8.1.1) a) $\gamma(\theta) = \mathbb{P}(X_1+X_2 \le 1 | \theta) = \begin{cases} .07 & ; \theta=1 \\ .33 & ; \theta=2 \\ .65 & ; \theta=3 \end{cases}$

b) $\sup_{\theta\in\{1,2\}} \gamma(\theta) = .33$

c) $\gamma_1(\theta) = \mathbb{P}(\max(X_1, X_2) \le 1 | \theta) = \begin{cases} .16 & ; \theta=1 \\ .49 & ; \theta=2 \\ .81 & ; \theta=3 \end{cases}$

d) Consider: $X_1+X_2+X_3 \le 2 \Rightarrow \gamma(\theta) = \begin{cases} .037 & ; \theta=1 \\ .279 & ; \theta=2 \\ .665 & ; \theta=3 \end{cases}$

8.1.2) a) $\mathbb{P}(X<c | p) = \Phi((c-100p)/\sqrt{100p(1-p)}) \xrightarrow[\text{at } p=.8]{\text{maximize}} c = 74.88 \simeq 74$

b) $\gamma(p) = \mathbb{P}(X \le 74 | p) = \mathbb{P}(X \le 74.5 | p) = \Phi((74.5-100p)/\sqrt{100p(1-p)})$

c) $\gamma(.65) \simeq .97$; $\gamma(.7) \simeq .83$; $\gamma(.75) \simeq .45$; $\gamma(.8) \simeq .08$; $\gamma(.85) \simeq .002$

8.1.6) a) $X_{(10)} = \max\{X_1, \dots, X_{10}\}$ test statistics;

$\mathbb{P}(X_{(10)} < c | \theta) \xrightarrow[\text{at } \theta=20]{\text{maximized}} \left(\frac{c}{20}\right)^{10} = .05 \Rightarrow c \simeq 14.8$

$\Rightarrow \gamma(\theta) = \mathbb{P}(X_{(10)} < 14.8 | \theta) = \left(\frac{14.8}{\theta}\right)^{10}$

b) $\mathbb{P}(X_{(10)} > c | \theta=20) = 1 - \left(\frac{c}{20}\right)^{10} = .05 \Rightarrow c \simeq 19.9$

$\Rightarrow$ We reject H_0 if $19.9 < X_{(10)} < 14.8$

and: $\gamma(\theta) = \left(\frac{14.8}{\theta}\right)^{10} + \left(1 - \left(\frac{19.9}{\theta}\right)^{10}\right)$

8.2.1) a) $\frac{L(\mu_1)}{L(\mu_0)} > k \iff \bar{X} > k_\alpha$. $\mathbb{P}(\bar{X} > k_\alpha | \mu=\mu_0) = \alpha \Rightarrow k_\alpha = \mu_0 + z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}}$

b) $\gamma(\mu) = \mathbb{P}(\bar{X} > \mu_0 + z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}} | \mu) = \Phi\left(\frac{(\mu-\mu_0)\sqrt{n}}{\sigma} - z_{1-\alpha}\right)$

c) This test is UMP by the NP Lemma.

d) The test that rejects H_0 if $\bar{X} < \mu_0 - z_{1-\alpha} \frac{\sigma}{\sqrt{n}}$ has power:

$\gamma_2(\mu) = \Phi\left(\frac{(\mu_0-\mu)\sqrt{n}}{\sigma} - z_{1-\alpha}\right) \xRightarrow{\mu<\mu_0} \gamma(\mu) < \gamma_2(\mu)$

e) $\frac{L(\sigma_1^2)}{L(\sigma_0^2)} > k \iff \sum (X_i-\mu)^2 > k_\alpha = \sigma_0^2 \chi^2_{n,1-\alpha}$

$\Rightarrow \gamma(\sigma^2) = \mathbb{P}\left(X > \frac{\sigma_0^2}{\sigma_1^2} \chi_{n,1-\alpha}\right) \xrightarrow[\sigma_0^2=100,\ \sigma_1^2=140]{n=25;\ \alpha=.05} \gamma(140) \simeq .36$

8.2.3) a) $\frac{\mathcal{L}(\lambda_1)}{\mathcal{L}(\lambda_0)} > k \iff X_1+X_2+X_3 > k_{.084}$

$\mathbb{P}\left(\sum_1^3 x_i \le k_{.084}\right) = .916 \Rightarrow k_{.084} \simeq 9$

b) $\gamma(\lambda) = \mathbb{P}(X_1+X_2+X_3 > 9 \mid \lambda)$

c) By NP lemma, this test is UMP.

