Seasonal Time Series Models and Their Application to the Modeling of River Flows

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Hydrology

by

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Abstract

This dissertation is concerned with the problem of modeling a periodic hydrological time series in general and river flows in particular. River flow series usually exhibit periodical stationarity; that is, their mean and covariance functions are periodic with respect to time. A class of models useful in such situations consists of periodic autoregressive moving average (PARMA) models, which are extensions of commonly used ARMA models that allow parameters to depend on season. The innovations algorithm can be used to obtain estimates of certain model parameters. Asymptotic distributions for the innovations estimates and model parameters provide us with a general technique for identifying PARMA models with the minimum number of parameters. A detailed simulation study is conducted to test the effectiveness of the innovation estimation procedures and asymptotics as a method for PARMA model identification. We use monthly river flow data for the Fraser River in British Colombia and the Salt River in Arizona so as to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows. In both cases, we are able to generate realistic synthetic river flows by developing a mixture probability model for the residuals.

Fitting the PARMA model to weekly or daily data, however, requires estimation of too many parameters which violates the principle of parsimony. In an effort to obtain a parsimonious model representing these periodically stationary series, we develop the asymptotic distribution of the discrete Fourier transform of the innovation estimates and PARMA model parameters, and then determine those statistically significant Fourier coefficients. We demonstrate the effectiveness of these techniques using simulated data from different PARMA models. An application of the techniques is demonstrated through the analysis of
a weekly river flow series for the Fraser River.

Hydrologic droughts are characterized by their duration and severity. We use PARMA models to model river flows and generate a large sequence of synthetic flows, and then drought properties are derived from the simulated samples based on the theory of runs. The applicability of these methods is demonstrated by using weekly river flow data for the Fraser River.
Dedication

To the memory of my father

To my mother, brothers and sisters
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1 Introduction

Time series analysis and modelling is an important tool in hydrology and water resources. It is used for building mathematical models to generate synthetic hydrologic records, to determine the likelihood of extreme events, to forecast hydrologic events, to detect trends and shifts in hydrologic records, and to fill in missing data and extend records. Synthetic river flow series are useful for determining the dimensions of hydraulic works, for flood and drought studies, for optimal operation of reservoir systems, for determining the risk of failure of dependable capacities of hydroelectric systems, for planning capacity expansion of water supply systems, and for many other purposes (Salas et al., 1985; Salas, 1993). For example, hydrologic drought properties (severity and duration as defined in Chapter 6) of various return periods are needed to assess the degree to which a water supply system will be able to cope with future droughts and, accordingly, to plan alternative water supply strategies. They can be determined from the historical record alone by using non-parametric methods but, because the number of drought events that can be drawn from the historical sample is generally small, the “historical” drought properties have a large degree of uncertainty. Other alternatives for finding drought properties include using stochastic models (such as PARMA models) that can represent the underlying river flows, simulating long records of such river flows, and then deriving droughts properties from the simulated samples based on the theory of runs (see Chapter 6 for a detailed discussion).

The main purposes of this dissertation are to fit a PARMA model to represent a given river flow data, estimate parameters, check for goodness of fit to the data, model the residuals, and to use the fitted model for generating synthetic river flows and apply them to extreme analysis such as droughts. The innovations algorithm can be used to obtain estimates of the PARMA model parameters. Asymptotic distributions for the innovations estimates
and model parameters (which are developed here) provide us with a general technique for identifying PARMA models with the minimum number of parameters. A detailed simulation study is conducted to test the effectiveness of the innovation estimation procedures and PARMA model identification. We use monthly river flow data for the Fraser River in British Colombia in order to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows. We apply the same methods to monthly flow data for the Salt River in Arizona, which has heavy tails. In both cases, we are able to generate realistic synthetic river flows by developing a mixture probability model for the residuals. In an effort to obtain a parsimonious model representing periodically stationary series, we develop the asymptotic distribution of the discrete Fourier transform of the innovation estimates and other model parameters, and then determine those statistically significant Fourier coefficients. We demonstrate the effectiveness of these techniques using simulated data from different PARMA models. An application of the techniques is demonstrated through the analysis of a weekly river flow series for the Fraser River. Finally, we demonstrate the application of synthetically generated weekly river flows using PARMA model for hydrologic drought analysis of the Fraser River.

1.1 Characteristics of Hydrological Time Series

The structure of hydrological time series consists of mainly one or more of these four basic structural properties and components (Salas, 1993):

- Over year trends and other deterministic changes (such as shifts in the parameters). In general, natural and human induced factors may produce gradual and instantaneous trends and shift in hydrological time series. See, for example, Salas (1980,1993) for a detailed discussion of trends and shifts in hydrological data and their removal.
• Intermittency in the processes, mainly consisting of the hydrology of intermittent sequences of zero and non-zero values.

• Seasonal or periodic changes of days, weeks or months within the annual cycle. Periodicity means that the statistical characteristic changes periodically within the year. For example, in hydrologic data concerning river flows, we expect high runoff periods in the spring and low flow periods in the summer. Thus the river flow correlations between spring months may be different from the correlations between summer months.

• Stochasticity or random variations.

All the rivers used as case studies in this research are perennial rivers. None of them revealed any significant trend or shift. Therefore, we consider only periodicity and stochasticity in the modeling of the river flow series.

Although basic hydrologic processes evolve on a continuous time scale, most analysis and modeling of such processes are made by defining them on a discrete time scale. In most cases, a discrete time processes is derived by aggregating the continuous time process within a given time interval while in other cases it is derived by sampling the continuous process at a discrete point in time. For example, a weekly river flow series can be derived by aggregating the continuous flow hydrograph on a weekly basis while a daily river flow series can be the result of simply sampling the flows of a river once daily or by integrating the continuous flow hydrograph on a daily basis. In any case, most time series of hydrological processes are defined at hourly, daily, weekly, monthly, bimonthly, quarterly, and annual time intervals. It must be emphasized at this point that the term period (also used interchangeably with season) applies for any time resolution smaller than one year.
The plot of a hydrological series gives a good indication of its main characteristics. For example, the daily, monthly, annual flows of the Truckee River at Farad (gauge 10346000), CA are given in Figure 1.1. The daily series (for this case, beginning October 1, 1912) shows a typical behavior in which during some days of the year the flows are low and in other days the flows are high. During the days of low flows the variability is small, on the other hand, during high flows the variability is large. This characteristic behavior of daily flows indicates that the daily means vary throughout the year. This is further substantiated if one observes the plot of monthly flows (ten years of data beginning October, 1912). In general, most hydrological time series exhibit periodic variations in mean, variance, covariance and skewness. The periodicity in mean, as can be seen from Figure 1.1, may be easily observed in the plot of the underlying hydrologic time series. However, the periodicity in higher order moments is not so easy to observe and usually requires further mathematical analysis (see, for example, Chapters 4 and 5 for a detailed discussion).

Autocorrelation analysis can also be used to identify cycles or periodic components of hydrological time series. Figure 1.1 (bottom right) shows the autocorrelation of monthly river flows for the Truckee River at Farad, CA. The non-decaying nature of the correlation function can be observed, indicating a correlation between flows at 12 months’ lag. Spectral analysis has also been used to detect cycles in hydrological data (Rosner and Yevejevich, 1966; Quimpo, 1967; Salas, 1993). In addition, periodicity in autocorrelation can be detected using asymptotic theory (see, for example, Anderson and Vecchia, 1993; Anderson and Meerschaert, 1997,1998).

Many river flow time series also exhibit occasional sharp spikes (also called heavy tail characteristics; that is, some observations are several times greater than the average), and as a result they are skewed. For example, Figure 1.2 shows a time series plot and histogram
for the 72 years (October 1914 to September 1986) of average monthly flows for the Salt River near Roosevelt, AZ. There are occasional sharp spikes in the time series plot of the data (for example, 14 observations are greater than $6 \times \text{Average}$, where $\text{Average} = 905.50$ cfs ), which indicates the heavy tail characteristics of the data. Since the data set exhibits heavy tails and skewness, it can be well described by a heavy tail distribution (for example, a stable distribution\(^1\)). The tail behavior of a non-normal stable distribution is asymptotically equivalent to a Pareto distribution (see Chapter 4). A Pareto random variable $X$, is defined by $P(X > x) = C x^{-\alpha}$ for some $C > 0$, $\alpha > 0$. The parameter $\alpha$ is called the ”tail index” and can be used as a measure of tail-heaviness. For data analysis applications, the actual

\(^1\)See the definition in Chapter 4
distribution is often unknown, so robust estimators (for instance, Hill’s estimator) are needed to fit the tail parameter \( \alpha \) of an unknown heavy tail distribution (see Chapter 4 for a detailed discussion of tail index estimation). Table 1.1 summarizes the Hill’s estimators for the average monthly flow data of Fraser, Salt, Truckee rivers. It must be noted that \( r \) is the number of order statistics.

Table 1.1: Hill estimators for monthly river flow data.

<table>
<thead>
<tr>
<th>( r + 1 )</th>
<th>( \hat{\alpha} )</th>
<th>( \hat{C} )</th>
</tr>
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<tbody>
<tr>
<td>Truckee</td>
<td>Salt</td>
<td>Fraser</td>
</tr>
<tr>
<td>5</td>
<td>2.548</td>
<td>4.825</td>
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<tr>
<td></td>
<td>3.163E+06</td>
<td>5.773E+16</td>
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<td>2.621</td>
</tr>
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</table>

Figure 1.2: Time series plot (left) and histogram (right) of monthly river flows (cfs) for the Salt River near Roosevelt, Arizona, from October 1914 to September 1986.
In time series modeling, the tail parameter $\alpha$ also dictates the modeling approach for data with a Pareto tail. If $\alpha > 4$, then the time series has finite fourth moments, and the classical approach based on normal asymptotics is appropriate. If $2 < \alpha < 4$, then the time series has finite variance but infinite fourth moment and the asymptotics of the sample autocorrelation functions (ACF) are governed by stable laws. Finally, if $\alpha < 2$, the time series has infinite second moments, and the asymptotics are governed by stable laws. Preliminary tail estimations from Table 1.1 (for small $r = 15$) indicates that for the Fraser River $\alpha > 4$, so the time series has light tails and normal asymptotics apply whereas $2 < \alpha < 4$ for the Salt and Truckee rivers, so that these time series have heavy tails with a finite variance but infinite fourth moment, and stable asymptotics apply.

1.2 Modeling and Simulation of Hydrological Series

A number of approaches have been suggested for modeling hydrological time series defined at time intervals less than a year (Salas, 1993; Hipel and McLeod, 1994). The common procedure in modeling such periodic river flow series is first to standardize or filter the series and then fit an appropriate stationary stochastic model to the reduced series (Salas, et al., 1980; Thompstone et al., 1985; Vecchia, 1985; Salas, 1993; Chen, 2002). However, standardizing or filtering most river flow series may not yield stationary residuals due to periodic autocorrelations. In these cases, the resulting model is misspecified (Tiao and Grupe, 1980). Periodic models can, therefore, be employed to remove the periodic correlation structure. An important class of periodic models useful in such situations consists of periodic autoregressive moving average (PARMA) models, which are extensions of commonly used ARMA models that allow periodic parameters. The PARMA modeling procedure involves iterative steps of model identification, parameter estimation, model diagnosis and fitting the
residuals (noise) with a probability distribution function (pdf). The opposite process to step-by-step modeling is the use of models to generate (simulate) new samples or a long sample of the process. One starts with the random noise and its pdf by generating its sample(s). Then generate the corresponding data samples by using the fitted PARMA model.

1.3 Literature Review

Early studies by Hazen (1914) and Sudler (1927) showed the feasibility of using statistics and probability theory in analyzing river flow sequences. Hurst (1951) in investigating the Nile River for the Aswan Dam project, reported studies of long records of river flows and other geophysical series, which years later tremendously impacted the theoretical and practical aspect of time series analysis of hydrologic and geophysical phenomena. Barnes (1954) extended the early empirical studies of Hazen and Sudler and introduced the idea of synthetic generation of stream flow by using a table of random numbers. However, it was not until the beginning of the 1960’s that the formal development of stochastic modeling started with introduction and application of autoregressive models for annual and seasonal stream flows (Thomas and Fiering, 1962; Yevejevich, 1963; Roesner and Yevejevich; 1966).

Since then, extensive research efforts have been made towards improving the earlier concepts and models, providing physical justification of some models, introducing alternative models, developing and applying improved estimation procedures as well as fitness tests for such models, and studying their impact in water resources systems planning and management. The concept of periodically correlated processes was introduced by Gladyshev (1961). He gave a formal definition of periodic stationarity for a general periodic process and showed that a necessary and sufficient condition for such stationarity is the stationarity of the so-called “lumped” vector process. The first application of periodic time series mod-

For example, Noakes et al. (1985) demonstrate the superiority of periodic autoregressive models among several other competitors in forecasting thirty monthly river flow time series. Troutman (1979) studies some properties of the periodic autoregressive models using a related multivariate autoregressive representation. Tiao and Grupe (1980) show how periodic autoregressive moving average models may be misspecified as homogeneous models. Because of the seasonal parameters, the estimation of periodic models is more difficult than that of the homogeneous models. Pangano (1978) dealt with moment estimation of parameters in periodic autoregressive (PAR) models. He showed that estimates of the seasonal parameters obtained by using the seasonal Yule-Walker equations possess many desirable properties, including maximum asymptotic efficiency under normality. Salas et al. (1982) suggest estimating the parameters of PARMA models using the seasonal Yule-Walker equations. Vecchia (1985b) proposed an algorithm for the maximum likelihood estimation for periodic ARMA models. Li (1988) developed an algorithm for exact likelihood of periodic moving average models. Anderson, Meerschaert and Vecchia (1999) developed the innovations algorithm for
estimation of PARMA model parameters. Lund and Basawa (2000) examine the recursive prediction and likelihood evaluation techniques for PARMA models, which are extensions of Ansley (1979) for ARMA series. Shao and Lund (2004) study correlation and partial autocorrelation properties PARMA time series models. Additional review of literature is given at the beginning of the remaining chapters.

1.4 Objectives of the Research

Hydrologic river flows can exhibit both heavy tails and/or nonstationarity. Therefore, the main goal of the research presented here is to fit a PARMA model to represent a given river flow data, estimate parameters, check for goodness of fit to the data, and possibly to use the fitted model for different purposes such as extreme analysis (flood and drought).

Specific objectives:

1. Anderson, Meerschaert and Vecchia (1999) developed a parameter estimation technique (known as innovations algorithm) for PARMA models with finite fourth moment as well as infinite fourth moment but finite variance. In addition, Anderson and Meerschaert (2003) provided a method for model identification when the time series have finite fourth moment. Therefore, in this part of the research, asymptotic distributions for model parameters are developed. These results provide us with a general technique for identifying PARMA models with minimum number of parameters. A detailed simulation study is conducted to test the effectiveness of the innovation estimation procedures and those asymptotics as method for PARMA model identification. We use monthly river flow data for the Fraser River at Hope in British Colombia so as to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows.
2. We describe the PARMA modeling procedure for time series with heavy tails. As an application, we use monthly flow data for the Salt River near Roosevelt in Arizona so as to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows in the heavy tail case.

3. For analysis and design of water resources systems, it is sometimes required to generate river flow data with high resolution (that is, weekly or daily values). Fitting the PARMA model to historical weekly or daily data, however, requires estimation of too many parameters which violates the principle of parsimony (model with minimum number of parameters). In an effort to obtain a parsimonious model representing these periodically stationary series, we develop the asymptotic distribution of the discrete Fourier transform of the innovation estimates and then determine those statistically significant Fourier coefficients. We also extend these results to other periodic model parameters. We demonstrate the effectiveness of the technique using simulated data from different PARMA models. An application of the technique is demonstrated through the analysis of a weekly river flow series for the Fraser River.

4. Hydrologic drought properties (such as duration and severity) of various return periods, for example, are needed to assess the degree to which a water supply system will be able to cope with future droughts and, accordingly, to plan alternative water supply strategies. They can be determined from the historical record alone by using non-parametric methods but, because the number of drought events that can be drawn from the historical sample is generally small, the “historical” drought properties have a large degree of uncertainty. In such cases, drought properties can be derived by
synthetically generating river flows at key points in the water supply system under consideration. In this part of the research, PARMA models are used to model river flows and to generate a large sequence of synthetic flows, and then drought properties are derived from the simulated samples based on the theory of runs. The applicability of these methods is demonstrated by using weekly river flow data for the Fraser River.

The remainder of the dissertation is organized as follows. Chapter 2 describes the general properties of PARMA models and the innovations algorithm for parameter estimation of PARMA models. Asymptotic distributions for the innovations estimates and model parameters are developed in Chapter 3, which also deals with PARMA model identification using those asymptotic distributions, and demonstrates the PARMA modeling techniques with a simulation study and with an example. Chapter 4 concerns modeling of river flows with heavy tails. In an effort to obtain a parsimonious model for periodically stationary series, asymptotic distributions of the discrete Fourier transform of the innovations estimates and PARMA model parameters are developed in Chapter 5 and used to determine statistically significant Fourier coefficients. In Chapter 5, we also included a simulation study and an example to demonstrate the effectiveness of the technique. Applications of PARMA models to analysis of droughts are presented in Chapter 6. Finally, conclusions are drawn and recommendations for further research are made in Chapter 7.
2 PARMA Models and Their Parameter Estimation

2.1 General

A time series is a set of observations \( \{X_t\} \), each one being recorded at a specific time \( t \). For example, \( \{X_t\} \) can be river flow (daily, monthly, etc.) measurements at time \( t \). In general, a collection of a random variables, indexed by \( t \) is referred to as a stochastic process. The observed values of a stochastic process are referred to as a realization of the stochastic process. A mathematical model representing a stochastic process is called a stochastic model or time series model. The model consists of a certain mathematical form or structure and a set of parameters. Such models are built to “reproduce” or to “resemble” the main statistical characteristics (mean, standard deviation, skewness, autocorrelation, range and run) of the time series.

Several stochastic models have been used for modeling hydrological time series in general and stream flow time series in particular (Salas et al., 1981). Unfortunately, the exact mathematical model of a hydrological time series is never known. The exact model parameters are also never known, they must be estimated from limited data. Identification of the models and estimation of their parameters from the available data are often referred to in the literature as time series modeling or stochastic modeling of hydrologic time series. A systematic approach to hydrologic time series modeling may be composed of six main phases (Salas et al, 1981):

1. Identification of model composition. The modeler has to decide whether the model(s) will be a univariate, a multivariate, or a combination of a univariate and a disaggregation, etc.

2. Identification of model type. Select a type of models among the various types available
in literature.

3. Identification of the model form. Determine, for example, the orders \( p \) and \( q \) of a PARMA model.


5. Testing the goodness of fit of the model. The model estimated in phase (4) needs to be checked to see whether it complies with certain assumptions about the model and to verify how well it represents the historic hydrologic time series.

6. Evaluation of uncertainties. It may be model uncertainty or parameter uncertainty.

The overall time series modeling is actually an iterative process with the feedback and interaction between each of the above-referred stages.

The selection or identification of the type of hydrologic stochastic model generally depends on (Salas et al, 1981):

- The modelers’ input such as judgment, experience, and personal preference;

- the physical basis of the process under study (physical or stochastic generating mechanisms of the data); and

- the statistical characteristics of a given time series.

