_{(\frac{n}{2})}$ and $X_{(\frac{n+1}{2})}$.

7.4.9) (a) $\ell(\beta x_i, \sigma^2 | y) = -n \log 2\pi - \frac{n}{2} \log \sigma^2 - \sum_{i=1}^{n} \frac{(y_i - \beta x_i)^2}{2\sigma^2}$

$$\Rightarrow \begin{cases} \frac{\partial \ell}{\partial \beta} = 0 \Rightarrow \hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \\ \frac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (y_i - \hat{\beta} x_i)^2}{n} \end{cases}$$

(b) $\hat{\beta} = 3 \quad ; \quad \hat{\sigma}^2 = 1$

(c) $(\hat{\beta}, \hat{\sigma}^2) = \left(\bar{Y}, \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n} \right)$

(d) $E\left(\frac{\sum x_i y_i}{\sum x_i^2} \right) = \frac{\sum x_i \cdot (\beta x_i)}{\sum x_i^2} = \beta$

(e) $Var(\hat{\beta}) = \frac{1}{(\sum x_i^2)^2} \cdot \left(\sum x_i^2 \cdot Var(y_i) \right) = \frac{\sigma^2}{\sum x_i^2}$

(f) $E(T) = E\left(\sum a_i y_i \right) = \sum a_i \cdot (\beta x_i) = \beta$

$\Rightarrow \sum a_i x_i = 1$: condition

(g) $k_i := \frac{x_i}{\sum x_i^2}$ and $d_i := k_i - a_i$

$\Rightarrow Var(T) = \sigma^2 \sum (k_i + d_i)^2 = \sigma^2 \left(\sum k_i^2 + 2 \sum k_i d_i + \sum d_i^2 \right)$

But : $\sum k_i d_i = \sum k_i a_i - \sum k_i^2 = 0$;(since $\sum x_i a_i = 1$)

$\Rightarrow Var(T) = \sigma^2 \sum k_i^2 + \sum d_i^2$

$= Var(\hat{\beta}) + \sum d_i^2$

$> Var(\hat{\beta})$ unless $T = \hat{\beta}$

7.5.2) $\mathbb{P}(|\hat{\theta}_{1n} - \theta_1| \geq \varepsilon) = 1 - \mathbb{P}(\hat{\theta}_{1n} < \theta_1 + \varepsilon)$

$$= \frac{(\theta_2 - \theta_1 - \varepsilon)^n}{(\theta_2 - \theta_1)^n} \xrightarrow[\text{as } n \to \infty]{} 0$$

$$\mathbb{P}(|\hat{\theta}_{2n} - \theta_2| \geq \varepsilon) = \left(\frac{\theta_2 - \varepsilon - \theta_1}{\theta_2 - \theta_1}\right)^n \xrightarrow[\text{as } n \to \infty]{} 0$$

7.5.4) (a) $\mathbb{P}(g(T_n) \leq g(\theta) - \varepsilon) = \left(\frac{1}{\theta}\right)^n \xrightarrow[\text{as } n \to \infty]{} 0$ for $\theta > 1$

$\mathbb{P}(g(T_n) \leq g(\theta) - \varepsilon) = 0$ for $\theta \in (0, 1]$

(b) for $\theta = 1$: $\mathbb{P}(g(T_n) \leq g(\theta) - \varepsilon) = \left(\frac{1}{\theta}\right)^n \underset{\text{as } n \to \infty}{\not\longrightarrow} 0$

(c) $\mathbb{P}(g(T_n) \leq g(\theta) - \varepsilon) = \left(\frac{\theta_0}{\theta}\right)^n \xrightarrow[\text{as } n \to \infty]{} 0$; for $\theta > \theta_0$

$\mathbb{P}(g(T_n) \leq g(\theta) - \varepsilon) = 0$ for $\theta < \theta_0$