d) $\mathbb{P}\left(\sum_1^{100} X_i > k \mid \lambda = 2\right) = \alpha \Rightarrow k = 200 + z_{1-\alpha}\sqrt{200} \xrightarrow{\alpha=.05} k \simeq 223.26$

8.2.5) a) $\frac{\mathcal{L}(\theta_1)}{\mathcal{L}(\theta_0)} > k \iff X_{(n)} > k_\alpha$; $\mathbb{P}(X_{(n)} > k_\alpha \mid \theta = \theta_0) = 1 - \left(\frac{k_\alpha}{\theta_0}\right)^n = \alpha$

$\Rightarrow k_{.1} \simeq 19.58$

b) $\gamma(\theta) = \mathbb{P}(X_{(5)} > k \mid \theta)$; $\gamma(24) \simeq .63$, $\gamma(22) \simeq .44$

c) This test is UMP.

d) reject H_0 if: $X_{(n)} < i_\alpha \Rightarrow \mathbb{P}(X_{(n)} < i_\alpha \mid \theta = \theta_0) = \left(\frac{i_\alpha}{\theta_0}\right)^n = \alpha \Rightarrow i_\alpha = \theta_0 \alpha^{\frac{1}{n}}$

$\gamma(\theta) = \left(\frac{\theta_0 \alpha^{1/5}}{\theta}\right)^5$; $\gamma(15) \simeq .21$

8.2.6) a) M_k := frequency of k. We reject H_0 if $M_1 \ln\left(\frac{.6}{.5}\right) + M_2 \ln\left(\frac{.3}{.3}\right) + M_3 \ln\left(\frac{.1}{.2}\right) \ge .721$

$\gamma(1) \simeq .9$

b) $58 \ln\left(\frac{.6}{.5}\right) + 15 \ln\left(\frac{.1}{.2}\right) \simeq .178 < .721 \Rightarrow$ not reject x.

c) n such that: $\frac{.721 - n(.04)}{\sqrt{n(.06)}} = -1.645$

8.3.2) a) $\frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\hat{\theta})} = \left(\frac{X_{(n)}}{\theta_0}\right)^n$ for $X_{(n)} < \theta_0$; 0 otherwise

b) $\mathbb{P}(X_{(n)} < c \mid \theta = \theta_0) = \left(\frac{c}{\theta_0}\right)^n = \alpha \iff c = \theta_0 \alpha^{1/n}$.

$$\Rightarrow \gamma(\theta) = \begin{cases} \mathbb{P}(X_{(n)} < \theta_0 \alpha^{1/n} \mid \theta) = 1 & ; \quad \theta \le \theta_0 \alpha^{1/n} \\ \mathbb{P}(X_{(n)} < \theta_0 \alpha^{1/n} \mid \theta) = \left(\frac{\theta_0 \alpha^{1/n}}{\theta}\right)^n & ; \quad \theta_0 \alpha^{\frac{1}{n}} < \theta < \theta_0 \\ \mathbb{P}(X_{(n)} > \theta_0 (1-\alpha)^{\frac{1}{n}} \mid \theta) = 1 - \left(\frac{\theta_0 (1-\alpha)^{\frac{1}{n}}}{\theta}\right)^n & ; \quad \theta \ge \theta_0 \end{cases}$$

Here $X_{(n)} = 13.1 > 20 \cdot (.05)^{1/7} \Rightarrow$ fail to reject.

c,d) like part b).

8.3.3) a) $\frac{\mathcal{L}(\Theta_0)}{\mathcal{L}(\hat{\theta})} = \exp\left\{-\frac{n}{2\sigma^2}(\bar{X}-\mu_0)^2\right\} = \Lambda(X_1,\dots,X_n) \Rightarrow -2\ln\Lambda(X_1,\dots,X_n) = \frac{n}{\sigma^2}(\bar{X}-\mu_0)^2$

b) $\frac{n}{\sigma^2}(\bar{X}-\mu_0)^2$ is asymptotically as χ_1^2 => we reject if $\frac{n}{\sigma^2}(\bar{X}-\mu_0)^2 \geq \chi^2_{1,1-\alpha}$

By taking square root: $\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu) \geq \sqrt{\chi_{1,1-\alpha}}$: left hand side is $N(0,1)$ =>