Based on the conceptual physical representation of a natural watershed, river flow processes are often modeled by the general class of ARMA processes (Salas et al, 1981). As discussed in Chapter 1, observed river flow series are characterized by seasonal variations in both the mean and covariance structure. Periodic ARMA models are, therefore, an appropriate class of models to represent the river flow series.
In this chapter and the following three chapters, we deal with the 3rd, 4th and 5th phases of hydrologic time series modeling. Parameter estimation for PARMA models is more complex than for normal ARMA models because of the higher number of parameters to be estimated. Pagano (1978) has used seasonal Yule-Walker equations (see equation (2.6)) to estimate the parameters of PAR models. Vecchia (1985) has used ML for parameter estimation of PARMA models, but this only works well for lower order models. With higher-order PARMA models it becomes difficult to obtain estimates using the ML algorithm. Adams and Goodwin (1995) have employed pseudo-linear regression algorithms for parameter estimation of PARMA models with finite fourth moment. Anderson, Meerschaert and Vecchia (1999) developed the innovations algorithm for parameter estimation of an infinite moving average representation of PARMA models, which is applicable for PARMA models with finite fourth moment as well as infinite fourth moment but finite variance. The innovations algorithm parameter estimation technique is the focus of this chapter.

2.2 Mathematical Formulation of PARMA Model

A stochastic process $X_t$ is periodically stationary (in the wide sense) if $\mu_t = EX_t$ and $\gamma_t(h) = \text{Cov}(X_t, X_{t+h}) = EX_tX_{t+h}$ for $h = 0, \pm 1, \pm 2, \ldots$ are all periodic functions of time $t$ with the same period $\nu \geq 1$. That is, for some integer $\nu$, for $i = 0, 1, ..., \nu - 1$, and for all integers $k$ and $h$, $\mu_i = \mu_{i+k\nu}$ and $\gamma_i(h) = \gamma_{i+k\nu}(h)$. If $\nu = 1$ then the process is stationary\(^3\).

The periodic ARMA process $\{\tilde{X}_t\}$ with period $\nu$ (denoted by PARMA\(_\nu\)\((p, q)\)) has representation

\(^2\)Maximum likelihood(ML) estimation, also called the maximum likelihood method, is the procedure of finding the value of one or more parameters for a given statistic which makes the known likelihood distribution a maximum.

\(^3\)See Definition A.2, Appendix A.
\[ X_t - \sum_{j=1}^{p} \phi_t(j) X_{t-j} = \varepsilon_t - \sum_{j=1}^{q} \theta_t(j) \varepsilon_{t-j} \]  

(2.1)

where \( X_t = \tilde{X}_t - \mu_t \) and \( \{ \varepsilon_t \} \) is a sequence of random variables with mean zero and scale \( \sigma_t \) such that \( \{ \delta_t = \sigma_t^{-1} \varepsilon_t \} \) is independent and identically distributed (iid). The autoregressive parameters \( \phi_t(j) \), the moving average parameters \( \theta_t(j) \), and the residual standard deviations \( \sigma_t \) are all periodic functions of \( t \) with the same period \( \nu \geq 1 \). The residual standard deviations parameters \( \sigma_t \), assumed to be strictly positive. We also assume:

(i) Finite variance: \( E \varepsilon_t^2 < \infty \).

(ii) We will consider models where \( E \varepsilon_t^4 < \infty \) (Finite Fourth Moment Case) and also models in which \( E \varepsilon_t^4 = \infty \) (Infinite Fourth Moment Case). In the latter case, we say that the iid sequence \( \{ \delta_t = \sigma_t^{-1} \varepsilon_t \} \) is RV(\( \alpha \)) if \( P[|\delta_t| > x] \) varies regularly\(^4\) with the index \(-\alpha\) and \( P[\delta_t > x]/P[|\delta_t| > x] \rightarrow p \) for some \( p \in [0, 1] \). In the case where the noise sequence has infinite fourth moment, it is assumed that the sequence is RV(\( \alpha \)) with \( \alpha > 2 \). This assumption implies that \( E|\delta_t|^\rho < \infty \) if \( 0 < \rho \leq \alpha \), in particular the variance of \( \varepsilon_t \) exists. With this technical condition, Anderson and Meerschaert (1997) showed that the sample autocovariance is a consistent estimator of the autocovariance, and asymptotically stable\(^5\) with tail index \( \alpha/2 \).

(iii) The model admits a causal representation

\[ X_t = \sum_{j=0}^{\infty} \psi_t(j) \varepsilon_{t-j} \]  

(2.2)

where \( \psi_t(0) = 1 \) and \( \sum_{j=0}^{\infty} |\psi_t(j)| < \infty \) for all \( t \). The absolute summability of the \( \psi \)-weights ensures that (2.2) converges almost surely for all \( t \), and in the mean-square to the same limit. The causality restriction imposes conditions on the seasonal autoregressive parameters.

\(^4\)See Definition 4.1, Section 4.3.

\(^5\)See Chapter 4 for the definition.
ters in (2.1). These conditions may be stated by expressing the PARMA model given by (2.1) in terms of an equivalent multivariate stationary ARMA process (Tiao and Grupe, 1980; Vecchia, 1985a). For example, if $p = 1$, the causality condition for the autoregressive parameters in (2.1) is given by $|\phi_0(1)\phi_1(1)\ldots\phi_{\nu-1}(1)| < 1$. It should be noted that $\psi_t(j) = \psi_{t+k\nu}(j)$ for all $j$.

(iv) The model also satisfies an invertibility condition

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_t(j)X_{t-j}$$

where $\pi_t(0) = 1$ and $\sum_{j=0}^{\infty} |\pi_t(j)| < \infty$ for all $t$. The invertibility condition places constraints on the moving average parameters in the same way that (2.2) places constraints on the autoregressive parameters. Again, $\pi_t(j) = \pi_{t+k\nu}(j)$ for all $j$.

2.3 Parameter Estimation for PARMA Model

The innovations algorithm, developed for periodic time series models by Anderson, Mee-erschaert and Vecchia (1999), is used to obtain parameters estimates for PARMA models. It can be used to estimate parameters for the PARMA models with finite fourth moments ($E\varepsilon_t^4 < \infty$) as well as infinite fourth moment but finite variance ($E\varepsilon_t^4 = \infty$). When $E\varepsilon_t^2 = \infty$, the autocovariance and autocorrelation functions are not defined.

Let $\hat{X}_{i+k} = P_{H_{k,i}}X_{i+k}$ denote the one-step predictors, where $H_{k,i} = \mathbb{P}\{X_i, \ldots, X_{i+k-1}\}$, $k \geq 1$, and $P_{H_{k,i}}$ is the orthogonal projection onto this space, which minimizes the mean squared error

$$v_{k,i} = \left\|X_{i+k} - \hat{X}_{i+k}^{(i)}\right\|^2 = E\left(X_{i+k} - \hat{X}_{i+k}^{(i)}\right)^2$$

Then

$$\hat{X}_{i+k}^{(i)} = \phi_{k,1}^{(i)}X_{i+k-1} + \ldots + \phi_{k,k}^{(i)}X_i, \quad k \geq 1,$$
where the vector of coefficients \( \phi_k^{(i)} = (\phi_{k,1}^{(i)}, \ldots, \phi_{k,k}^{(i)})' \) solves the prediction equations

\[
\Gamma_{k,i} \phi_k^{(i)} = \gamma_k^{(i)} \tag{2.6}
\]

with \( \gamma_k^{(i)} = (\gamma_{i+k-1}(1), \gamma_{i+k-2}(2), \ldots, \gamma_i(k))' \) and

\[
\Gamma_{k,i} = \left[\gamma_{i+k-l}(\ell - m)\right]_{\ell,m=1,...,k} \tag{2.7}
\]

is the covariance matrix of \((X_{i+k-1}, \ldots, X_i)\) for each \( i = 0, \ldots, \nu - 1 \). Because the process is nonstationary, the Durbin-Levinson algorithm (Brockwell and Davis, 1991) for computing \( \hat{\phi}_{k,j}^{(i)} \) does not apply. However, the innovations algorithm still applies to a nonstationary process. Writing

\[
\hat{X}_{i+k}^{(i)} = \sum_{j=1}^{k} \theta_{k,j}^{(i)} \left( X_{i+k-j} - \hat{X}_{i+k-j}^{(i)} \right) \tag{2.8}
\]

yields the one step predictors in terms of the innovations \( X_{i+k-j} - \hat{X}_{i+k-j}^{(i)} \). If \( \sigma_i^2 > 0 \) for \( i = 0, \ldots, \nu - 1 \), then for a causal PARMA \( \nu(p,q) \) process the covariance matrix \( \Gamma_{k,i} \) is non-singular for every \( k \geq 1 \) and each \( i \) (see Lund and Basawa, 2000). Anderson, Meerschaert, and Vecchia (1999) showed that if \( EX_t = 0 \) and \( \Gamma_{k,i} \) is non-singular for each \( k \geq 1 \), then the one-step predictors \( \hat{X}_{i+k} \), \( k \geq 0 \), and their mean-square errors \( v_{k,i} \), \( k \geq 1 \), are given by

\[
v_{0,i} = \gamma_i(0)
\]

\[
\theta_{k,k-l}^{(i)} = \left( v_{k,i} \right)^{-1} \left[ \gamma_{i+k-l}(k-l) - \sum_{j=0}^{l-1} \theta_{k,l-j}^{(i)} \theta_{k,k-j}^{(i)} v_{j,i} \right] \tag{2.9}
\]

\[
v_{k,i} = \gamma_{i+k}(0) - \sum_{j=0}^{k-1} \left( \theta_{k,k-j}^{(i)} \right)^2 v_{j,i}
\]

where (2.9) is solved in the order \( v_{0,i}, \theta_{1,1}^{(i)}, v_{1,i}, \theta_{2,1}^{(i)}, \theta_{2,2}^{(i)}, v_{2,i}, \theta_{3,1}^{(i)}, \theta_{3,2}^{(i)}, \theta_{3,3}^{(i)}, v_{3,i}, \ldots \). The
results in Anderson, Meerschaert and Vecchia (1999) show that

\[ \theta_{k,j}^{(i-k)} \rightarrow \psi_i(j) \]

\[ v_{k,(i-k)} \rightarrow \sigma_i^2 \]  
(2.10)

\[ \phi_{k,j}^{(i-k)} \rightarrow -\pi_i(j) \]

as \( k \rightarrow \infty \) for all \( i, j \), where

\[ \langle j \rangle = \begin{cases} 
  j - \nu \lfloor j/\nu \rfloor & \text{if } j = 0, 1, \ldots \\
  \nu + j - \nu \lfloor j/\nu + 1 \rfloor & \text{if } j = -1, -2, \ldots 
\end{cases} \]

and \( \lfloor \cdot \rfloor \) is the greatest integer function. For example, for monthly data \( X_t \), \( \langle t \rangle \) is the month of the year= 0, 1, \ldots , 11. Given \( N_y \) years of data, \( N = N_y \nu \), define the sample mean

\[ \hat{\mu}_i = N_y^{-1} \sum_{k=0}^{N_y-1} \tilde{X}_{kv+i} \]  
(2.11)

the sample autocovariance

\[ \hat{\gamma}_i(\ell) = (N_y - m)^{-1} \sum_{j=0}^{N_y-1-m} X_{j\nu+i}X_{j\nu+i+\ell} \]  
(2.12)

and the sample autocorrelation

\[ \hat{\rho}_i(j) = \frac{\hat{\gamma}_i(j)}{\sqrt{\hat{\gamma}_i(0)\hat{\gamma}_{i+j-m\nu}(0)}} \]  
(2.13)

where \( X_t = \tilde{X}_t - \hat{\mu}_t \), \( m = [(i + \ell)/\nu] \) and \( \lfloor \cdot \rfloor \) is the greatest integer function. If we replace the autocovariances in the prediction equation (2.6) with their corresponding sample autocovariances, we obtain the estimator \( \hat{\phi}_{k,j}^{(i)} \) of \( \phi_{k,j}^{(i)} \). If we also replace the autocovariances in (2.9) with corresponding sample autocovariances (2.12), we obtain the innovations estimates \( \hat{\theta}_{k,l}^{(i)} \). The consistency of these estimators was also established in the following result due to Anderson, Meerschaert and Vecchia (1999).
Theorem 2.1 (Finite Fourth Moment Case) Suppose that \( \{X_t\} \) be the mean zero PARMA process with period \( \nu \) given by (2.1) and that \( E\varepsilon_t^4 < \infty \). Assume that the spectral density matrix, \( f(\lambda) \), of its equivalent vector ARMA process is such that \( mzz' \leq z'f(\lambda)z \leq Mzz' \), \(-\pi \leq \lambda \leq \pi\), for some \( m \) and \( M \) such that \( 0 < m \leq M < \infty \) and for all \( z \) in \( \mathbb{R}^\nu \). If \( k \) is chosen as a function of the sample size \( N_y \) so that \( k^2/N_y \to 0 \) as \( N_y \to \infty \) and \( k \to \infty \), then

\[
\begin{align*}
\hat{\theta}_{k,j}^{(i-k)} & \xrightarrow{P} \psi_i(j) \\
\hat{\nu}_{k,(i-k)} & \xrightarrow{P} \sigma_i^2 \\
\hat{\phi}_{k,j}^{(i-k)} & \xrightarrow{P} -\pi_i(j)
\end{align*}
\]

(2.14)

for all \( i, j \) where \( \xrightarrow{P} \) denotes convergence in probability\(^7\).

PROOF. See Anderson, Meerschaert and Vecchia (1999) for a proof.

This yields a practical method for estimating the model parameters, in the classical case of finite fourth moments. The asymptotic results in Chapter 3 can then be used to determine which of these model parameters are statistically significantly different from zero.

Define

\[
a_{N_y} = \inf\{x : P(|\delta_t| > x) < 1/N_y\},
\]

where

\[
a_{N_y}^{-2} = \sum_{t=0}^{N_y-1} \delta_{t+1}^2 \Rightarrow S^{(i)},
\]

\( S^{(i)} \) is \( \alpha/2 \)-stable law, and \( \Rightarrow \) denotes convergence in distribution\(^8\).

---

\(^6\)See Section A-3, Appendix A.

\(^7\)See Definition A.10, Appendix A.

\(^8\)See Definition A.11, Appendix A.
Theorem 2.2 (Infinite Fourth Moment Case) Suppose that \( \{X_t\} \) be the mean zero \( \text{PARMA} \) process with period \( \nu \) given by (2.1) and \( \{\delta_t\} \) is \( RV(\alpha) \) with \( 2 < \alpha \leq 4 \). Assume that the spectral density matrix, \( f(\lambda) \), of its equivalent vector \( \text{ARMA} \) process is such that \( \text{mzz}' \leq z'f(\lambda)z \leq M\text{zz}' \), \(-\pi \leq \lambda \leq \pi\), for some \( m \) and \( M \) such that \( 0 < m \leq M \) and for all \( z \) in \( \mathbb{R}^\nu \). If \( k \) is chosen as a function of the sample size \( N_y \) so that \( k^{5/2}a_N^2/N_y \to 0 \) as \( N_y \to \infty \) and \( k \to \infty \), then

\[
\hat{\theta}_{k,j}^{(i-k)} \xrightarrow{P} \psi_i(j)
\]

\[
\hat{\nu}_{k,(i-k)} \xrightarrow{P} \sigma_i^2
\]

\[
\hat{\phi}_{k,j}^{(i-k)} \xrightarrow{P} -\pi_i(j)
\]

for all \( j \) and every \( i = 0, 1, \ldots, \nu - 1 \), where \( \xrightarrow{P} \) denotes convergence in probability.

PROOF. See Anderson, Meerschaert and Vecchia (1999) for a proof.

Theorem 2.2 yields a practical method for estimating the model parameters, in the case of infinite fourth moments.
3 PARMA Model Identification: Finite Fourth Moment Case

3.1 General

The PARMA modeling procedure usually involves three main steps: model identification, parameter estimation and diagnostic checking. The goal of PARMA model identification is to identify the order of the PARMA model. Ula (2003) has used the cut-off property of the periodic autocorrelation function for the identification of the orders of periodic MA (PMA) models, which is an extension of Box-Jenkins (Box and Jenkins, 1976) identification techniques for ordinary MA and AR models. Anderson and Vecchia (1993) computed the asymptotic distributions of the moment estimates of the periodic autocovariance and autocorrelation functions of a PARMA process and used them for detecting periodicity in autocorrelations and model parameters. Also, they derived the asymptotic distribution of the Fourier-transformed sample periodic autocorrelation function and applied those asymptotics for identification of a parsimonious PARMA model.

Anderson and Meerschaert (2003) provided an asymptotic distribution for innovation estimates of PARMA models in the case where the innovations have finite fourth moment. These results are used for the identification of the order of PARMA models. However, such techniques often do not adequately identify the order of the model when the model is mixed in the sense that both the autoregressive (AR) and moving average (MA) components are present. If the innovations algorithm is supplemented by modeler experience or the Akaike’s information criterion⁹ (AIC), then an approximate order of the mixed PARMA model can be obtained. This chapter, which is an extension of previous work by Anderson and Meerschaert,

⁹See Section A-5, Appendix A.
has three objectives. The first is to develop asymptotic distributions for PARMA model parameters for the finite fourth moment case. These asymptotic results provide us with a general technique for computing asymptotic confidence regions for the model parameters from the innovation algorithm estimates, and for identifying PARMA models with minimum number of parameters. The second is to demonstrate the effectiveness of the innovation estimation procedures and those asymptotics as a method for PARMA model identification by using simulated data from different PARMA models for the finite fourth moment case. Finally we use monthly river flow data for the Fraser River at Hope in British Colombia so as to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows.

3.2 Asymptotic Distributions for Moving Average Processes

Any mean-centered finite variance periodic ARMA process (2.1) can be expressed in terms of a periodic moving average

\[ X_t = \sum_{j=-\infty}^{\infty} \psi_t(j) \varepsilon_{t-j} \]  

(3.1)

where \( \delta_t = \sigma_t^{-1} \varepsilon_t \) is a sequence of iid random variables with mean zero and variance one. We also assume that the moving average parameters \( \psi_t(j) \) are periodic in \( t \) with the same period \( \nu \), and that \( \sum_{j=-\infty}^{\infty} |\psi_t(j)| < \infty \) for all \( t \). Also, we consider a finite fourth moment case: \( E \varepsilon_t^4 < \infty \).

For any periodically stationary time series, we can construct an equivalent (stationary) vector moving average process in the following way: Let \( Z_t = (\delta_t^{(1)}, ..., \delta_{(t+1)\nu-1})' \) and \( Y_t = \)
\[(X_{\nu}, \ldots, X_{(t+1)\nu-1})', \text{ so that}\]

\[Y_t = \sum_{j=-\infty}^{\infty} \Psi_j Z_{t-j}\]  

(3.2)

where \(\Psi_j\) is the \(\nu \times \nu\) matrix with \(ij\) entry \(\sigma_i \psi_i (t\nu + i - j)\), and we number the rows and columns 0, 1, ..., \(\nu - 1\) for ease of notation.

