7.8.1) a) $\operatorname{Var}_\theta(T(X)) \geq \frac{1}{nI(\theta)}$ where $I(\theta) = \frac{1}{\theta^2}$

b) $\mathcal{L}(\theta) = \theta^n e^{-\theta \sum x_i} \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}$

c) $E\left(\frac{1}{\bar{X}}\right) = \frac{n}{n-1}\theta \Rightarrow c(n) = \frac{n-1}{n}$

$\operatorname{Var}(\hat{\theta}_2) = \frac{\theta^2}{n-2} \geq \frac{\theta^2}{n}$; $n > 2$

d) $\sqrt{nI(\theta)}\,(\hat{\theta} - \theta) \xrightarrow{d} N(0,1)$

$\Rightarrow d(\theta) = \frac{\theta}{n}$

e) $\mathbb{P}\left(\hat{\theta} - z_{1-\alpha/2} \cdot \frac{\hat{\theta}}{\sqrt{n}} \leq \theta \leq \hat{\theta} + z_{1-\alpha/2} \cdot \frac{\hat{\theta}}{\sqrt{n}}\right) \approx 1 - \alpha$

7.8.3) a) $\frac{\partial^2}{\partial \theta^2} \ln f_{X_i}(x_i) = -\frac{1}{\theta^2} - \frac{x_i - 1}{(1-\theta)^2} \Rightarrow I(\theta) = \frac{1}{\theta^2(1-\theta)}$

b) $\operatorname{Var}_\theta(T(X)) \geq \frac{\theta^2(1-\theta)}{n}$

c) $\mathcal{L}(\theta) = \theta^n (1-\theta)^{\sum x_i - n} \Rightarrow$ (MLE) $\hat{\theta} = \frac{1}{\bar{X}}$

$E(X_i) = \frac{1}{\theta} = \bar{X} \Rightarrow \theta = \frac{1}{\bar{X}}$ (MME)

d) $h_1(X) = h(X_0) + h'(X_0)(X - X_0)$ where $h(x) = \frac{1}{x}$

and $X_0 = \bar{X}$

$\Rightarrow \operatorname{Var}(\hat{\theta}) \approx (h'(X_0))^2 \sigma^2 = \frac{1}{\bar{X}^4} \cdot \frac{1-\theta}{\theta^2}$

e) $\frac{1}{\bar{X}} \pm 1.96 \sqrt{\operatorname{Var}\left(\frac{1}{\bar{X}}\right)}$

7.8.3) a) $\frac{\partial^2}{\partial \eta^2} \ln f_{X_i}(x_i) = \frac{1}{2\eta^2} - \frac{(x_i-\mu)^2}{\eta^3} \Rightarrow I(\eta) = \frac{1}{2\eta^2}$

$\Rightarrow$ Cramer-Rao bound $= \frac{2\eta^2}{n}$

b) $\mathcal{L}(\eta) = (2\pi\eta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\eta} \sum (X_i-\mu)^2\right)$

$\Rightarrow$ (MLE) $\hat{\eta} = \frac{1}{n} \sum (X_i-\mu)^2$

c) $\eta = E(\hat{\eta}_2) = c(n)\, E(\hat{\eta}) = c(n) \frac{n-1}{n} E(S^2) = c(n) \frac{n-1}{n} \eta$

$\Rightarrow \quad c(n) = \frac{n}{n-1}$

d) $\sqrt{\frac{n}{2\eta^2}}\,(\hat{\eta}-\eta) \xrightarrow{d} N(0,1) \Rightarrow d(\eta) = \frac{\sqrt{2}\,\eta}{n}$

e) $\hat{\eta} \pm 1.96 \frac{\sqrt{2}\,\hat{\eta}}{\sqrt{n}}$

7.9.2) a) $f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \theta^n (1-\theta)^{\sum x_i - n}$

$T = \sum X_i \sim NegBin(n,\theta) \Rightarrow f_T(t) = \binom{t-1}{n-1} \theta^n (1-\theta)^{t-n}$

$\Rightarrow \frac{f_{X_1,\dots,X_n}(x_1,\dots,x_n)}{f_T(t)} = \frac{1}{\binom{t-1}{n-1}}$ not depend on $\theta \Rightarrow T$ is sufficient for θ.