$\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu_0) \geq Z_{1-\alpha}$

8.3.8) By factorization theorem: $f(x_1,\dots,x_n|\theta) = g(T(x_1,\dots,x_n)|\theta).\underbrace{h(x_1,\dots,x_n)}_{\text{not depend on }\theta}$

$$\Rightarrow \Lambda(X_1,\dots,X_n) = \frac{\sup_{\theta\in\Theta_0}\mathcal{L}(\theta|x_1,\dots,x_n)}{\sup_{\theta\in\Theta}\mathcal{L}(\theta|x_1,\dots,x_n)}$$

$$= \frac{\sup_{\theta\in\Theta_0} f(x_1,\dots,x_n|\theta)}{\sup_{\theta\in\Theta} f(x_1,\dots,x_n|\theta)}$$

$$= \frac{\sup_{\theta\in\Theta_0} g(T(x_1,\dots,x_n)|\theta)\cancel{h(x_1,\dots,x_n)}}{\sup_{\theta\in\Theta} g(T(x_1,\dots,x_n)|\theta)\cancel{h(x_1,\dots,x_n)}}$$

$$= \frac{\sup_{\theta\in\Theta_0}\mathcal{L}^*(\theta|T(x_1,\dots,x_n))}{\sup_{\theta\in\Theta}\mathcal{L}^*(\theta|T(x_1,\dots,x_n))}$$

$$= \Lambda^*(T(x_1,\dots,x_n)).$$

8.4.2) a) $\mathbb{P}(T_2>k|\lambda=3)=.08 \Rightarrow \mathbb{P}(T_2\leq 9) = .916$ => we reject if $T_2 \geq 10$.

b) $\alpha = \mathbb{P}(T_2\geq 12|\lambda=3) \simeq .02$

c) $\gamma(4) = \mathbb{P}(T_2>9|\lambda=4) \simeq .28$

d) $\mathbb{P}(T_{100}>k|\lambda=3)=.08 \Leftrightarrow k=324.3$ => we reject if $T_{100}\geq 325$.

e) $1-\Phi\left(\frac{329-300}{\sqrt{300}}\right) \simeq .05$

f) $\gamma(\lambda) = 1-\Phi\left(\frac{325-100\lambda}{\sqrt{100\lambda}}\right) \Rightarrow \gamma(3.3) \simeq .619$

8.4.3) a) $\hat{\alpha} = 2\left(1-\Phi\left(\frac{84-80}{8/5}\right)\right) = .012$

b) we reject if $\frac{25}{64}(\bar{X}-80)^2 > c \Leftrightarrow |\bar{X}-80|>k$.

$\xRightarrow{\alpha=.05}$ We reject if $\bar{X}$ is outside $[80-1.96(\frac{8}{5}), 80+1.96(\frac{8}{5})] = [78.77, 81.22]$

c) Exchanging μ_0 for $\bar{X}$ in b), gives us 95% CI for μ: $[80.86, 87.13]$

8.4.4) a) we reject H_0 if $X_{(n)}$ is <u>outside</u> $[\theta_0 \alpha^{\frac{1}{n}}, \theta_0]$.

b) Rearrange the above interval, gives us $100(1-\alpha)\%$ CI for θ which is:

$$[X_{(n)} \alpha^{-\frac{1}{n}}, X_{(n)}].$$

c) $\mathbb{P}(X_{(n)} > 46 \mid \theta = 50) = 1 - (\frac{46}{50})^{10} = .566$

d) $\hat{\alpha} = \mathbb{P}(X_{(n)} < 39.2 \mid \theta = 46) = (\frac{39.2}{46})^{10} = .202$

8.4.6) a) $\Lambda(X_1, \dots, X_{25}) = \begin{cases} 1 & ; X_{(1)} < 10 \\ e^{-5(X_{(1)}-10)} & X_{(1)} > 10 \end{cases}$

We reject if $X_{(1)} \geq 10 - \frac{\ln(c)}{5}$

$\mathbb{P}(X_{(1)} \geq 10 - \frac{\ln(c)}{5} \mid \eta = 10) = \alpha = .1$

$(X_{(1)} = \eta + Y_{(1)}) \Rightarrow \mathbb{P}(Y_{(1)} \geq -\frac{\ln(c)}{5}) = .1 \Leftrightarrow c = .1$

$\Rightarrow$ We reject if $X_{(1)} \geq 10 - \frac{\ln(.1)}{5} = 10.46$

b) $X_{(1)} = 10.53$; $\hat{\alpha} = \mathbb{P}(X_{(1)} \geq 10.53) = \mathbb{P}(Y_{(1)} \geq .53)$

$= e^{-\frac{.53}{5}}$

$\simeq .9$.