We define the sample autocovariance matrix by \(\hat{\Gamma}(h) = N_y^{-1} \sum_{t=0}^{N_y-1} (Y_t - \hat{\mu}) (Y_{t+h} - \hat{\mu})'\), where \(\hat{\mu} = N_y^{-1} \sum_{t=0}^{N_y-1} Y_t\) is the sample mean and the autocovariance matrix by \(\Gamma(h) = E(Y_t - \mu) (Y_{t+h} - \mu)'\), where \(\mu = EY_t\). Note that the \(ij\) entry of \(\Gamma(h)\) is \(\gamma_i(h \nu + j - i)\) and likewise for \(\hat{\Gamma}(h)\). The autocorrelation matrix \(R(h)\) has \(ij\) entry equal to \(\rho_i(h \nu + j - i)\) and likewise for the sample autocorrelation matrix \(\hat{R}(h)\). The \(ij\) term of \(R(h)\) is also called the cross correlation of the \(i\) and \(j\) components of the vector process at lag \(h\). In this application it represents the correlation between season \(i\) at year \(t\) and season \(j\) at year \(t + h\).

**Theorem 3.1** Let \(X_t = \tilde{X}_t - \mu_t\), where \(\{X_t\}\) is the periodic moving average process (3.1) and \(\mu_t\) is a periodic mean function with period \(\nu\). Then, for \(\sigma_i^2 = E\xi_i^2 < \infty\), \(\hat{\mu}_t\) is a consistent estimator of \(\mu_t\) and

\[N_y^{1/2} \{\hat{\mu} - \mu\} \Rightarrow \mathcal{N}(0, C)\]  

(3.3)

where \(N = N_y \nu\), \(\mathcal{N}(0, C)\) is a Gaussian random vector with mean zero and covariance

\[C = (\sum_j \Psi_j)(\sum_j \Psi_j)'\]

\[\hat{\mu} = [\hat{\mu}_0, \hat{\mu}_1, \cdots, \hat{\mu}_{\nu-1}]^T\]

\[\mu = [\mu_0, \mu_1, \cdots, \mu_{\nu-1}]^T\]  

(3.4)

**PROOF.** See Anderson and Meerschaert (1998).

This theorem can be used for constructing confidence regions for \(\mu\). We adapt the procedure of Brockwell and Davis (1991, p. 407) to obtain the simultaneous \((1 - \alpha)100\%\) confidence region for \(\mu_t\). By Theorem 3.1, we have \(N_y^{1/2} \{\hat{\mu}_t - \mu_t\} \Rightarrow \mathcal{N}(0, v_i)\), where \(v_i\) is a
variance

\[ \upsilon_i = 2\pi f_i(0) = \sum_{j=-\infty}^{\infty} \gamma_i(j) \]  

(3.5)

and \( f_i(\omega) \) is the spectral density for the \( i \)th seasonal component. The lag window estimator

\[ 2\pi f_i(0) = \sum_{|h| \leq r} \left( 1 - \frac{|h|}{r} \right) \hat{\gamma}_i(h) \]  

(3.6)

for \( \upsilon_i \) is consistent when \( r = r_n \to \infty \) in such a way that \( r_n/n \to 0 \). Then, with probability \( 1 - \alpha \),

\[ |\hat{\mu}_i - \mu_i| \leq \Phi_{1-\alpha/(2\nu)} \sqrt{2\pi \hat{f}_i(0)/N_y} \]  

(3.7)

is approximately true for large \( r \) and \( n \). Here \( \Phi_p \) is the \( p \) percentile of a standard normal distribution.

For finite fourth moment case, Anderson and Vecchia (1993) compute the asymptotic distributions of the moment estimates of the periodic autocovariance function \( \gamma_t(\ell) \) and of the periodic autocorrelation function \( \rho_t(\ell) \), where \( \rho_t(\ell) = \gamma_t(\ell)/\{\gamma_t(0)\gamma_{t+\ell}(0)\}^{1/2} \). The joint asymptotic distribution of the estimated periodic autocovariances are given in Theorem 3.2.

**Theorem 3.2** Let \( X_t = \tilde{X}_t - \mu_t \), where \( X_t \) is the periodic moving average process (3.1) and \( \mu_t \) is a periodic mean function with period \( \nu \). Then, for any nonnegative integers \( j \) and \( h \) with \( j \neq h \),

\[ N^{1/2} \begin{pmatrix} \gamma(j) - \gamma(j) \\ \hat{\gamma}(h) - \gamma(h) \end{pmatrix} \to \mathcal{N} \left( 0, \begin{pmatrix} O_{jj} & O_{jh} \\ O_{hj} & O_{hh} \end{pmatrix} \right) \]  

(3.8)

where \( N = N_y\nu \), \( \mathcal{N}(\mu, O) \) denotes the multivariate normal distribution with mean \( \mu \) and covariance matrix \( O \),

\[ \hat{\gamma}(\ell) = [\hat{\gamma}_0(\ell), \hat{\gamma}_1(\ell), \cdots, \hat{\gamma}_{\nu-1}(\ell)]^T \]  

\[ \gamma(\ell) = [\gamma_0(\ell), \gamma_1(\ell), \cdots, \gamma_{\nu-1}(\ell)]^T \]  

(3.9)

and, for \( i, \ell = 0, 1, \ldots, \nu - 1 \),
\[(O_{jh})_{it} = \sum_{n=-\infty}^{\infty} \{\gamma_i(n\nu + \ell - i)\gamma_{i+j}(n\nu + \ell - i - j + h) + \\
\gamma_i(n\nu + \ell - i + h)\gamma_{i+j}(n\nu + \ell - i - j)\} (3.10)\]

\[O_{jh} = \sum_{n=-\infty}^{\infty} \{G_n\Pi^j G_{n+h-j}\Pi^{n-j} + G_{n+h}\Pi^j G_{n-j}\Pi^{n-j}\}\]

where

\[G_n = \text{diag}\{\gamma_0(n), \gamma_1(n), \ldots, \gamma_{\nu-1}(n)\} (3.11)\]

and \(\Pi\) an orthogonal \(\nu \times \nu\) cyclic permutation matrix

\[
\Pi = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\] (3.12)

Note that

\[\Pi^j\text{diag}\{a_0, a_1, \ldots, a_{\nu-1}\}\Pi^{-j} = \text{diag}\{a_j, a_{j+1}, \ldots, a_{j+\nu-1}\} \] (3.13)

where \(\{a_t\}\) is any sequence of constants satisfying \(a_{t+\nu} = a_t\) for all \(t\).

PROOF. See Anderson and Vecchia (1993).

**Theorem 3.3** Under the assumption of Theorem 3.2,

\[
N^{1/2}_y \begin{pmatrix}
\hat{\rho}(j) - \rho(j) \\
\hat{\rho}(h) - \rho(h)
\end{pmatrix} \Rightarrow \mathcal{N} \left( 0, \begin{pmatrix}
W_{jj} & W_{jh} \\
W_{hj} & W_{hh}
\end{pmatrix} \right) \] (3.14)

where \(N = N_y\nu\), \(j\) and \(h\) are positive integers with \(j \neq h\),

\[
\hat{\rho}(\ell) = [\hat{\rho}_0(\ell), \hat{\rho}_1(\ell), \ldots, \hat{\rho}_{\nu-1}(\ell)]^T
\]

\[
\rho(\ell) = [\rho_0(\ell), \rho_1(\ell), \ldots, \rho_{\nu-1}(\ell)]^T
\] (3.15)
and, for $i, \ell = 0, 1, \ldots, \nu - 1$,

$$W_{jh} = \sum_{n=-\infty}^{\infty} \{L_n \Pi^j L_{n+h-j} \Pi^{n-j} + L_{n+h} \Pi^j L_{n-j} \Pi^{n-j} - L_j (I + \Pi^j) L_n L_{n+h} \Pi^n
$$

$$-L_n \Pi^j L_{n-j} \Pi^{n-j} (I + \Pi^{-h}) L_h + \frac{1}{2} L_j (I + \Pi^j) L_n \Pi^n (I + \Pi^{-h}) L_h \}$$

where

$$L_n = \text{diag}\{\rho_0(n), \rho_1(n), \ldots, \rho_{\nu-1}(n)\}$$

and $\Pi$ an orthogonal $\nu \times \nu$ cyclic permutation matrix.

PROOF. See Anderson and Vecchia (1993) for a proof.

For $\nu = 1$, the expression for $W_{jh}$ in (3.16) reduces to

$$W_{jh} = \sum_{n=-\infty}^{\infty} \{\rho(n)\rho(n + h - j) + \rho(n + h)\rho(n - j) - 2\rho(j)\rho(n)\rho(n + h)
$$

$$-2\rho(n)\rho(n - j)\rho(h) + 2\rho(j)\rho(h)\rho^2(n)\}$$

See also Theorem A.15, Appendix A.

Anderson and Meerschaert (2003) also provide the asymptotic distribution for the innovations estimates of the parameters in a periodically stationary series (2.2) with period $\nu \geq 1$ under the classical assumption that the innovations have finite fourth moment.

**Theorem 3.4** Suppose that a periodically stationary moving average (2.2) is causal, invertible, $E\varepsilon_t^4 < \infty$, and that for some $0 < g \leq G < \infty$ we have $g'z \leq z'f(\lambda)z \leq Gz'$, for all $-\pi \leq \lambda \leq \pi$, and for all $z$ in $\mathbb{R}^\nu$, where $f(\lambda)$ is the spectral density matrix of equivalent vector moving average process (3.2). If $k = k(N_y) \to \infty$ as $N_y \to \infty$ with $k^3/N_y \to 0$ and

$$N^{1/2}_y \sum_{j=1}^{\infty} |\pi_\ell(k + j)| \to 0 \quad \text{for } \ell = 0, 1, \ldots, \nu - 1$$

then for any fixed positive integer $D$

$$N^{1/2}_y (\hat{\theta}_{k,u}^{(i-k)} - \psi_i(u) : u = 1, \ldots, D, i = 0, \ldots, \nu - 1) \Rightarrow \mathcal{N}(0, V)$$
where

\[ V = A \text{ diag } (\sigma_0^2 D^{(0)}, \ldots, \sigma_{\nu-1}^2 D^{(\nu-1)}) A' \]  

(3.21)

\[ A = \sum_{n=0}^{D-1} E_n \Pi^{D-n(D+1)} \]  

(3.22)

\[ D^{(i)} = \text{ diag } (\sigma_{i-1}^{-2}, \sigma_{i-2}^{-2}, \ldots, \sigma_{i-D}^{-2}) \]  

(3.23)

\[ E_n = \text{ diag } \{0, \ldots, 0, \psi_0(n), \ldots, \psi_0(n), 0, \ldots, 0, \psi_1(n), \ldots, \psi_1(n), \ldots, 0, \ldots, 0, \psi_{\nu-1}(n), \ldots, \psi_{\nu-1}(n)\} \]  

(3.24)

and \( \Pi \) an orthogonal \( D\nu \times D\nu \) cyclic permutation matrix.

PROOF. See Anderson and Meerschaert (2003) for a proof.

Note that \( \Pi^0 \) is \( D\nu \times D\nu \) identity matrix and \( \Pi^{-\ell} \equiv (\Pi')^\ell \). Matrix multiplication yields the following corollary.

**Corollary 3.5** Regarding Theorem 3.4, in particular, we have that

\[ N_{y}^{1/2}(\hat{\theta}_{k,u} - \psi(u)) \Rightarrow N\left(0, \sum_{n=0}^{u-1} \frac{\sigma_{i-n}^2}{\sigma_{i-u}^2} \psi_i^2(n)\right) \]  

(3.25)

PROOF. See Anderson and Meerschaert (2003) for a proof.

**Remark 3.6** Corollary 3.5 also holds the asymptotic result for second-order stationary process where the period is just \( \nu = 1 \). In this case \( \sigma_i^2 = \sigma^2 \) so (3.25) becomes

\[ N_{y}^{1/2}(\hat{\theta}_{k,u} - \psi(u)) \Rightarrow N\left(0, \sum_{n=0}^{u-1} \psi_i^2(n)\right) \]  

(3.26)

The \( \alpha \)-level test statistic rejects the null hypothesis \( H_0 : \psi_i(u) = 0 \) in favor of the alternative \( (H_a : \psi_i(u) \neq 0) \) if \( |Z| > z_{\alpha/2} \). The \( p \)-value of this test is

\[ p = P(\{|Z| > |z|\}) \quad \text{where} \quad Z \sim N(0, 1) \]  

(3.27)
and

\[ Z = \frac{N_y^{1/2} \hat{\theta}_{k,u}^{(i-k)}}{W}, \quad W^2 = \sum_{n=0}^{u-1} \hat{\psi}_{k,(i-k-n)}^2 \left( \hat{\theta}_{k,n}^{(i-k)} \right)^2 \] (3.28)

The joint asymptotic distribution of the innovation estimates are given in Theorem 3.7.

**Theorem 3.7** average process (2) and \( \mu_t \) is a Let \( \hat{\theta}_{k,t}^{(i-k)} = \hat{\psi}_i(\ell) \), under the assumption of Theorem 3.4, for any nonnegative integers \( j \) and \( h \) with \( j \neq h \),

\[
N_y^{1/2} \begin{pmatrix} \hat{\psi}(j) - \psi(j) \\ \hat{\psi}(h) - \psi(h) \end{pmatrix} \Rightarrow \mathcal{N} \begin{pmatrix} 0, & V_{jj} \\ V_{jh} & V_{hh} \end{pmatrix} \] (3.29)

where \( \mathcal{N}(\mu, V) \) denotes the multivariate normal distribution with mean \( \mu \) and covariance matrix \( V \),

\[
\hat{\psi}(\ell) = [\hat{\psi}_0(\ell), \hat{\psi}_1(\ell), \ldots, \hat{\psi}_{\nu-1}(\ell)]^T \quad (3.30)
\]

\[
\psi(\ell) = [\psi_0(\ell), \psi_1(\ell), \ldots, \psi_{\nu-1}(\ell)]^T
\]

and for \( x = \text{Min}(h, j) \)

\[
V_{jh} = \sum_{n=0}^{x-1} \left\{ F_{j-1-n}^{(\nu-1)(j-1-n)} B_{n+1}^{(\nu-1)(\nu-1)} \right\} \] (3.31)

where

\[
F_n = \text{diag}\{\psi_0(n), \psi_1(n), \ldots, \psi_{\nu-1}(n)\} \quad (3.32)
\]

\[
B_n = \text{diag}\{\sigma_0^2, \sigma_1^2, \ldots, \sigma_{\nu-1}^2\}
\]

and \( \Pi \) an orthogonal \( \nu \times \nu \) cyclic permutation matrix.

**PROOF.** This is simply a rearrangement of rows and columns of the matrix \( V \) in (3.21).

**Vector Difference Equation for the \( \psi \)-weights of the PARMA process**

For a PARMA\(_\nu\)(\( p, q \)) model given by (2.1), we develop a vector difference equation for the \( \psi \)-weights of the PARMA process so that we can determine feasible values of \( p \) and \( q \). For fixed \( p \) and \( q \) (\( p + q = m \); assuming \( m \) statistically significant values of \( \psi_1(j) \)), to determine
the parameters $\phi_t(\ell)$ and $\theta_t(\ell)$, (2.2) is substituted into (2.1) to obtain

$$
\sum_{j=0}^{\infty} \psi_t(j) \varepsilon_{t-j} - \sum_{\ell=1}^{p} \phi_t(\ell) \sum_{j=0}^{\infty} \psi_{t-\ell}(j) \varepsilon_{t-\ell-j} = \varepsilon_t - \sum_{\ell=1}^{q} \theta_t(\ell) \varepsilon_{t-\ell}
$$

(3.33)

and then the coefficients on both sides are equated in order to calculate the $\phi_t(\ell)$ and $\theta_t(\ell)$.

$$
\begin{align*}
\psi_t(0) &= 1 \\
\psi_t(1) - \phi_t(1) \psi_{t-1}(0) &= -\theta_t(1) \\
\psi_t(2) - \phi_t(1) \psi_{t-1}(1) - \phi_t(2) \psi_{t-2}(0) &= -\theta_t(2) \\
&\vdots \\
\psi_t(3) - \phi_t(1) \psi_{t-1}(2) - \phi_t(2) \psi_{t-2}(1) - \phi_t(3) \psi_{t-3}(0) &= -\theta_t(3) \\
&\vdots = \vdots
\end{align*}
$$

(3.34)

where we take $\phi_t(\ell) = 0$ for $\ell > p$, and $\theta_t(\ell) = 0$ for $\ell > q$. Hence the $\psi$-weights satisfy the homogeneous difference equation given by

$$
\begin{align*}
\psi_t(j) - \sum_{k=1}^{p} \phi_t(k) \psi_{t-k}(j-k) &= 0 & j \geq \text{Max}(p, q+1) \\
\psi_t(j) - \sum_{k=1}^{j} \phi_t(k) \psi_{t-k}(j-k) &= -\theta_t(j) & 0 \leq j \leq \text{Max}(p, q+1)
\end{align*}
$$

(3.35)

Defining

$$
\begin{align*}
A_t &= \text{diag}\{\phi_0(\ell), \phi_1(\ell), \ldots, \phi_{\nu-1}(\ell)\} \\
\psi(j) &= (\psi_0(j), \psi_1(j), \ldots, \psi_{\nu-1}(j))^T \\
\theta(j) &= (\theta_0(j), \theta_1(j), \ldots, \theta_{\nu-1}(j))^T
\end{align*}
$$

(3.36)

$$
\psi_{jk}(j - k) = (\psi_{j-k}(j - k), \psi_{j-k+1}(j - k), \ldots, \psi_{j-k+\nu-1}(j - k))^T
$$

then (3.35) becomes the vector difference equation

$$
\begin{align*}
\psi(j) - \sum_{k=1}^{p} A_{jk} \psi_{j-k}(j-k) &= 0 & j \geq \text{Max}(p, q+1) \\
\psi(j) - \sum_{k=1}^{j} A_{jk} \psi_{j-k}(j-k) &= -\theta(j) & 0 \leq j \leq \text{Max}(p, q+1)
\end{align*}
$$

(3.37)

Note that $\psi_t(0) = 1$
We can obtain a difference equation for \( \psi_{j0}(j - 0) = \psi(j) \) by introducing the orthogonal \( \nu \times \nu \) cyclic permutation matrix, and observing that \( \psi_{jk}(j - k) = \Pi^{-k} \psi(j - k) \). So, (3.37) becomes

\[
\begin{align*}
\psi(j) - \sum_{k=1}^{p} A_k \Pi^{-k} \psi(j - k) &= 0 & j \geq \text{Max}(p, q + 1) \\
\psi(j) - \sum_{k=1}^{j} A_k \Pi^{-k} \psi(j - k) &= -\theta(j) & 0 \leq j \leq \text{Max}(p, q + 1)
\end{align*}
\]  

(3.38)

The vector difference equation (3.38) can be helpful for the analysis of higher order PARMA models using matrix algebra.

(a) Periodic Moving Average Processes

The vector difference equation for the PMA\(_\nu\)(q) process given by (3.38) is

\[
\begin{align*}
\psi(j) &= -\theta(j) & 0 \leq j \leq q \\
\psi(j) &= 0 & j > q
\end{align*}
\]  

(3.39)

We see that the innovation estimates of a periodic moving average process of order \( q \) has a “cut-off” at lag \( q \). Therefore, Corollary 3.5 can be directly applied to identify the order of the PMA process.

\[
N_g^{1/2}(\hat{\theta}(j) + \theta(j)) \Rightarrow N\left(0, V_{jj}\right)
\]  

(3.40)

where \( V_{jj} \) is obtained from (3.31).