$T_2 := \sum_{i=2}^{n} X_i \Rightarrow f_{X_1,T_2}(x_1,t_2) = \binom{t_2-1}{n-2} \theta^n (1-\theta)^{t_2+x_1-n}$

$\Rightarrow \frac{f_{X_1,\dots,X_n}(x_1,\dots,x_n)}{f_{X_1,T_2}(x_1,t_2)} = \frac{1}{\binom{t_2-1}{n-2}}$ not depend on $\theta \Rightarrow (X_1, T_2)$ is sufficient for θ

b) $h(x_1,\dots,x_n) = 1$ & $g(t) = \theta^n (1-\theta)^{t-n}$

7.9.4) a) $h(x_1,\ldots,x_n) = \prod_1^n \frac{1}{x_i!}$ & $g(t) = \lambda^t e^{-n\lambda}$

$\Rightarrow T = \sum_1^n X_i$ is sufficient for λ.

b) $\hat{\theta} = I_{\{X_1 = 0\}}$

$$\Rightarrow \hat{\theta}^* = E(\hat{\theta} \mid T) = \mathbb{P}\left(X_1 = 0 \mid \sum_1^n X_i = t\right)$$

$$= \left(1 - \frac{1}{n}\right)^{n\bar{X}} \approx e^{-\bar{X}} \quad \text{; for large } n$$

c) $E\left(e^{t\sum X_i}\right) = e^{n\lambda(e^t - 1)}$

$$\Rightarrow Var(\hat{\theta}^*) = e^{n\lambda(e^{-\frac{2}{n}} - 1)} - e^{2n\lambda(e^{-\frac{1}{n}} - 1)}$$

d) $\mathbb{P}\left(\left| I_{\{X_1 = 0\}} - e^{-\lambda}\right| \geq \varepsilon\right) = e^{-\lambda} \overset{n}{\not\longrightarrow} 0 \Rightarrow \hat{\theta}$ is not consistent for θ

$\mathbb{P}(|\bar{X} - \lambda| \geq \varepsilon) \leq \frac{\lambda/n}{\varepsilon^2} \xrightarrow{n} 0 \Rightarrow \bar{X}$ is consistent for λ

$\Rightarrow e^{-\bar{X}}$ is consistent for $e^{-\lambda}$

e) $\bar{X}$ is MLE for $\lambda \Rightarrow e^{-\bar{X}}$ is MLE for $e^{-\lambda}$.

7.9.6) a) $p(x) = \dfrac{e^{\mu(x)}}{1 + e^{\mu(x)}} \Rightarrow \ln\left(\dfrac{p(x)}{1 - p(x)}\right) = \mu(x)$

b) $h(y_1,\ldots,y_k) = \prod_1^n \binom{n_i}{y_i}$ & $g(t,s) = e^{\beta_0 t + \beta_1 s} \cdot \prod_1^k \left(1 + e^{\beta_0 + \beta_1 x_i}\right)^{-n_i}$

$\Rightarrow \left(\sum_1^k y_i, \sum_1^k x_i y_i\right)$ is sufficient for (β_0, β_1)

c) $h(y_1,\ldots,y_k) = (2\pi\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(\sum y_i^2 + n\beta_0^2 + 2\beta_0\beta_1 \sum x_i + \beta^2 \sum x_i^2\right)\right.$

and $g(t,s) = \exp\left(-\frac{1}{\sigma^2}\left(\beta_0 \sum y_i + \beta_1 \sum x_i y_i\right)\right)$

$\Rightarrow \left(\sum_1^k y_i, \sum_1^k x_i y_i\right)$ is sufficient for (β_0, β_1)