(b) Periodic Autoregressive Processes

The vector difference equation for the PAR\(_\nu\)(p) process given by (3.38) is

\[
\psi(j) = \sum_{k=1}^{p} A_k \Pi^{-k} \psi(j - k) \quad j \geq p
\]  

(3.41)

For PAR\(_\nu\)(1), \( \psi(1) = A_1 \Pi^{-1} \psi(0) = \phi \) where \( \phi = \{\phi_0, \phi_1, \ldots, \phi_{\nu-1}\}^T \).

(c) PARMA\(_\nu\)(1, 1) Process

For higher order PAR or PARMA models, it is difficult to obtain the explicit solution for \( \phi(\ell) \) and \( \theta(\ell) \), and hence the model identification is a complicated problem. However, we develop herein an asymptotic distribution for PARMA\(_\nu\)(1, 1) model parameters.
Consider a PARMA \( \nu (1,1) \) model

\[
X_t = \phi_t X_{t-1} + \varepsilon_t - \theta_t \varepsilon_{t-1}
\]  
(3.42)

where \( \{\sigma_i^{-1} \varepsilon_{k+i}\} \) form an iid sequences of normal random variables with mean zero and standard deviation one. Then from (3.34) given the innovations coefficients, we can solve for \( \phi_t \) and \( \theta_t \) using the following equations.

\[
\psi_t(0) = 1
\]  
(3.43)

\[
\theta_t = \phi_t - \psi_t(1)
\]  
(3.44)

\[
\psi_t(2) = \phi_t \psi_{t-1}(1)
\]  
(3.45)

We now use Theorem 3.7 to derive the asymptotic distribution of the estimates of the autoregressive as well as the moving average parameters \( (\phi_t, \theta_t) \).

**Proposition 3.8** Suppose that \( X_n \) is \( AN (\mu, c_n^2 \Sigma) \) where \( \Sigma \) is a symmetric non-negative definite matrix and \( c_n \to 0 \) as \( n \to \infty \). If \( g(X) = (g_1(X), \ldots, g_m(X))' \) is a mapping from \( \mathbb{R}^k \) into \( \mathbb{R}^m \) such that each \( g_i(.) \) is continuously differentiable in a neighborhood of \( \mu \), and if \( D \Sigma D' \) has all of its diagonal elements non-zero, where \( D \) is the \( m \times k \) matrix \( [\partial g_i/\partial x_j] (\mu) \), then \( g(X_n) \) is \( AN (g(\mu), c_n^2 D \Sigma D') \)

**PROOF.** See Proposition 6.4.3 of Brockwell and Davis (1991) for a proof.

**Theorem 3.9** Under the assumption of Theorem 3.7, we have

\[
N_{y}^{1/2} \left( \hat{\phi} - \phi \right) \Rightarrow \mathcal{N}(0, Q)
\]  
(3.46)

where

\[
\hat{\phi} = [\hat{\phi}_0, \hat{\phi}_1, \cdots, \hat{\phi}_{\nu-1}]^T
\]

\[
\phi = [\phi_0, \phi_1, \cdots, \phi_{\nu-1}]^T
\]  
(3.47)
and the $\nu \times \nu$ matrix $Q$ is defined by

$$Q = \sum_{k,\ell=1}^{2} H_{\ell} V_{\ell k} H_{k}^{T}$$

(3.48)

where the matrix $H$ is a $\nu \times 2\nu$ matrices of partial derivatives

$$H = (H_{1}, H_{2}) = \left( \frac{\partial \phi_{\ell}}{\partial \psi_{m}(1)}, \frac{\partial \phi_{\ell}}{\partial \psi_{m}(2)} \right) \ell, m = 0, 1, \ldots, \nu - 1$$

(3.49)

$H$ can be expressed using $\nu \times \nu$ permutation matrix as

$$H = (-F_{2}\Pi^{-1}F_{1}^{-2}, \Pi^{-1}F_{1}^{-1}\Pi)$$

(3.50)

and for $\ell, k = 1, 2$, $V_{\ell k}$ is given by

$$V_{11} = B_{1}$$

$$V_{12} = B_{1}\Pi^{-(\nu-1)}F_{1} = B_{1}\Pi F_{1}$$

$$V_{21} = F_{1}\Pi^{-1}B_{1} = F_{1}\Pi^{-1}B_{1}$$

$$V_{22} = B_{2} + F_{1}\Pi^{-1}B_{1}\Pi^{-(\nu-1)}F_{1} = B_{2} + F_{1}\Pi^{-1}B_{1}\Pi F_{1}$$

(3.51)

in which $B_{n}$ and $F_{n}$ are obtained from 3.32.

PROOF. Using Proposition 3.8, together with the relation $\phi_{\ell} = \frac{\psi_{\ell}(2)}{\psi_{\ell+1}(1)}$ and Theorem 3.7 for $j = 1$ and $h = 2$ we have that (3.46) holds with

$$Q = HVH^{T}$$

where

$$\Sigma = V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

$$\mathbf{g}(\mu) = \phi \quad \text{where} \quad \mu = \begin{pmatrix} \psi(1) \\ \psi(2) \end{pmatrix}$$
\[
g \begin{pmatrix}
    \psi_0(1) \\
    \vdots \\
    \psi_{\nu-1}(1) \\
    \psi_0(2) \\
    \vdots \\
    \psi_{\nu-1}(2)
\end{pmatrix}
= 
\begin{pmatrix}
    \phi_0 \\
    \vdots \\
    \phi_{\nu-1}
\end{pmatrix}
= 
\begin{pmatrix}
    \frac{\psi_0(2)}{\psi_{\nu-1}(1)} \\
    \vdots \\
    \frac{\psi_{\nu-1}(2)}{\psi_{\nu-2}(1)}
\end{pmatrix}
= 
\begin{pmatrix}
    g_0 \\
    \vdots \\
    g_{\nu-1}
\end{pmatrix}
\]

and \( g(X_n) = \hat{\phi} \) where \( X_n = \begin{pmatrix} \hat{\psi}(1) \\ \hat{\psi}(2) \end{pmatrix} \)

Then,
\[
D = H = 
\begin{pmatrix}
\frac{\partial \phi_0}{\partial \psi_0(1)} & \cdots & \frac{\partial \phi_0}{\partial \psi_{\nu-1}(1)} & \frac{\partial \phi_0}{\partial \psi_0(2)} & \cdots & \frac{\partial \phi_0}{\partial \psi_{\nu-1}(2)} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \phi_{\nu-1}}{\partial \psi_{\nu-1}(1)} & \cdots & \frac{\partial \phi_{\nu-1}}{\partial \psi_{\nu-1}(1)} & \frac{\partial \phi_{\nu-1}}{\partial \psi_0(2)} & \cdots & \frac{\partial \phi_{\nu-1}}{\partial \psi_{\nu-1}(2)}
\end{pmatrix}
= 
\begin{pmatrix}
\left[ \frac{\partial \phi_\ell}{\partial \psi_m(1)} \right]_{\ell,m=0,\ldots,\nu-1} & \left[ \frac{\partial \phi_\ell}{\partial \psi_m(2)} \right]_{\ell,m=0,\ldots,\nu-1}
\end{pmatrix}
= 
\begin{pmatrix} H_1 & H_2 \end{pmatrix}
\]
same as (3.49)

Therefore,
\[
Q = HVH^T
= 
\begin{pmatrix} H_1 & H_2 \end{pmatrix}
\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}
\begin{pmatrix} H_1^T \\ H_2^T \end{pmatrix}
\]
\[
= 
\begin{pmatrix} H_1V_{11} + H_2V_{21} & H_1V_{12} + H_2V_{22} \end{pmatrix}
\begin{pmatrix} H_1^T \\ H_2^T \end{pmatrix}
\]
\[
= H_1V_{11}H_1^T + H_2V_{21}H_1^T + H_1V_{12}H_2^T + H_2V_{22}H_2^T
\]
\[
= \sum_{k,\ell=1}^{2} H_{\ell}V_{\ell k}H_{k}^T
\]
same as (3.48)

where \( V_{\ell k} \) is from (3.31).
For $\phi_i = \frac{\psi_i(2)}{\psi_{i-1}(1)}$, we obtain

$$H_1 = \begin{pmatrix}
0 & 0 & \ldots & 0 & \frac{-\psi_0(2)}{\psi_{i-1}(1)} \\
-\frac{\psi_1(2)}{\psi_0(1)} & 0 & \ldots & 0 & 0 \\
: & : & \ldots & : & : \\
0 & 0 & \ldots & \frac{\psi_{i-2}(2)}{\psi_{i-3}(1)} & 0
\end{pmatrix} = -F_1 \Pi^{-1} F_1^{-2} \quad \text{same as (3.50)}$$

and

$$H_2 = \begin{pmatrix}
\frac{1}{\psi_{i-1}(1)} & 0 & \ldots & 0 \\
0 & \frac{1}{\psi_0(1)} & \ldots & 0 \\
: & : & \ldots & : \\
0 & 0 & \ldots & \frac{1}{\psi_{i-2}(1)}
\end{pmatrix} = \Pi^{-1} F_1^{-1} \Pi \quad \text{same as (3.50)}$$

From (3.32)

$$B_1 = \begin{pmatrix}
\frac{\sigma^2_0}{\sigma^2_{i-1}} & 0 & \ldots & 0 \\
0 & \frac{\sigma^2_1}{\sigma^2_0} & \ldots & 0 \\
: & : & \ldots & : \\
0 & 0 & \ldots & \frac{\sigma^2_{i-2}}{\sigma^2_{i-3}}
\end{pmatrix} \quad F_1 = \begin{pmatrix}
\psi_0(1) & 0 & \ldots & 0 \\
0 & \psi_1(1) & \ldots & 0 \\
: & : & \ldots & : \\
0 & 0 & \ldots & \psi_{i-1}(1)
\end{pmatrix}$$

and

$$B_2 = \begin{pmatrix}
\frac{\sigma^2_0}{\sigma^2_{i-2}} & 0 & \ldots & 0 \\
0 & \frac{\sigma^2_1}{\sigma^2_{i-1}} & \ldots & 0 \\
: & : & \ldots & : \\
0 & 0 & \ldots & \frac{\sigma^2_{i-3}}{\sigma^2_{i-4}}
\end{pmatrix}$$

Using (3.31), we obtain (3.51) and then we can compute

$$V_{11} = B_1 = \begin{pmatrix}
\frac{\sigma^2_0}{\sigma^2_{i-1}} & 0 & \ldots & 0 \\
0 & \frac{\sigma^2_1}{\sigma^2_0} & \ldots & 0 \\
: & : & \ldots & : \\
0 & 0 & \ldots & \frac{\sigma^2_{i-2}}{\sigma^2_{i-3}}
\end{pmatrix}$$
\[ V_{12} = B_1 F_1 = \begin{pmatrix} 0 & \psi_1(1) \sigma_2^2 \sigma_{\nu-1} & 0 & \ldots & 0 \\ 0 & 0 & \psi_2(1) \sigma_2^2 \sigma_0^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_0(1) \sigma_{\nu-1}^2 \sigma_{\nu-2}^2 & 0 & 0 & \ldots & 0 \end{pmatrix} \]

\[ V_{21} = F_1 \Pi^{-1} B_1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & \psi_0(1) \sigma_{\nu-1}^2 \\ \psi_1(1) \sigma_2^2 \sigma_{\nu-1}^2 & 0 & 0 & \ldots & 0 \\ 0 & \psi_2(1) \sigma_2^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_0(1) \sigma_{\nu-1}^2 \sigma_{\nu-2}^2 & 0 & 0 & \ldots & 0 \end{pmatrix} \]

\[ V_{22} = B_2 + F_1 \Pi^{-1} B_1 F_1 \]

Corollary 3.10 Regarding Theorem 3.9, in particular, we have that

\[ N_{y}^{1/2} \left( \hat{\phi}_i - \phi_i \right) \Rightarrow N \left( 0, w^2_{\phi_i} \right) \quad (3.52) \]

where

\[ w^2_{\phi_i} = \psi_{i-1}^{-4} \left\{ \psi_1^2(2) \sigma_{i-2}^2 \sigma_{i-1}^2 \left( 1 - \frac{2 \psi_1(1) \psi_{i-1}(1)}{\psi_1(2)} \right) + \psi_{i-1}^2(1) \sigma_{i-2}^{-2} \sum_{n=0}^{1} \sigma_{i-n}^2 \psi_1^2(n) \right\} \quad (3.53) \]

PROOF. Using the formulas from the proof of Theorem 3.9, we obtain

\[ H_1 V_{11} H_1^T = \begin{pmatrix} \sigma_{i-1}^2 \psi_1^2(2) & 0 & \ldots & 0 \\ \sigma_{i-2}^2 \psi_1^2(2) \psi_{i-1}^2(1) & \sigma_{i-1}^2 \psi_1^2(2) & \ldots & 0 \\ 0 & \sigma_{i-2}^2 \psi_1^2(2) \psi_{i-1}^2(1) & \sigma_{i-2}^2 \psi_1^2(2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{i-1}^2 \psi_1^2(2) & 0 & \ldots & \sigma_{i-2}^2 \psi_1^2(2) \psi_{i-1}^2(1) \end{pmatrix} \]

\(i^{th}\) diagonal element is \(\frac{\sigma_{i-1}^2 \psi_1^2(2)}{\sigma_{i-2}^2 \psi_1^2(2) \psi_{i-1}^2(1)}\)
Corollary 3. Theorem 3.11

Under the assumption of Theorem 3.7, we have

\[ N_y^{1/2} \left( \hat{\theta} - \theta \right) \Rightarrow \mathcal{N}(0, S) \] (3.54)
where

\[ \hat{\theta} = [\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_{\nu-1}]^T \]

\[ \theta = [\theta_0, \theta_1, \ldots, \theta_{\nu-1}]^T \]

and the \(\nu \times \nu\) matrix \(S\) is defined by

\[ S = \sum_{k,\ell=1}^{2} M_{\ell} V_{tk} M_{k}^T \]

(3.55)

where \(V_{tk}\) is given in (3.51) and \(M\) is a \(\nu \times 2\nu\) matrix of partial derivatives

\[ M = (M_1, M_2) = \left( \frac{\partial \theta_\ell}{\partial \psi_m(1)}, \frac{\partial \theta_\ell}{\partial \psi_m(2)} \right) \ell, m = 0, 1, \ldots, \nu - 1 \]

(3.56)

\(M\) can be expressed using \(\nu \times \nu\) permutation matrix (22) as

\[ M = (-I - F_2 \Pi^{-1} F_1^{-2}, \Pi^{-1} F_1^{-1} \Pi) . \]

(3.57)

PROOF. Using proposition 3.8, the relation \(\theta_j = \psi_{j-1}(2) - \psi_j(1)\), and Theorem 3.7 for \(j = 1\) and \(h = 2\), we have that (3.55) holds with

\[ S = MVM^T \]

where

\[ g(\mu) = \theta \text{ where } \mu = \begin{pmatrix} \psi(1) \\ \psi(2) \end{pmatrix} \]

\[ g = \begin{pmatrix} \psi_0(1) \\ \vdots \\ \psi_{\nu-1}(1) \\ \psi_0(2) \\ \vdots \\ \psi_{\nu-1}(2) \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \vdots \\ \theta_{\nu-1} \end{pmatrix} = \begin{pmatrix} \frac{\psi_{\nu}(2)}{\psi_{\nu-1}(1)} - \psi_0(1) \\ \vdots \\ \frac{\psi_{\nu-1}(2)}{\psi_{\nu-2}(1)} - \psi_{\nu-1}(1) \end{pmatrix} = \begin{pmatrix} g_0 \\ \vdots \\ g_{\nu-1} \end{pmatrix} \]

and \( g(X_n) = \hat{\theta} \text{ where } X_n = \begin{pmatrix} \hat{\psi}(1) \\ \hat{\psi}(2) \end{pmatrix} \)
Following the same procedure as in the proof of Theorem 3.9, we obtain

\[
M = \left( \begin{bmatrix} \frac{\partial \ell}{\partial \theta_{\circ \circ \, m(1)}}_{\ell,m=0,\ldots,\nu-1} & \frac{\partial \ell}{\partial \psi_{\circ \circ \, m(2)}}_{\ell,m=0,\ldots,\nu-1} \end{bmatrix} \right)
\]

\[
= \begin{pmatrix} M_1 & M_2 \end{pmatrix} \quad \text{same as (3.56)}
\]

Therefore,

\[
S = MVMT
\]

\[
= \begin{pmatrix} M_1 & M_2 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} M_1^T \\ M_2^T \end{pmatrix}
\]

\[
= \begin{pmatrix} M_1 V_{11} + M_2 V_{21} & M_1 V_{12} + M_2 V_{22} \end{pmatrix} \begin{pmatrix} M_1^T \\ M_2^T \end{pmatrix}
\]

\[
= M_1 V_{11} M_1^T + M_2 V_{21} M_1^T + M_1 V_{12} M_2^T + M_2 V_{22} M_2^T
\]

\[
= \sum_{k,\ell=1}^{2} M_{\ell} V_{tk} M_{k}^T \quad \text{same as (3.55)}
\]

where \( V_{tk} \) is from (3.31).

A comparison to the proof of Theorem 3.9 shows that \( M_1 = -I + H_1 \) and \( M_2 = H_2 \) and then (3.57) follows.

**Corollary 3.12** Regarding Theorem 3.11, in particular, we have that

\[
N_y^{1/2} \left( \hat{\theta}_i - \theta_i \right) \Rightarrow \mathcal{N}(0, w_{\theta_i}^2)
\]

(3.58)

where

\[
w_{\theta_i}^2 = \psi_i^{-4}(1) \left\{ \psi_i^2(2) \sigma_i^{-2} \sigma_{i-1}^{-2} \left( 1 - \frac{2\psi_i(1)\psi_{i-1}(1)}{\psi_i(2)} \right) + \sum_{j=1}^{2} \psi_{i-j}(1) \sigma_{i-j}^{-2} \sum_{n=0}^{j-1} \sigma_{i-n}^2 \psi_i^2(n) \right\}
\]

**PROOF.** A comparison to the proof of Theorem 3.9 shows that \( S = Q + \bar{S} \) where

\[
\bar{S} = V_{11} - H_1 V_1 - V_{11} H_1^T - H_2 V_{21} - V_{12} H_2^T
\]

and the \( i^{th} \) diagonal element of \( \bar{S} \) is therefore

\[
\frac{\sigma_i^2}{\sigma_{i-1}^2} = \psi_i^{-4}(1) \left\{ \frac{\psi_i^4(1)\sigma_i^2}{\sigma_{i-1}^2} \right\}
\]
and hence the \( i^{th} \) diagonal element of \( S \) is
\[
= \psi_{i-1}^{-4}(1) \left\{ \frac{\psi_{i-1}^2(2)}{\sigma_{i-2}^2} - \frac{2\psi_i(2)\psi_i(1)\psi_{i-1}(1)\sigma_{i-1}^2}{\sigma_{i-2}^2} + \frac{\psi_i^2(1)\psi_{i-1}(1)\sigma_{i-1}^2}{\sigma_{i-2}^2} + \frac{\psi_{i-1}(1)\sigma_{i}^2}{\sigma_{i-1}^2} \right\}.
\]

Corollary 3.12 is useful for finding approximate large-sample confidence interval for \( \theta \). An \( \alpha \)-level confidence interval for \( \theta \) is \((\hat{\theta}_i - z_{\alpha/2}N_y^{-1/2}w_{\theta_i}, \hat{\theta}_i + z_{\alpha/2}N_y^{-1/2}w_{\theta_i})\), where \( Z \sim \mathcal{N}(0, 1) \).

Note that non-overlapping confidence intervals give statistically significant conclusions.

### 3.3 Simulation Study

A detailed simulation study was conducted to investigate the practical utility of the innovations algorithm for model identification in the presence of seasonally correlated data. Data for several different PARMA\(_p\)\(_q\)\(_\nu\) models with finite fourth moment were simulated.

For each model, individual realizations of \( N_y = 50, 100, 300, \) and 500 years of data were simulated and the innovations algorithm was used to obtain parameter estimates for each realization. In each case, estimates were obtained for \( k = 10, k = 15 \) and \( k = 20 \) iterations of the innovations algorithm in order to examine the convergence, and \( p \)-values were computed using (3.27) to identify those estimates that were statistically significant (\( p < 0.05 \)).

A FORTRAN program was used to simulate the PARMA samples as well as to make all the necessary calculations. Some general conclusions can be drawn from this study. We found that 10 to 20 iterations of the innovations algorithm are usually sufficient to obtain reasonable estimates of the model parameters. We also found that \( N_y = 50 \) years of monthly or quarterly data give only rough estimates of the model parameters, while \( N_y = 100 \) years generally is enough to give good estimates. For the data between 50 and 100 years, the estimates are less accurate but generally adequate for practical applications.

In order to illustrate the general quality of those results, we present the results from two
of the models. The first model is a Gaussian PARMA$_4(0,1)$ model:

\[ X_{kv+i} = \varepsilon_{kv+i} + \theta_i \varepsilon_{kv+i-1} \]  

(3.59)

where \( \{ \delta_{kv+i} = \sigma_i^{-1} \varepsilon_{kv+i} \} \) is an iid sequence of normal random variables with mean zero and standard deviation one. The periodic notation \( X_{kv+i} \) refers to the (mean zero) simulated data for season \( i \) of year \( k \). From the above model, a single realization with \( N_y = 500 \) years of quarterly data (sample size of \( N = N_y \nu = 500 \cdot 4 = 2000 \)) was generated. Table 3.1 shows the results after \( k = 15 \) iterations of the innovations algorithm.

Table 3.1: Moving average parameter estimates and \( p \)-values after \( k = 15 \) iterations of the innovations algorithm applied to \( N_y = 500 \) years of simulated PARMA$_4(0,1)$ data.

<table>
<thead>
<tr>
<th>Lag ( \ell )</th>
<th>( \hat{\psi}_0(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_1(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_2(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_3(\ell) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.265</td>
<td>.00</td>
<td>0.640</td>
<td>.00</td>
<td>0.918</td>
<td>.00</td>
<td>0.348</td>
<td>.00</td>
</tr>
<tr>
<td>2</td>
<td>-0.104</td>
<td>.24</td>
<td>-0.026</td>
<td>.70</td>
<td>-0.058</td>
<td>.50</td>
<td>-0.065</td>
<td>.03</td>
</tr>
<tr>
<td>3</td>
<td>-0.015</td>
<td>.52</td>
<td>0.132</td>
<td>.46</td>
<td>-0.066</td>
<td>.29</td>
<td>-0.090</td>
<td>.15</td>
</tr>
<tr>
<td>4</td>
<td>0.003</td>
<td>.95</td>
<td>0.063</td>
<td>.18</td>
<td>0.149</td>
<td>.36</td>
<td>-0.071</td>
<td>.12</td>
</tr>
<tr>
<td>5</td>
<td>0.010</td>
<td>.76</td>
<td>0.049</td>
<td>.61</td>
<td>0.056</td>
<td>.19</td>
<td>0.118</td>
<td>.31</td>
</tr>
<tr>
<td>6</td>
<td>-0.015</td>
<td>.87</td>
<td>-0.113</td>
<td>.10</td>
<td>0.037</td>
<td>.67</td>
<td>-0.003</td>
<td>.92</td>
</tr>
<tr>
<td>7</td>
<td>0.013</td>
<td>.57</td>
<td>-0.202</td>
<td>.26</td>
<td>-0.115</td>
<td>.07</td>
<td>-0.091</td>
<td>.15</td>
</tr>
<tr>
<td>8</td>
<td>-0.009</td>
<td>.84</td>
<td>0.042</td>
<td>.37</td>
<td>-0.131</td>
<td>.42</td>
<td>-0.054</td>
<td>.23</td>
</tr>
<tr>
<td>9</td>
<td>-0.033</td>
<td>.34</td>
<td>0.071</td>
<td>.47</td>
<td>0.053</td>
<td>.21</td>
<td>0.304</td>
<td>.01</td>
</tr>
</tbody>
</table>

The innovation estimates in Table 3.1 cut-off at lag 1 is significantly different from zero, implying that a PARMA$_4(0,1)$ model is adequate to fit the simulated data, which agrees
with the model used to simulate the data. The parameter estimates for this model, obtained from the relationship (3.39), are summarized in Table 3.2.

Table 3.2: Model parameters and estimates for simulated PARMA$_4(0, 1)$ data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.265</td>
<td>0.640</td>
<td>0.918</td>
<td>0.348</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.854</td>
<td>1.753</td>
<td>0.458</td>
<td>1.182</td>
</tr>
</tbody>
</table>

The model selected above needs to be checked in order to verify whether it complies with certain assumptions about the model, and to verify how well it represents the generated time series. The model assumptions to be checked are usually the independence and normality of residuals of the model. For PARMA$_4(0, 1)$, the standardized residuals can be calculated using the following relationship:

$$\hat{\delta}_t = \frac{X_t + \sum_{j=1}^{\infty} (-1)^j \hat{\theta}_t \hat{\theta}_{t-1} \ldots \hat{\theta}_{t-j+1} X_{t-j}}{\sqrt{\hat{\sigma}^2_t}}$$

(3.60)

Inspection of the time series plot of the residuals (see Figure 3.1) indicates no obvious pattern or trend. A normal probability plot of the residuals (see Figure 3.2) also supports the approximate normality of residuals.

The Anderson-Darling (AD) test rejects $H_0$ (the data follows a normal distribution) if the test statistic is greater than the critical value (or $p < 0.05$). Since the critical value 0.752 is greater than the test statistic 0.627 (and the $p$-value 0.102 is larger than the critical value 0.05), we conclude that there is no reason to reject the hypothesis of normality.
The plots of the sample autocorrelation functions (ACF)\textsuperscript{10} and partial autocorrelation functions (PACF)\textsuperscript{11} for the residuals are shown in Figure 3.3. All values of ACF and PACF fall inside or close to the 95% confidence bounds ($\pm 1.96/\sqrt{N}$) indicating iid sequence.

In summary, these plots show that the residuals look like iid normal sequence with mean zero and variance close to one, which agrees with the assumption made at the beginning of this section (see also Figure C-1, Appendix C for $N_y = 50, 100, 300$ years).

\textsuperscript{10}See Section A-1, Appendix A.

\textsuperscript{11}See Section A-1, Appendix A.
Figure 3.3: The sample ACF and PACF for the residuals, showing the 95% confidence bounds 
\( \pm 1.96/\sqrt{N} \), indicating iid sequence.

The second model for which simulation result will be presented is a PARMA\(_4\)(1, 1) model

\[
X_{k\nu+i} = \phi_i X_{k\nu+i-1} + \varepsilon_{k\nu+i} + \theta_i \varepsilon_{k\nu+i-1}
\]

(3.61)

where \( \{\delta_{k\nu+i} = \sigma_i^{-1} \varepsilon_{k\nu+i}\} \) is an iid sequence of normal random variables with mean zero and standard deviation one. The periodic notation \( X_{k\nu+i} \) refers to the (mean zero) simulated data for season \( i \) of year \( k \). From the above model, a single realization with \( N_y = 500 \) years of quarterly data (sample size of \( N = N_y \nu = 500 \cdot 4 = 2000 \)) was generated.

Table 3.3 shows the results after \( k = 15 \) iterations of the innovations algorithm. For season 0 the first four lags are statistically significant, for season 1 and 3 the first three lags are significant, while for season 2 only the first two are significant. Since parameter estimates do not generally cut-off to (statistically) zero at a certain lag, it is advantageous to seek a parsimonious mixed moving average/autoregressive model. The PARMA\(_\nu\)(1, 1) model is the simplest mixed model, and thus is preferred so long as diagnostic plots of the residual autocorrelation (ACF) and/or partial autocorrelation (PACF) indicate no significant serial dependence. The parameter estimates for this model, obtained using equations (3.44) and
Table 3.3: Moving average parameter estimates and \( p \)-values after \( k = 15 \) iterations of the innovations algorithm applied to \( N_y = 500 \) years of simulated PARMA\(_{4}(1,1) \) data.

<table>
<thead>
<tr>
<th>Lag ( \ell )</th>
<th>( \hat{\psi}_0(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_1(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_2(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_3(\ell) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.606 .00</td>
<td></td>
<td>1.231 .00</td>
<td></td>
<td>1.710 .00</td>
<td></td>
<td>0.617 .00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.620 .00</td>
<td></td>
<td>-0.360 .00</td>
<td></td>
<td>1.047 .00</td>
<td></td>
<td>0.387 .00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.329 .00</td>
<td></td>
<td>-0.250 .20</td>
<td></td>
<td>-0.346 .00</td>
<td></td>
<td>0.329 .00</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.322 .00</td>
<td></td>
<td>-0.037 .48</td>
<td></td>
<td>-0.120 .69</td>
<td></td>
<td>-0.009 .87</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.041 .40</td>
<td></td>
<td>-0.055 .62</td>
<td></td>
<td>0.058 .47</td>
<td></td>
<td>0.087 .52</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.004 .98</td>
<td></td>
<td>0.136 .08</td>
<td></td>
<td>0.128 .46</td>
<td></td>
<td>0.001 .98</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.017 .61</td>
<td></td>
<td>0.148 .45</td>
<td></td>
<td>0.187 .11</td>
<td></td>
<td>0.086 .27</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.024 .74</td>
<td></td>
<td>0.042 .42</td>
<td></td>
<td>0.199 .50</td>
<td></td>
<td>-0.003 .95</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.017 .72</td>
<td></td>
<td>-0.006 .96</td>
<td></td>
<td>0.049 .55</td>
<td></td>
<td>0.034 .80</td>
<td></td>
</tr>
</tbody>
</table>

(3.45), are summarized in Table 3.4. It must be noted that \( \theta_t \) in (3.44) must be replaced with \( -\theta_t \) so as to be consistent with the representation in (2.1).

Residuals for this PARMA\(_{4}(1,1) \) model can be computed using the equation

\[
\hat{\delta}_t = \hat{\sigma}^{-1}_t \left\{ X_t - \left( \hat{\phi}_t + \hat{\theta}_t \right) X_{t-1} + \sum_{j=2}^{\infty} (-1)^j \left( \hat{\phi}_{t-j+1} + \hat{\theta}_{t-j+1} \right) \hat{\theta}_t \hat{\theta}_{t-1} \ldots \hat{\theta}_{t-j+2} X_{t-j} \right\} \quad (3.62)
\]

which was obtained by solving (2.1) for the innovations and substituting the estimated model parameters for their true values. The analysis of residuals also indicates that they are iid normal sequence with mean zero and variance close to one, which agrees with the assumption in (3.61). For more complex mixed PARMA models, Akaike’s information criterion (AIC) could also be a useful guide for the selection process (see Vecchia, 1985b; Brockwell and
Table 3.4: Model parameters and estimates (with 95% confidence interval) for simulated PARMA_{1}(1, 1) data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta)</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>(\hat{\theta})</td>
<td>0.400 ± 0.265</td>
<td>0.636 ± 0.264</td>
<td>0.859 ± 0.127</td>
<td>0.391 ± 0.204</td>
</tr>
<tr>
<td>(\phi)</td>
<td>-0.90</td>
<td>0.50</td>
<td>0.80</td>
<td>0.25</td>
</tr>
<tr>
<td>(\hat{\phi})</td>
<td>-1.005 ± 0.259</td>
<td>0.595 ± 0.201</td>
<td>0.850 ± 0.125</td>
<td>0.226 ± 0.032</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
<tr>
<td>(\hat{\sigma})</td>
<td>0.832</td>
<td>1.771</td>
<td>0.482</td>
<td>1.215</td>
</tr>
</tbody>
</table>

3.4 Application to Modeling of Natural River Flows

Next we model a monthly river flow time series from the Fraser River at Hope, British Columbia. The Fraser River is the longest river in BC, travelling almost 1400 km and sustained by a drainage area covering 220,000 km\(^2\). It rises in the Rocky Mts., at Yellowhead Pass, near the British Columbia-Alta. line and flows northwest through the Rocky Mt. Trench to Prince George, thence south and west to the Strait of Georgia at Vancouver. Its main tributaries are the Nechako, Quesnel, Chilcotin, and Thompson rivers.

The data are obtained from daily discharge measurements, in cubic meter per second, averaged over each of the respective months to obtain the monthly series. The series contains 72 years of data from October 1912 to September 1984. In the following analysis, \(\nu = 0\) corresponds to October and \(\nu = 11\) corresponds to September. A partial plot of the original data, given Figure 3.4, shows the cyclic behavior of the monthly flows. The sample
mean, standard deviation and autocorrelations at lag 1 and lag 2 are given in Table 3.5 (see also Figure 3.11). The nonstationarity of the series is apparent since the mean, standard deviation and correlation functions vary significantly from month to month. For example, the confidence intervals for the means are non-overlapping, indicating statically significant difference. Removing the periodicity in mean and variance will not yield a stationary series. Therefore, a periodically stationary time series model is appropriate. After \( k = 20 \) iterations, the innovations algorithm yields the innovations estimates and associated \( p \)-values found in Table 3.6.

![Figure 3.4: Average monthly flows (cms) for the Fraser River at Hope, BC, indicate a seasonal pattern.](image)

Since the \( \hat{\psi}_i \) weights do not generally cut-off to (statistically) zero at a certain lag, we choose a parsimonious mixed model that captures the periodic behavior as well as the exponential decay evidenced in the autocorrelation function. We find that a PARMA\(_{12}(1,1)\) model

\[
X_{kv+i} - \phi_i X_{kv+i-1} = \varepsilon_{kv+i} + \theta_i \varepsilon_{kv+i-1}
\]  

(3.63)

is sufficient in adequately capturing the series autocorrelation structure. The Akaike’s information criterion (AIC) or physical basis of the river flow process could also be helpful
Table 3.5: Sample mean, standard deviation and autocorrelation at lag 1 and 2 of average monthly flow series for the Fraser River at Hope, BC.

<table>
<thead>
<tr>
<th>month</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\rho}(1)$</th>
<th>$\hat{\rho}(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCT</td>
<td>69763 ± 11542</td>
<td>19997</td>
<td>0.688</td>
<td>0.517</td>
</tr>
<tr>
<td>NOV</td>
<td>56000 ± 11497</td>
<td>17698</td>
<td>0.731</td>
<td>0.581</td>
</tr>
<tr>
<td>DEC</td>
<td>40352 ± 7987</td>
<td>12817</td>
<td>0.715</td>
<td>0.531</td>
</tr>
<tr>
<td>JAN</td>
<td>33135 ± 7420</td>
<td>9252</td>
<td>0.787</td>
<td>0.691</td>
</tr>
<tr>
<td>FEB</td>
<td>30861 ± 7271</td>
<td>8845</td>
<td>0.779</td>
<td>0.385</td>
</tr>
<tr>
<td>MAR</td>
<td>29709 ± 6492</td>
<td>8834</td>
<td>0.510</td>
<td>0.224</td>
</tr>
<tr>
<td>APR</td>
<td>59293 ± 5766</td>
<td>20268</td>
<td>0.302</td>
<td>-0.294</td>
</tr>
<tr>
<td>MAY</td>
<td>171907 ±15616</td>
<td>40200</td>
<td>0.272</td>
<td>-0.047</td>
</tr>
<tr>
<td>JUN</td>
<td>248728 ±22226</td>
<td>45120</td>
<td>0.563</td>
<td>0.496</td>
</tr>
<tr>
<td>JUL</td>
<td>199118 ±20410</td>
<td>42543</td>
<td>0.779</td>
<td>0.462</td>
</tr>
<tr>
<td>AUG</td>
<td>127157 ±14149</td>
<td>28070</td>
<td>0.718</td>
<td>0.320</td>
</tr>
<tr>
<td>SEP</td>
<td>86552 ± 11354</td>
<td>20052</td>
<td>0.635</td>
<td>0.454</td>
</tr>
</tbody>
</table>

in choosing the appropriate model (Vecchia, 1985b; Salas et al., 1981,1992; Brockwell and Davis, 1991). The parameter estimates for this model, obtained using equations (3.44) and (3.45), are summarized in Table 3.7. The confidence intervals are narrower and non-overlapping, indicating strong statistical significance, which can also verified by small $p$-values (not shown). Model residuals were estimated using equation (3.62). It must be noted that $\theta_t$ in (3.44) must be replaced with $-\theta_t$ so as to be consistent with the representation in (2.1). Although the model is periodically stationary, the standard residuals should be
Table 3.6: Moving average parameter estimates $\hat{\psi}_i(\ell)$ at season $i$ and lag $\ell = 1, 2, \ldots, 6$, and $p$-values, after $k = 20$ iterations of the innovations algorithm applied to average monthly flow series for the Fraser River at Hope, BC.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{\psi}_i(1)$</th>
<th>$p$</th>
<th>$\hat{\psi}_i(2)$</th>
<th>$p$</th>
<th>$\hat{\psi}_i(3)$</th>
<th>$p$</th>
<th>$\hat{\psi}_i(4)$</th>
<th>$p$</th>
<th>$\hat{\psi}_i(5)$</th>
<th>$p$</th>
<th>$\hat{\psi}_i(6)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.885 .00</td>
<td></td>
<td>0.134 .28</td>
<td></td>
<td>0.105 .10</td>
<td></td>
<td>0.163 .01</td>
<td></td>
<td>0.006 .93</td>
<td></td>
<td>0.038 .78</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.625 .00</td>
<td></td>
<td>0.625 .00</td>
<td></td>
<td>0.085 .46</td>
<td></td>
<td>0.140 .02</td>
<td></td>
<td>0.077 .17</td>
<td></td>
<td>-0.004 .94</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.508 .00</td>
<td></td>
<td>0.350 .00</td>
<td></td>
<td>0.419 .00</td>
<td></td>
<td>0.032 .72</td>
<td></td>
<td>0.097 .03</td>
<td></td>
<td>0.019 .65</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.515 .00</td>
<td></td>
<td>0.287 .00</td>
<td></td>
<td>0.140 .07</td>
<td></td>
<td>0.239 .00</td>
<td></td>
<td>0.034 .60</td>
<td></td>
<td>0.030 .37</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.791 .00</td>
<td></td>
<td>0.165 .10</td>
<td></td>
<td>0.295 .00</td>
<td></td>
<td>0.112 .12</td>
<td></td>
<td>0.160 .03</td>
<td></td>
<td>0.045 .43</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.567 .00</td>
<td></td>
<td>0.757 .00</td>
<td></td>
<td>0.057 .61</td>
<td></td>
<td>0.250 .00</td>
<td></td>
<td>0.062 .40</td>
<td></td>
<td>0.139 .06</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.076 .01</td>
<td></td>
<td>0.711 .11</td>
<td></td>
<td>0.856 .01</td>
<td></td>
<td>0.415 .13</td>
<td></td>
<td>0.241 .17</td>
<td></td>
<td>0.112 .52</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.522 .03</td>
<td></td>
<td>0.684 .41</td>
<td></td>
<td>0.988 .28</td>
<td></td>
<td>1.095 .09</td>
<td></td>
<td>0.350 .51</td>
<td></td>
<td>0.198 .56</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.451 .00</td>
<td></td>
<td>-1.014 .00</td>
<td></td>
<td>-0.062 .66</td>
<td></td>
<td>-0.745 .50</td>
<td></td>
<td>0.128 .87</td>
<td></td>
<td>-0.635 .31</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.618 .00</td>
<td></td>
<td>-0.041 .77</td>
<td></td>
<td>-0.746 .01</td>
<td></td>
<td>-1.083 .26</td>
<td></td>
<td>-0.047 .97</td>
<td></td>
<td>0.514 .50</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.448 .00</td>
<td></td>
<td>0.409 .00</td>
<td></td>
<td>0.026 .78</td>
<td></td>
<td>-0.241 .20</td>
<td></td>
<td>-1.125 .08</td>
<td></td>
<td>0.799 .26</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.677 .00</td>
<td></td>
<td>0.159 .01</td>
<td></td>
<td>0.194 .00</td>
<td></td>
<td>0.050 .46</td>
<td></td>
<td>-0.190 .17</td>
<td></td>
<td>-0.402 .38</td>
<td></td>
</tr>
</tbody>
</table>

stationary, so the standard $1.96/\sqrt{N}$ 95% confidence limits still apply. Figure 3.6 shows the ACF and PACF of the model residuals. Although a few values lie slightly outside of the 95% confidence bands, there is no apparent pattern, providing some evidence that the PARMA$_{12}(1, 1)$ model is adequate.

One reason for carefully modeling the river flow time series is to develop the ability to generate synthetic river flows for further analysis. This requires a realistic distributional model for the residuals that can be used to simulate the innovations. After exploring a num-
ber of possible distributions, we found that a three-parameter lognormal fits the residuals fairly well. A histogram of the residuals showing the best fitting lognormal density curve (scale = 0.217, location = 1.656 and threshold = −5.363), as well as the corresponding probability plot, are shown in Figure 3.7. On the probability plot, points along the diagonal line (model percentiles equal data percentiles) indicate a good fit. According to this lognormal model, residuals follow the distribution of a random variable $R = −5.363 + e^{(1.656+0.217Z)}$ where $Z \sim \mathcal{N}(0, 1)$.

The histogram in Figure 3.7 (b) shows that the three parameter lognormal gives an
Table 3.7: Parameter estimates (with 95% confidence interval) for PARMA model (3.63) of average monthly flow series for the Fraser River at Hope, BC.

<table>
<thead>
<tr>
<th>month</th>
<th>$\hat{\phi}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCT</td>
<td>0.198 ± 0.319</td>
<td>0.687 ± 0.392</td>
<td>11875.479</td>
</tr>
<tr>
<td>NOV</td>
<td>0.568 ± 0.251</td>
<td>0.056 ± 0.337</td>
<td>11598.254</td>
</tr>
<tr>
<td>DEC</td>
<td>0.560 ± 0.228</td>
<td>-0.052 ± 0.271</td>
<td>7311.452</td>
</tr>
<tr>
<td>JAN</td>
<td>0.565 ± 0.233</td>
<td>-0.050 ± 0.299</td>
<td>5940.845</td>
</tr>
<tr>
<td>FEB</td>
<td>0.321 ± 0.307</td>
<td>0.470 ± 0.347</td>
<td>4160.214</td>
</tr>
<tr>
<td>MAR</td>
<td>0.956 ± 0.240</td>
<td>-0.389 ± 0.351</td>
<td>4610.209</td>
</tr>
<tr>
<td>APR</td>
<td>1.254 ± 1.494</td>
<td>-0.178 ± 1.677</td>
<td>15232.867</td>
</tr>
<tr>
<td>MAY</td>
<td>0.636 ± 1.451</td>
<td>-0.114 ± 1.526</td>
<td>31114.514</td>
</tr>
<tr>
<td>JUN</td>
<td>-1.942 ± 2.362</td>
<td>2.393 ± 2.374</td>
<td>32824.370</td>
</tr>
<tr>
<td>JUL</td>
<td>-0.092 ± 0.621</td>
<td>0.710 ± 0.655</td>
<td>29712.190</td>
</tr>
<tr>
<td>AUG</td>
<td>0.662 ± 0.191</td>
<td>-0.213 ± 0.226</td>
<td>15511.187</td>
</tr>
<tr>
<td>SEP</td>
<td>0.355 ± 0.227</td>
<td>0.322 ± 0.289</td>
<td>12077.991</td>
</tr>
</tbody>
</table>

acceptable overall fit, but the probability plot in Figure 3.7 (a) reveals a lack of fit at both tails. This is important for practical applications, since tail behavior of the residuals (or the innovations) determines the extreme values of the times series, which govern both droughts and floods. None of the standard probability plots we tried (normal, lognormal, Weibull, gamma, etc.) gave an adequate fit at the tails. To check for a power law probability tail we constructed a Mandelbrot plot of each tail (Figure 3.8) as described in Mandelbrot (1963) and Anderson and Meerschaert (1998). Suppose that $X_1, \ldots, X_n$ are iid Pareto with distribution function $F(x) = Cx^{-\alpha}$. Then $F(x) = P[X > x] = Cx^{-\alpha}$ and so $\ln F(x) = \ln C - \alpha \ln x$. 
Sorting the data in decreasing order so that $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$ (order statistics) we should have approximately that $x = X_{(r)}$ when $F(x) = r/n$. Then a plot of $\ln X_{(r)}$ versus $\ln(r/n)$ should be approximately linear with slope $-\alpha$. In Figure 3.8, the downward curve indicates that a simple power law model is not appropriate. However, the shape of the plots is consistent with many examples of truncated Pareto distributions found in the geophysics literature (see for example Aban, Meerschaert and Panorska, 2004; Burroughs and Tebbens, 2001a,2001b,2002). This distribution is appropriate when a power law model is affected by an upper bound on the observations. In hydrology it is commonly believed that there is an upper bound on precipitation and therefore river flows (see, for example, Maidment, 1993). A truncated Pareto random variable $X$ has distribution function

$$F_X(x) = P(X \leq x) = \frac{1 - (\gamma/x)^\alpha}{1 - (\gamma/\beta)^\alpha} \quad (3.64)$$

and density

$$f_X(x) = \frac{\alpha \gamma^\alpha x^{-\alpha-1}}{1 - (\gamma/\beta)^\alpha} \quad (3.65)$$

with $0 < \gamma \leq x \leq \beta < \infty$ and $\gamma < \beta$. Aban, Meerschaert and Panorska (2004) develop maximum likelihood estimators (MLE) for the parameters of the truncated Pareto distribu-
tion. When a truncated Pareto is fit to the tail of the data, the parameters are estimated by obtaining the conditional maximum likelihood estimate based on the largest order statistics, representing only the portion of the tail where the truncated Pareto model holds.

**Theorem 3.13** When $X(r) > X(r+1)$, the conditional maximum likelihood estimator for the parameters of the upper-truncated Pareto in (3.64) based on the $r + 1$ largest order statistics is given by

\[
\hat{\beta} = X(1) \quad (3.66)
\]

\[
\hat{\gamma} = r^{1/\hat{\alpha}} X(r+1) \left[ n - (n - r)(X(r+1)/X(1))^{\hat{\alpha}} \right]^{-1/\hat{\alpha}} \quad (3.67)
\]

and $\hat{\alpha}$ solves the equation

\[
\frac{r}{\hat{\alpha}} + \frac{r(X(r+1)/X(1))^{\hat{\alpha}} \ln(X(r+1)/X(1))}{1 - (X(r+1)/X(1))^{\hat{\alpha}}} - \sum_{i=1}^{r} \left[ \ln X(i) - \ln X(r+1) \right] = 0 \quad (3.68)
\]

The truncated Pareto distribution given by (3.64) with $0 < \gamma < \infty$ reduces to the Pareto distribution in the limit $\beta = \infty$. An asymptotic $q$-level test (see Aban et al., 2004) is used to distinguish whether a given data set is more appropriately modelled by a Pareto or truncated Pareto distribution. The asymptotic level $q$ test ($0 < q < 1$) rejects the null hypothesis $H_0 : \beta = \infty$ (Pareto) in favor of the alternative $H_a : \beta < \infty$ (truncated Pareto) if $X(1) < x_q$ where $q = \text{Exp} \left( -nC x_q^{-\alpha} \right)$. This test rejects $H_0$ if and only if

\[
X(1) < \left( \frac{nC}{-\ln q} \right)^{1/\alpha} \quad (3.69)
\]

and the $p$-value of this test is given by

\[
p = \text{Exp} \left( -nC X^{-\alpha}_{(1)} \right) \quad (3.70)
\]

Hill’s estimator\(^{12}\) (4.12) gives $\hat{\alpha} = 2.450$ and $\hat{C} = 0.282$ for the upper tail ($r = 39$ residuals), and $\hat{\alpha} = 4.184$ and $\hat{C} = 0.348$ for the lower tail ($r = 34$ residuals). Using (3.70),

---

\(^{12}\)See Section 4.4 for a detailed discussion.
the $p$-values are 0.018 and 0.041, respectively. The small $p$-values as well as the visual evidence of the probability plot in Figure 3.8 suggests that the Pareto model is not a good fit for the tails of the estimated residuals. Therefore, we decided to fit a truncated Pareto to roughly the upper 5% and the lower 5% of the residuals. Then we computed $-1.697 = \text{the 5th percentile}$ and $2.122 = \text{the 95th percentile}$ of the three parameter lognormal distribution we fit to the body of the residuals. Next we determined that $r = 39$ residuals exceed the 95th percentile, and $r = 34$ residuals fall below the 5th percentile. Then the MLE from Theorem 3.13 was used to estimate the parameters $(\hat{\beta} = 5.336, \hat{\gamma} = 0.072, \hat{\alpha} = 0.722)$ of the best fitting truncated Pareto distribution, and the theoretical distribution tail $P(R > r)$ was plotted over the 39 largest positive residuals in Figure 3.8(left). In Figure 3.8 (right), we used the same method to fit a truncated Pareto $(\hat{\beta} = 2.961, \hat{\gamma} = 0.291, \hat{\alpha} = 1.560)$ to the 34 largest negative residuals, after a change of sign. Both of the plots in Figures 3.8 indicate an adequate fit.

Figure 3.8: Log-log plot of upper(left) and lower(right) residual tails and fitted Pareto and truncated Pareto distributions, Fraser river at Hope, BC.

A mixture distribution with lognormal body and truncated Pareto tails was used to
simulate the innovations. The mixture has cumulative distribution function (cdf)

\[ P(\delta \leq r) = \begin{cases} F_-(r) & \text{if } r < -1.697 \\ F_0(r) & \text{if } -1.697 \leq r \leq 2.122 \\ F_+(r) & \text{if } r > 2.122 \end{cases} \]  

where \( F_0 \) is the cdf of the lognormal, and \( F_+, F_- \) are truncated Pareto cdfs of the positive and negative tails, respectively. The truncated Pareto distributions were slightly shifted (by \( s = 0.172 \) on the positive tail and \( s = 0.174 \) on the negative tail) to make the mixture cdf continuous. Now innovations could be simulated by the inverse cumulative distribution function method \( \delta = F^{-1}(U) \) where \( U \) is a pseudorandom number uniformly distributed on the unit interval \((0, 1)\). However, this is impractical in the present case since the lognormal cdf is not analytically invertible. Instead, we used the Box-Müller method\(^\text{13}\) to generate standard normal random variates \( Z \) (see Gentle, 2003). Then lognormal random variates were calculated using \( \delta = -5.363 + \exp(1.656 + 0.217Z) \). If \( R > 2.122 \), the 95th percentile of the lognormal, we generated another uniform \((0, 1)\) random variate \( U \) and substituted \( \delta = F_+^{-1}(0.95 + 0.05U) \). If \( R < -1.697 \), the 5th percentile of the lognormal, we substituted \( \delta = F_-^{-1}(0.05U) \). This gives simulated innovations \( \delta \) with the mixture distribution (3.71).

Figure 3.9 shows a probability plot for \( N = N_y \nu \) simulated innovations (for \( \nu = 12 \) months and \( N_y = 100 \) years) from the mixture distribution (3.71). Comparison with Figure 3.7 (a) shows that the simulated innovations are statistically identical to the computed model residuals in terms of distribution. Substituting the simulated innovations into the model (3.63) generates \( N_y \) years of simulated river flow. It is advantageous to simulate several extra years of river flows and throw out the initial years (100 years in this case), since we did not simulate \( X_t \) for \( t < 0 \). This ensures that the simulated series is periodically stationary.

\(^{13}\)See Section A-4, Appendix A.
Figure 3.9: Probability plot of simulated innovations using the mixed three parameter log-normal and truncated Pareto distributions. Compare Figure 3.7 (a).

(see Figure 3.10). Figure 3.11 shows the main statistical characteristics (mean, standard deviation and autocorrelations) of a typical synthetic river flow time series obtained by this method, as well as the same statistical measures for the observed time series. It is apparent that this procedure closely reproduces the main statistical characteristics, indicating that the modeling procedure is trustworthy for generating synthetic river flows. Such synthetic river flows are useful for design of hydraulic structures, for optimal operation of reservoir systems, for calculating the risk of failure of water supply systems, etc.

Figure 3.10: Plot of (a) observed and (b) simulated monthly river flows for the Fraser River at Hope, BC, indicating similarity.
Figure 3.11: Comparison of mean, standard deviation, and autocorrelations for simulated vs. observed monthly river flow data for the Fraser River at Hope, BC.
4 Modeling River Flows with Heavy Tails

4.1 General

Many river flow time series exhibit occasional sharp spikes (heavy tail characteristics). A model that captures these heavy tail characteristics is, therefore, important in adequately describing the series. Typically, a time series with heavy tails is appropriately transformed so that the normal asymptotics apply. The normal distribution is widely applicable because of the central limit theorem, which states that the sum of a large number of independent, identically distributed variables from a finite-variance distribution will tend to be normally distributed. An alternative model (based on stable asymptotics) allows a more faithful representation of the river flow without preliminary transformation. In this chapter, we fit a periodic ARMA model to river flow data without moment assumptions (Anderson and Meerschaert, 1997, 1998), and model the innovations process using stable distributions or a mixture of stable with appropriate tail distributions. Although there are other heavy tailed alternatives to a normal law, stable distributions have been proposed herein mainly because of the generalized central limit theorem, which states that stable laws are the only possible limit distributions for properly normalized and centered sums of independent, identically distributed random variables.


In this chapter we will discuss the application of heavy tails models to river flow data. As an application, we consider the average monthly flow of the Salt River near Roosevelt, Arizona.

4.2 Definition of Heavy Tails

In the literature, different definitions of heavy tailed distributions exist (see, for example, Werner and Upper, 2002). We say that a probability distribution function has heavy tails if the tails of the distribution diminish \( P (|X| > x) \rightarrow 0 \) at an algebraic rate (like some power of \( x \)) rather than at an exponential rate. In this case some of the moments of this probability distribution will fail to exist. The \( k \)th moment of a probability distribution function \( F(x) \) with density function \( f(x) \) is defined by \( \mu_k = \int x^k f(x) dx \). The mean \( \mu_1 \) and the variance \( \sigma^2 \) are related to the first two moments by familiar equations \( \mu = \mu_1 \) and the variance \( \sigma^2 = \mu_2 - \mu^2_1 \). Examples of probability distributions with heavy tails are stable (for instance, Cauchy and Lévy distribution), Student’s \( t \), and Pareto distributions.

In the Gaussian case, \( X \sim \mathcal{N}(0,1) \), there is a well known formula to approximate the tail probability: as \( x \rightarrow \infty \),

\[
P(X > x) \sim \frac{e^{-x^2}}{x \sqrt{2\pi}}
\]  

(4.1)

(see Feller, 1971).
If $X$, for example, is standard Cauchy, then

$$F(x) = P[X \leq x] = \frac{1}{2} + \frac{1}{\pi} \arctan(x) \quad \text{and} \quad f(x) = \frac{dF(x)}{dx} = \frac{1}{\pi(1 + x^2)} \quad (4.2)$$

Although the bell-shaped of the density of the Cauchy distribution appears to be similar to that of normal law, the tails are heavier, see Table 4.1. While the density of the normal law diminishes at an exponential rate, for the Cauchy we have $f(x) \sim \pi^{-1}x^{-2}$ as $|x| \to \infty$. This causes the integral defining the $k$th moment to diverge when $k \geq 1$, and hence the mean and standard deviations of the Cauchy are undefined. Figure 4.1 shows a plot of Normal, Cauchy and Lévy densities.

Table 4.1: Comparison of tail probabilities for standard normal, Cauchy and Lévy distributions.

<table>
<thead>
<tr>
<th>c</th>
<th>Normal</th>
<th>Cauchy</th>
<th>Lévy</th>
</tr>
</thead>
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<tr>
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<td>0.5000</td>
<td>1.0000</td>
</tr>
<tr>
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<td>0.1587</td>
<td>0.2500</td>
<td>0.6827</td>
</tr>
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<td>0.1476</td>
<td>0.5205</td>
</tr>
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<td>3</td>
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<td>0.1024</td>
<td>0.4363</td>
</tr>
<tr>
<td>4</td>
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<td>0.3829</td>
</tr>
<tr>
<td>5</td>
<td>0.0000002866</td>
<td>0.0628</td>
<td>0.3453</td>
</tr>
</tbody>
</table>

### 4.3 Heavy Tailed Time Series Model

Suppose that $\{\delta_t\}$ are independent, identically distributed random variables with common distribution function $F(x) = P[\delta_t \leq x]$. Regular variation is a technical condition that is required for the generalized central limit theorem to apply.
Definition 4.1 A Borel measurable function $R : \mathbb{R}^+ \to \mathbb{R}^+$ said to be regularly varying at infinity with index $\rho \in \mathbb{R}$ if

$$\lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} = \lambda^\rho \text{ for all } \lambda > 0.$$  

(4.3)

$R$ is slowly varying iff $\rho = 0$ in (4.3).

For example, the following are some regularly varying functions on $\mathbb{R}^+$.

- Constant functions are slowly varying, and therefore regularly varying.
- If $f \sim g$ (meaning that $f(x)/g(x) \to 1$ as $x \to \infty$) and $f$ is regularly varying, then $g$ is regularly varying with the same index.
- $f(x) = ax^b$ is regularly varying with index $b$.

We say that the distribution $F(x)$ belongs to the domain of attraction of some nondegenerate random variable $Y$ with distribution $G(x)$ if there exist real constants $a_n > 0$ and $b_n$ such that

$$a_n^{-1}(\delta_1 + \ldots + \delta_n - nb_n) \Rightarrow Y$$

(4.4)
where $\Rightarrow$ indicates convergence in distribution. The generalize central limit theorem states that (4.4) holds with $Y$ normal if and only if the truncated second moment function

$$
\mu(x) = \int_{|u| \leq x} u^2 dF(u)
$$

is slowly varying; (4.4) holds with $Y$ nonnormal if and only if the tail function $T(x) = P[|\delta_t| > x] = F(-x) + 1 - F(x)$ varies regularly with index $-\alpha$ for some $0 < \alpha < 2$ and the tails satisfy the balancing condition

$$
\lim_{x \to \infty} \frac{1 - F(x)}{T(x)} = p \quad \text{and} \quad \lim_{x \to \infty} \frac{F(-x)}{T(x)} = q
$$

for some $0 \leq p, q \leq 1$ with $p + q = 1$; see, for example, Feller (1971). The norming constants can be chosen to satisfy $nP[|\delta_t| > a_nx] \to C$ and are always of the form $a_n = n^{1/\alpha}\ell_n$, where $\ell_n$ is slowly varying. The domain of normal attraction, in which we assume that $a_n = n^{1/\alpha}$ is strictly smaller class of distributions. If $\alpha > 1$, then the mean $E\delta_t$ exists, and we can take $b_n = E\delta_t$. If $\alpha < 1$, the mean fails to exist, the norming constant $a_n \to \infty$ faster than $n \to \infty$, and we can let $b_n = 0$. The Pareto distribution $F(x) = 1 - Cx^{-\alpha}$ ($x \geq C^{1/\alpha}$) as well as the type II extreme value distribution $F(x) = \exp(-Cx^{-\alpha})$ ($x \geq 0$) belong to the stable domain of attraction when $0 < \alpha < 2$. Sums of iid random variables with these distributions are approximately stable. Stable distributions are rich class of distributions that include the Gaussian and Cauchy distributions in a family that allow skewness and heavy tails. The class was characterized by Paul Lévy (1925) in his study of normalized sums of iid terms.

The general stable distribution is described by four parameters: an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta$, a scale parameter $\gamma$ and a location parameter $\eta$ (see Nolan, 1999). We will use $\gamma$ for the scale parameter and $\eta$ for the location parameter to avoid confusion with the symbols $\sigma$ and $\mu$, which are used exclusively for the standard deviation.
and mean. The parametrization most often used now (see Samorodnitsky and Taqqu, 1994) is the following: \( Y \sim S1(\alpha, \beta, \gamma, \eta) \) if the characteristic function of \( Y \) is given by

\[
E\{\exp(itY)\} = \begin{cases} 
\exp\{\gamma\alpha|t|^{\alpha}(1 - i\beta\frac{\text{sign}(t)\tan\frac{\pi}{2}}{2}) + i\eta t\} & \alpha \neq 1 \\
\exp\{-\gamma|t|(1 + i\beta\frac{\text{sign}(t)\ln|t|}{\pi}) + i\eta t\} & \alpha = 1
\end{cases}
\]

(4.7)

where \( 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \gamma > 0, \) and \( \eta \in \mathbb{R} \). For a more detailed discussion of stable distributions and their properties, the reader is referred to Nolan (1999). If we choose the norming constant \( a_n \) in (4.4) to satisfy \( nP[|\delta_t| > a_n x] \to C \), then the stable limit \( Y \) will have a scale factor satisfying \( \gamma^\alpha = CT(1 - \alpha)\cos(\pi\alpha/2) \) when \( \alpha \neq 1 \) and \( \gamma = C\pi/2 \) when \( \alpha = 1 \) (Theorem 7.3.5 on p.265 of Meerschaert and Scheffler, 2001). The parameter \( C \) is called the dispersion and is sometimes used as an alternative scale parameter for stable laws.

The parameters in the balance equations (4.6) are related to the skewness of the limit \( Y \) by \( p = (1 + \beta)/2 \) and \( q = (1 - \beta)/2 \). If \( \alpha > 1 \) and \( b_n = E\delta_t \) or \( \alpha < 1 \) and \( b_n = 0 \), then the location parameter of the limit \( Y \) in (4.4) is \( \eta = 0 \). If \( \alpha \leq 1 \), then \( EY \) does not exist, but if \( \alpha > 1 \), then \( EY = \eta \), and, in particular, we obtain \( EY = 0 \) when we center to zero expectation. If \( Y_n \) are iid stable with the same distribution as \( Y \), then \( \sum k_n Y_n \) is also stable with the same stable index and skewness parameters. The location parameter of \( \sum k_n Y_n \) is \( \gamma(\sum |k_n|^\alpha)^{1/\alpha} \), and so, in particular, the scale parameter of \( kY \) is \( k\gamma \). The dispersion of \( \sum k_n Y_n \) is \( C \sum |k_n|^\alpha \).

Suppose that the iid sequence \( \{\delta_t\} \) represents the innovations process of a time series (2.1). If \( \delta_t \) has finite fourth moment, then normal asymptotics apply (see Chapter 3), but if \( \delta_t \) has an infinite fourth moment, the asymptotics are governed by stable laws. Infinite fourth moments occur when the tail function \( T(x) = P[|\delta_t| > x] \) varies regularly with index \(-\alpha \) for some \( 0 < \alpha < 4 \). Then \( T(x) \to 0 \) about as fast as \( x^{-\alpha} \to 0 \) as \( x \to \infty \). See Anderson
and Meerschaert (1998) for a review of literature in heavy tail modeling in hydrology.

Suppose that \( \{\delta_t\} \) are iid and that their common distribution \( F(x) \) has a regularly varying tails with index \(-\alpha\) for some \( 2 < \alpha < 4 \). Then \( \sigma_t^2 = E\varepsilon_t^2 < \infty \), and the autocovariance and autocorrelation of the infinite moving average (3.1) at season \( i \) and lag \( \ell \) are

\[
\gamma_i(\ell) = \sum_j \sigma_t \psi_t(j) \sigma_{t+\ell} \psi_{t+\ell}(j + \ell) \\
\rho_i(\ell) = \frac{\gamma_i(\ell)}{\sqrt{\gamma_i(0)\gamma_{i+\ell}(0)}} = \frac{\sum_j \psi_t(j) \psi_{t+\ell}(j + \ell)}{\sqrt{\sum_j \psi_t(j)^2 \sum_j \psi_{t+\ell}(j)^2}}
\]

(4.8)

Given a sample size of \( N = N_y \nu \), the sample statistics are obtained using equations (2.11), (2.12) and (2.13).

We will say that the iid sequence \( \{\delta_t\} \) is \( RV(\alpha) \) if \( P[|\delta_t| > x] \) varies regularly with index \(-\alpha\) and \( P[\delta_t > x]/P[|\delta_t| > x] \to p \) for some \( p \in [0, 1] \). If \( \delta_t \) is \( RV(\alpha) \), then the equivalent vector process \( Z_t \) in (3.2) has iid components with regularly varying tails, and we will also say \( Z_t \) is \( RV(\alpha) \). If \( \alpha > 2 \), then \( Z_t \) belongs to the domain of attraction of a multivariate normal law whose components are iid univariate normal. Loretan and Philips (1994) find that the price fluctuations of currency exchange rates and stock prices often follow an \( RV(\alpha) \) model with \( 2 < \alpha < 4 \). In this case, \( E\delta_t^2 < \infty \), but \( E\delta_t^4 = \infty \). Since the innovation \( \varepsilon_t \) has a finite variance, the sample autocorrelations for the stationary moving average model are asymptotically normal; see for example, Brockwell and Davis (1991, Proposition 7.3.8). The following results show that when \( 2 < \alpha < 4 \), the sample autocorrelations of the periodic moving average model are asymptotically stable.

**Theorem 4.2** Suppose that \( X_t \) is periodic moving average (3.1) of an \( RV(\alpha) \) sequence \( \delta_t = \sigma_t^{-1}\varepsilon_t \) with mean zero, variance one, and \( 2 < \alpha < 4 \). Then \( \sigma_t^2 = E\varepsilon_t^2 < \infty \), \( \hat{\mu}_i, \hat{\gamma}_i(\ell), \hat{\rho}_i(\ell) \)
are consistent estimators of $\mu_i, \gamma_i(\ell), \rho_i(\ell)$, respectively, and

$$N_y^{-1/2}(\hat{\mu} - \mu) \Rightarrow N(0, C) \quad (4.9)$$

where $C = \left(\sum_j \Psi_j \right) \left(\sum_j \Psi_j \right)'$; for some $a_{N_y} \to \infty$ we have

$$N_y a_{N_y}^{-2} [\hat{\gamma}_i(\ell) - \gamma_i(\ell)] \Rightarrow C_{it} = \sum_{r=0}^{\nu-1} C_r(i, \ell) S_r$$

$$N_y a_{N_y}^{-2} [\hat{\rho}_i(\ell) - \rho_i(\ell)] \Rightarrow D_{it} = \sum_{r=0}^{\nu-1} D_r(i, \ell) S_r \quad (4.10)$$

jointly in $i = 0, \ldots, \nu - 1$ and $\ell = 0, \ldots, h$, where

$$D_r(i, \ell) = \frac{\rho_i(\ell)}{\gamma_i(\ell)} C_r(i, \ell) - \frac{\rho_i(\ell)}{2\gamma_i(0)} C_r(i, 0) - \frac{\rho_i(\ell)}{2\gamma_{i+\ell}(0)} C_r(i + \ell, 0) \quad (4.11)$$

and $C_r(i, \ell) = \sum_j \sigma_i \psi_j (jv + i - r) \sigma_{i+\ell} \psi_{i+\ell} (jv + i + \ell - r)$. We can always choose $a_N$ so that $N_y P[|\delta_1| > a_{N_y}] \to C$ for some $C > 0$. Then $S_0, S_1, \ldots, S_{\nu-1}$ are iid $\alpha/2$ stable with mean zero, skewness 1, and dispersion $C$; $C_{it}$ is $\alpha/2$ with mean zero, skewness 1, and dispersion $C \sum_r |C_r(i, \ell)|^{\alpha/2}$; and $D_{it}$ is $\alpha/2$ stable with mean zero, skewness 1, and dispersion $C \sum_r |D_r(i, \ell)|^{\alpha/2}$.


### 4.4 Checking Heavy Tails in Time Series Data

When $\alpha \in (0, 2)$, we have stable distributions with $f(x) \sim C_0 x^{-\alpha - 1}$ as $x \to -\infty$ and $f(x) \sim C_0 x^{-\alpha - 1}$ as $x \to +\infty$, where $C > 0$ and $0 \leq p, q \leq 1$ with $p + q = 1$. Therefore, it is reasonable to model a stable law as having Pareto tails. In this section we consider in detail the problem of detecting heavy tails in time series data and estimating the tail parameter $\alpha$. We illustrate the problem with a data set representing monthly flows of the Salt River near Roosevelt, Arizona.
For data analysis applications, the actual distribution is often unknown, so robust estimators are needed to fit the tail parameter $\alpha$ of an unknown heavy tail distribution. Many robust estimators of the tail index have been proposed in the past several years. Hill (1975) proposed a robust estimator based on the asymptotics of extreme values. Hill calculated that

$$\hat{H} = r^{-1} \sum_{i=1}^{r} \ln X_{(i)} - \ln X_{(r+1)}$$

(4.12)

is the conditional maximum likelihood estimator $1/\alpha$ conditional on $X_{(r+1)} \geq D$, where $X_{(1)} \geq X_{(2)} \geq \ldots$ are the order statistics of a random sample $X_1 \ldots X_n$. We can approximate $\alpha$ by $\hat{\alpha}_r = 1/\hat{H}_r$.

The Hill’s estimator for $C$ is given by

$$\hat{C} = \frac{r + 1}{n} X_{(r+1)}^{\hat{\alpha}_r}$$

(4.13)

For data which are approximately Pareto in the tails, one should choose $r$ small enough so that only the Pareto-like tail is represented.

Resnick and Stărică (1995) show that Hill’s estimation procedure yields a consistent estimator of the tail index $\alpha$ for stationary moving average models where the innovations have regularly varying probability tails with index $-\alpha$. Their result can be understood by noting that since the largest observations in a heavy tail time series model tend to be widely spaced in time, they resemble iid observations. Hill’s estimator can be applied to a wide variety of distributions, such as the stable and type II extreme value distributions, whose tails are approximately Pareto. Thus, for example, applying Hill estimator to the largest $r = 15$ order statistics of the Salt River data, we obtain $\hat{\alpha} = 2.925$ and $C = 1.40 \times 10^9$ (see Table 1.1). This indicate that the Salt River data have heavy tails with infinite fourth moment and finite
variance. Figure 4.2a shows the plot of $\hat{\alpha}_r$ versus $r$. Hall (1982) shows that Hill’s estimator is consistent and asymptotically normal with variance approximately $\sigma^2 = \alpha^2/r$. For $r = 15$ and $\hat{\alpha} = 2.925$, we obtain $\sigma = 0.755$, and the $z$ test of $H_0 : \alpha = 4$ versus $H_0 : \alpha < 4$ or $H_0 : \alpha = 2$ versus $H_a : \alpha > 2$ has $p$-value 0.08 and 0.11, respectively, indicating that the null hypothesis is rejected. Increasing $r$ yields a smaller standard deviation, but Hall’s theorem assumes that $r/n \to 0$ as $n \to \infty$, so that one should base Hill’s estimator on a vanishingly small percentage of the data.

The simplest and most straightforward method of estimating the tail index is to plot the

![Figure 4.2: (a) Hill’s estimator, $\hat{\alpha}$ versus $r$, and (b) log-log plot for the upper tail of the monthly flows for the Salt River near Roosevelt, AZ.](image)

Figure 4.2: (a) Hill’s estimator, $\hat{\alpha}$ versus $r$, and (b) log-log plot for the upper tail of the monthly flows for the Salt River near Roosevelt, AZ.
right tail of the empirical cdf on a log-log scale (see, Figure 3.8, Section 3.4).

Figure 4.2 (b) uses a log-log plot to illustrate the heavy tail distribution of the Salt river data. It shows the Salt River data along with the line that fits a Pareto model with parameters $\alpha = 2.92$ and $C = 1.40 \times 10^9$ (from Hill’s estimator for $r=20$). The graph indicates that the true $\alpha$ lies between 2.85 and 3.0. This method is very sensitive to the size of the sample and the choice of the number of observations used in the regression.

Other tail estimators have also been proposed, see for example Dekkers et al. (1989), Kratz and Resnick (1996) and Wei (1995), but Hill’s estimator continues to be the most popular in practice. McCulloch (1997), Resnick (1997) and Fofack and Nolan (1999) observed that Hill’s estimator performs poorly for stable data with $1.5 < \alpha < 2$. Though Hill’s estimator is scale invariant, it is not shift invariant. Aban and Meerschaert (2001) modified Hill’s estimator to make it shift invariant.

Meerschaert and Scheffler (1998) introduced another robust estimator (hereafter the MS estimator) based on the asymptotics of the sample variance.

Suppose that $\{X_i\}$ is a sequence of $n$ iid random variables that are in the domain of attraction of some stable random variable with $\alpha < 2$. The MS estimator, given in Meerschaert and Scheffler (1998) is

$$
\hat{\beta}_n = \frac{\gamma + \ln \left( \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right)}{2(\gamma + \ln n)}
$$

where $\ln^+ (x) = \max(\ln (x), 0)$, $\bar{X}$ is the sample mean and $\gamma = 0.5772$. $\hat{\beta}_n$ is the MS estimator for $1/\alpha$. This quadratic estimator performs well in many cases where Hill’s estimator is ineffective, such as stable distributions with tail index $1.5 < \alpha < 2$. The quadratic estimator is shift invariant, but not scale invariant. In practical applications, one has to manually scale the data before applying this estimator in order to get useful results. Bianchi and Meerschaert (2004) applied two modifications of the quadratic estimator that makes it both
shift and scale variant.

When $0 < \alpha < 2$, the above formula (4.14) yields a consistent estimator for $1/\alpha$ for a broad range of time series models, including periodic moving averages, whose innovations have regularly varying tails with index $\alpha$. If $\alpha > 2$, then the estimator $\hat{\gamma} \to \frac{1}{2}$ in probability as $n \to \infty$. Since we believe that $2 < \alpha < 4$, we apply the estimator to the squared data, which has tail index $\alpha/2$. The MS estimator is unbiased when the data are Pareto-like with $C = 1$. For Pareto this can be accomplished by dividing by $C^{1/\alpha}$, so we divide the raw data by $C^{1/\alpha} = 1338$, obtained from Hill’s estimator with $r = 15$. Then we applied the above estimator of $\alpha$ and doubled the result (since the index of the squared data is $\alpha/2$) to get 2.623 as the MS estimate of $\hat{\alpha}$ for the raw Salt River data.

These tail estimations indicates that $2 < \alpha < 4$, so the probability distribution of $\tilde{X}_t$ has heavy tails, with a finite variance but infinite fourth moment.

### 4.5 Application to Modeling of Natural River Flows

In this section we illustrate the application of heavy tail time series methods by fitting a periodic ARMA model to the Salt River flow data. Let $\tilde{X}_t$ denote the average flow rate in cubic feet per second $t$ months after October 1914 of the Salt River near Roosevelt, Arizona.

Preliminary estimates of the tail index in the previous section indicate that $2 < \alpha < 4$, so the probability distribution of $\tilde{X}_t$ has heavy tails, with a finite variance but infinite fourth moment. A partial plot of the original data is given in Figure 4.3, clearly shows the cyclic behavior of the monthly flows. There are occasional sharp spikes in the time series plot of the data (see Figure 4.3), which indicates the heavy tail characteristics of the data.

The sample mean, standard deviation and autocorrelations at lag 1 and lag 2 are given in Table 4.2. The nonstationarity of the series is apparent since the mean, standard deviation
and correlation functions vary widely from month to month. Removing the periodicity in mean and variance will not yield a stationary series. Therefore, a periodically stationary series model is appropriate.

In the following analysis, \( i = 0 \) corresponds to October and \( i = 11 \) corresponds to September. Since \( 2 < \alpha < 4 \), Theorem 2.2 is applicable here. After \( k = 15 \) iterations, the innovations algorithm yields the innovations estimates found in Table 4.3. Following the same procedure as in Section 3.4, we selected a PARMA\(_{12}(1, 1)\) model to the Salt River flow data.

The parameter estimates for this model, obtained using equations (3.44) and (3.45), are summarized in Table 4.4. It must be noted that \( \theta_t \) in (3.44) must be replaced with \(-\theta_t\) so as to be consistent with the representation in (2.1). Model residuals were estimated using equation (3.62). Figure 4.4 shows the ACF and PACF of the model residuals. Although a few values lie slightly outside of the 95% confidence bands, there is no apparent pattern, providing some evidence that the PARMA\(_{12}(1, 1)\) model is adequate.

One reason for carefully modeling the river flow time series is to develop the ability to generate synthetic river flows for further analysis. This requires a realistic distributional
Table 4.2: Sample mean, standard deviation and autocorrelation at lag 1 and 2 of average monthly flow series for the Salt River near Roosevelt, AZ.

<table>
<thead>
<tr>
<th>month</th>
<th>ˆµ</th>
<th>ˆσ</th>
<th>ˆρ(1)</th>
<th>ˆρ(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCT</td>
<td>464.43</td>
<td>796.38</td>
<td>0.315</td>
<td>0.173</td>
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<tr>
<td>NOV</td>
<td>373.96</td>
<td>361.90</td>
<td>0.740</td>
<td>0.230</td>
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<td>DEC</td>
<td>802.15</td>
<td>1289.14</td>
<td>0.273</td>
<td>0.336</td>
</tr>
<tr>
<td>JAN</td>
<td>1012.64</td>
<td>2055.94</td>
<td>0.354</td>
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</tr>
<tr>
<td>FEB</td>
<td>1372.51</td>
<td>1792.97</td>
<td>0.486</td>
<td>0.479</td>
</tr>
<tr>
<td>MAR</td>
<td>1910.04</td>
<td>1800.47</td>
<td>0.619</td>
<td>0.562</td>
</tr>
<tr>
<td>APR</td>
<td>2107.85</td>
<td>1697.76</td>
<td>0.897</td>
<td>0.871</td>
</tr>
<tr>
<td>MAY</td>
<td>1064.12</td>
<td>1059.41</td>
<td>0.939</td>
<td>0.280</td>
</tr>
<tr>
<td>JUN</td>
<td>371.80</td>
<td>303.08</td>
<td>0.298</td>
<td>-0.120</td>
</tr>
<tr>
<td>JUL</td>
<td>341.93</td>
<td>396.27</td>
<td>0.340</td>
<td>0.143</td>
</tr>
<tr>
<td>AUG</td>
<td>589.57</td>
<td>481.41</td>
<td>0.415</td>
<td>-0.015</td>
</tr>
<tr>
<td>SEP</td>
<td>449.97</td>
<td>332.06</td>
<td>0.140</td>
<td>0.207</td>
</tr>
</tbody>
</table>

model for the residuals that can be used to simulate the innovations. A number of possible standard distributions (normal, lognormal, gamma, Weibull, etc.) were tried to model the residuals. It seems that none of them gave an adequate fit both at the center and the tails. For example, a summary (histogram) of the residuals as well as probability plot for a three parameter lognormal distribution are shown in Figure 4.5. The plots clearly indicate that normal and lognormal are not a good fit for the residuals.

The Hill’s estimators for the PARMA_{12}(1, 1) Model Residuals are shown in Figure 4.6. Although Hill’s estimator indicates 2 < α < 4, we investigate use of a stable distribution
Table 4.3: Moving average parameter estimates $\hat{\psi}_i(\ell)$ at season $i$ and lag $\ell = 1, 2, \ldots, 10$, after $k = 15$ iterations of the innovations algorithm applied to average monthly flow series for the Salt River near Roosevelt, AZ.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{\psi}_i(1)$</th>
<th>$\hat{\psi}_i(2)$</th>
<th>$\hat{\psi}_i(3)$</th>
<th>$\hat{\psi}_i(4)$</th>
<th>$\hat{\psi}_i(5)$</th>
<th>$\hat{\psi}_i(6)$</th>
<th>$\hat{\psi}_i(7)$</th>
<th>$\hat{\psi}_i(8)$</th>
<th>$\hat{\psi}_i(9)$</th>
<th>$\hat{\psi}_i(10)$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.410</td>
<td>-0.122</td>
<td>0.002</td>
<td>2.637</td>
<td>0.347</td>
<td>-0.131</td>
<td>0.007</td>
<td>0.014</td>
<td>0.051</td>
<td>0.103</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>1</td>
<td>0.168</td>
<td>0.182</td>
<td>0.019</td>
<td>0.544</td>
<td>0.072</td>
<td>-0.164</td>
<td>-0.032</td>
<td>0.088</td>
<td>0.016</td>
<td>0.000</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>2</td>
<td>3.064</td>
<td>0.221</td>
<td>0.507</td>
<td>1.103</td>
<td>1.631</td>
<td>-0.636</td>
<td>-0.070</td>
<td>0.285</td>
<td>0.021</td>
<td>$\ldots$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.368</td>
<td>1.060</td>
<td>-0.189</td>
<td>-0.489</td>
<td>-0.032</td>
<td>1.673</td>
<td>4.882</td>
<td>0.381</td>
<td>0.227</td>
<td>0.140</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>4</td>
<td>0.147</td>
<td>0.084</td>
<td>1.454</td>
<td>0.136</td>
<td>0.529</td>
<td>0.032</td>
<td>2.283</td>
<td>4.711</td>
<td>-0.924</td>
<td>0.309</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>5</td>
<td>0.459</td>
<td>0.440</td>
<td>0.379</td>
<td>1.898</td>
<td>0.283</td>
<td>-0.488</td>
<td>-0.170</td>
<td>0.801</td>
<td>3.354</td>
<td>-0.131</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>6</td>
<td>0.236</td>
<td>0.489</td>
<td>0.366</td>
<td>0.423</td>
<td>2.724</td>
<td>0.225</td>
<td>0.309</td>
<td>-0.088</td>
<td>0.142</td>
<td>4.074</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>7</td>
<td>0.542</td>
<td>0.111</td>
<td>0.322</td>
<td>0.162</td>
<td>0.211</td>
<td>1.580</td>
<td>0.444</td>
<td>-0.036</td>
<td>-0.225</td>
<td>0.210</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>8</td>
<td>0.265</td>
<td>0.123</td>
<td>0.032</td>
<td>0.097</td>
<td>0.060</td>
<td>0.075</td>
<td>0.457</td>
<td>0.085</td>
<td>-0.019</td>
<td>-0.023</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>9</td>
<td>1.056</td>
<td>-0.055</td>
<td>0.126</td>
<td>0.009</td>
<td>0.025</td>
<td>0.003</td>
<td>0.026</td>
<td>0.051</td>
<td>-0.006</td>
<td>-0.056</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>10</td>
<td>0.559</td>
<td>0.785</td>
<td>0.082</td>
<td>-0.064</td>
<td>-0.057</td>
<td>-0.031</td>
<td>-0.011</td>
<td>-0.033</td>
<td>-0.161</td>
<td>-0.045</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>11</td>
<td>0.256</td>
<td>0.184</td>
<td>0.096</td>
<td>0.049</td>
<td>-0.029</td>
<td>-0.031</td>
<td>-0.004</td>
<td>0.011</td>
<td>0.017</td>
<td>-0.158</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

($\alpha < 2$). It has been observed that Hill’s estimator can overestimate $\alpha$ for stable data (see, for example, McCulloch, 1997).

There are several methods for estimating stable parameters, but we focus on the three general methods (see, for example, Nolan, 1999): the maximum likelihood (ML) approach maximizes the likelihood numerically, the quantile method (QM) tries to match certain data quantiles with those of stable distributions, the characteristic function based methods fit the empirical characteristic function.

The model residuals were then fitted with a stable distribution $S1(\alpha, \beta, \gamma, \eta)$ and the
Table 4.4: Parameter estimates for \( \text{PARMA}_{12}(1, 1) \) model of average monthly flow series for the Salt River near Roosevelt, AZ.

<table>
<thead>
<tr>
<th>month</th>
<th>( \hat{\phi} )</th>
<th>( \hat{\theta} )</th>
<th>( \hat{\sigma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCT</td>
<td>-0.476</td>
<td>0.886</td>
<td>696.20</td>
</tr>
<tr>
<td>NOV</td>
<td>0.443</td>
<td>-0.275</td>
<td>229.67</td>
</tr>
<tr>
<td>DEC</td>
<td>1.315</td>
<td>1.749</td>
<td>779.23</td>
</tr>
<tr>
<td>JAN</td>
<td>0.346</td>
<td>0.022</td>
<td>1781.03</td>
</tr>
<tr>
<td>FEB</td>
<td>0.227</td>
<td>-0.080</td>
<td>1263.85</td>
</tr>
<tr>
<td>MAR</td>
<td>3.000</td>
<td>-2.541</td>
<td>1291.93</td>
</tr>
<tr>
<td>APR</td>
<td>1.065</td>
<td>-0.829</td>
<td>1044.32</td>
</tr>
<tr>
<td>MAY</td>
<td>0.472</td>
<td>0.071</td>
<td>378.06</td>
</tr>
<tr>
<td>JUN</td>
<td>0.226</td>
<td>0.039</td>
<td>75.54</td>
</tr>
<tr>
<td>JUL</td>
<td>-0.208</td>
<td>1.265</td>
<td>346.75</td>
</tr>
<tr>
<td>AUG</td>
<td>0.743</td>
<td>-0.184</td>
<td>409.58</td>
</tr>
<tr>
<td>SEP</td>
<td>0.329</td>
<td>-0.074</td>
<td>295.17</td>
</tr>
</tbody>
</table>

Parameters were estimated using three methods: ML, QM and the characteristic function based methods. The results are summarized in Table 4.5. The computer program STABLE 3.04 have been used for all the analysis (Nolan, 1999).

Since the estimates from the three methods are close for \( \text{PARMA}_{12}(1, 1) \) model residuals, the residuals can be considered as stably distributed (see Nolan, 1999) for diagnostics of assessing stability). The next step is to do the smoothed density, \( q-q \) plot and the variance stabilized \( p-p \) plot based on the ML estimates. Figure 4.7 shows the smoothed density, \( q-q \) plot and the variance stabilized \( p-p \) plot for the model residuals. The \( q-q \) plots shows that the extreme tails of the data are lighter than the stable model. The \( p-p \) plot shows that the
Figure 4.4: (a) ACF and (b) PACF for PARMA$_{12}(1,1)$ model residuals, showing the 95% confidence bounds $\pm 1.96/\sqrt{N}$.

Figure 4.5: PARMA$_{12}(1,1)$ Model residuals of the Salt River near Roosevelt, AZ: Statistical summary (left) and three parameter lognormal probability plot (right).

data look stable. The plots show a very close stable fit at the center (body), but not good at the tails. This suggest for the use of a mixture of stable with appropriate tail distributions.

Hill’s estimators (4.12) and (4.13) give $\hat{\alpha} = 1.661$ and $\hat{C} = 0.163$ for the upper tail ($r = 36$ residuals), and $\hat{\alpha} = 2.320$ and $\hat{C} = 0.043$ for the lower tail ($r = 15$ residuals). Using (3.60), the p-values are 0.018 and 0.393, respectively. The small p-value as well as the visual evidence of the probability plot in Figure 4.8 suggest that the Pareto model is not a good fit for the upper tail of the estimated residuals. Therefore, we decided to fit a truncated Pareto to the upper tail ($x > 2.30$) of the residuals. Since $p > 0.05$, Pareto is a good fit for the lower
Figure 4.6: Hill’s estimator, $\hat{\alpha}$ versus $r$, of (a) upper tail and (b) lower tail for the PARMA$_{12}(1,1)$ model residuals.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\eta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantile</td>
<td>1.1762</td>
<td>0.6319</td>
<td>0.3220</td>
<td>0.3388</td>
</tr>
<tr>
<td>Sample Characteristic</td>
<td>1.2061</td>
<td>0.6264</td>
<td>0.3306</td>
<td>0.2238</td>
</tr>
<tr>
<td>Maximum Likelihood</td>
<td>1.2081</td>
<td>0.6129</td>
<td>0.3260</td>
<td>0.2414</td>
</tr>
</tbody>
</table>

tail. However, we chose a truncated Pareto for the lower tails ($x < -1.52$) of the residuals because river flows are always bounded from below (i.e. river flows are always greater than zero). Note that this means $Ez_i^4 < \infty$ so that results from chapter 3 also apply here. Next we determined that $r = 36$ residuals exceed 2.30 (which is the 96 percentile of the stable distribution), and $r = 15$ residuals fall below -1.52 (which equals the 2.26 percentile of the stable distribution). Then the MLE from Theorem 3.13 was used to estimate the parameters ($\hat{\beta} = 8.499$, $\hat{\gamma} = 0.003$, $\hat{\alpha} = 0.345$) of the best fitting truncated Pareto distribution, and the theoretical distribution tail $P(R > r)$ was plotted over the 36 largest positive residuals in Figure 4.8 (left). In Figure 4.8 (right), we used the same method to fit a truncated Pareto ($\hat{\beta} = 4.886$, $\hat{\gamma} = 0.093$, $\hat{\alpha} = 1.401$) to the 15 largest negative residuals, after a change of sign.
Figure 4.7: Density, variance stabilized p-p plot and q-q plot for the PARMA_{12}(1,1) model residuals.

Both of the plots in Figure 4.8 indicate an adequate fit.

A mixture distribution with stable body and truncated Pareto tails was used to simulate the innovations. The mixture has cumulative distribution function (cdf)

\[
P(\delta \leq r) = \begin{cases} 
F_-(r) & \text{if } r < -1.52 \\
F_0(r) & \text{if } -1.52 \leq r \leq 2.30 \\
F_+(r) & \text{if } r > 2.30
\end{cases}
\] (4.15)

where \(F_0\) is the cdf of the stable, and \(F_+, F_-\) are truncated Pareto cdfs of the positive and negative tails, respectively. The truncated Pareto distributions were slightly shifted (by \(s = -0.057\) on the positive tail and \(s = 0.271\) on the negative tail) to make the mixture cdf continuous. Now stable innovations are simulated from \(S1(1.2081, 0.6129, 0.3260, 0.2414)\)
Figure 4.8: Log-log plot of upper (left) and lower (right) residual tails and fitted Pareto and truncated Pareto distributions, Salt River near Roosevelt, AZ.

by using STABLE 3.04 and innovations at the tails are obtained by using the inverse cumulative distribution function method. First, stable random variates were calculated from $S_1(1.2081,0.6129,0.3260,0.2414)$ using STABLE 3.04. If $R > 2.30$, the 96% quantile of stable, we generated a uniform $(0, 1)$ random variate $U$ and substituted $\delta = F_+^{-1}(0.96 + 0.04U)$. If $R < -1.52$, the 2.26% quantile of the stable, we substituted $\delta = F_-^{-1}(0.0226U)$. This gives simulated innovations $\delta$ with the mixture distribution (4.15).

Figure 4.9: Probability plot of simulated innovations using the mixed stable and truncated Pareto distributions. Compare Figure 4.5 (right).

Figure 4.9 shows a probability plot for $N = N_y \nu$ simulated innovations (for $\nu = 12$ months and $N_y = 100$ years) from the mixture distribution (4.15). Comparison with Figure
4.5 (a) shows that the simulated innovations are statistically identical to the computed model residuals in terms of distribution. Substituting the simulated innovations into the model (3.63) generates $N_y$ years of simulated river flow. In rare cases, a simulated flow comes out negative. When this occurs, we resample (draw a different random innovation) for that period. It is advantageous to simulate several extra years of river flows and throw out the initial years (100 years in this case), since we did not simulate $X_t$ for $t < 0$. This ensures that the simulated series is periodically stationary. Figure 4.10 gives a side-by-side comparison of the observed and simulated river flow time series, illustrating that the synthetic river flow appears statistically similar to the observed flow. Figure 4.11 shows the main statistical characteristics (mean, standard deviation and autocorrelations) of a typical synthetic river flow time series obtained by this method, as well as the same statistical measures for the observed time series. It is apparent that this procedure closely reproduces the main statistical characteristics, indicating that the modeling procedure is trustworthy for generating synthetic river flows. Such synthetic river flows are useful for design of hydraulic structures, for optimal operation of reservoir systems, for calculating the risk of failure of water supply systems, etc.

![Figure 4.10: Plot of (a) observed and (b) simulated monthly river flows for the Salt River near Roosevelt, AZ, indicating statistical similarity.](image-url)
Figure 4.11: Comparison of mean, standard deviation, and autocorrelations for simulated vs. observed monthly river flow data for the Salt River near Roosevelt, AZ.
5 Estimation of Periodic Parameters by Fourier Series

5.1 Overview

In most cases, PARMA models have been applied to monthly or quarter time series. However, when the number of periods is large (for instance, in weekly or daily data), PARMA models require an estimation of too many parameters, which violates the principle of statistical parsimony (model with minimum number of parameters). In this chapter, model parsimony is achieved by expressing the periodic model parameters in terms of their discrete Fourier transforms.

In most of the water resources systems design and operation studies the periodic phenomena have been represented by Fourier functions. Quimpo (1967) has applied Fourier analysis to daily river flow sequences in order to detect significant harmonic components embodied within the sequence considered. Since then Fourier analysis has become a standard tool in any hydrologic study concerning periodicity. Salas et al. (1980) proposed a Fourier series approach for reducing the number of parameters in PAR or PARMA models. Vecchia (1985a) also adopted the same approach but used Akaike’s information criterion (AIC) for the selection of significant harmonics. Experience in using Fourier analysis for estimating periodic parameters of hydrologic time series shows that for small time interval series, such as daily and weekly series, only the first few harmonics are necessary for a good Fourier series fit in the periodic estimate of model parameters (Salas et al., 1980). This practical criteria should be supplemented by more precise analysis and tests. For instance, Anderson and Vecchia (1993) uses asymptotic properties of the discrete Fourier transform of the estimated periodic autocovariance and autocorrelation function for selecting the harmonics in the PARMA model parameters.
In an effort to obtain a parsimonious model for periodically stationary series, we develop the asymptotic distribution of the discrete Fourier transform of the innovations estimates ($\psi$-weights of the periodic moving average processes) and then determine those statistically significant Fourier coefficients. We also extend these results to other periodic model parameters. We demonstrate the effectiveness of the technique using simulated data from different PARMA models. An application of the technique is demonstrated through the analysis of a weekly river flow series for the Fraser River, British Columbia.

5.2 Selection of Significant Harmonics and Fourier Coefficients

The PARMA$_\nu(p, q)$ model (1) has $(p + q + 1)\nu$ total parameters (see Lund and Basawa, 2000; Salas et al., 1980). For example, for a monthly series ($\nu = 12$) where $p = q = 1$, there are 36 parameters. When the period $\nu$ is large it is reasonable to assume that the model parameters vary smoothly with respect to time and can therefore be explained by only a few of their non-zero discrete Fourier coefficients.

Experience in using Fourier analysis for estimating periodic parameters of hydrologic time series shows that for small time interval series, such daily and weekly series, only the first few harmonics are necessary for a good Fourier series fit in the periodic estimate of $\theta_t(j)$, $\theta_t(j)$, and $\sigma_t$ (Salas et al., 1980). This practical criteria should be supplemented by more precise analysis and tests.

More precise analysis for selecting the harmonics in Fourier series fit of a periodic estimate is based on a theorem for detecting periodic variation in the model parameters in (2.1) and (2.2). This theorem is derived by first obtaining the asymptotic distribution of the discrete Fourier transform of the parameters from the innovations algorithm.
Write the moving average parameters in (2.2) at lag \( j \) in the form
\[
\psi_t(j) = c_0(j) + \sum_{r=1}^{k} \left\{ c_r(j) \cos \left( \frac{2\pi rt}{\nu} \right) + s_r(j) \sin \left( \frac{2\pi rt}{\nu} \right) \right\}
\] (5.1)
where \( c_r(j) \) and \( s_r(j) \) are the Fourier coefficients, \( r \) is the harmonic and \( k \) is the total number of harmonics which is equal to \( \nu/2 \) or \( (\nu - 1)/2 \) depending on whether \( \nu \) is even or odd, respectively.

Write the vector of Fourier coefficients at lag \( j \) in the form
\[
f(j) = \begin{cases} 
[c_0(j), c_1(j), s_1(j), \ldots, c_{(\nu-1)/2}(j), s_{(\nu-1)/2}(j)]^T & (\nu \text{ odd}) \\
[c_0(j), c_1(j), s_1(j), \ldots, s_{(\nu/2-1)}(j), c_{(\nu/2)}(j)]^T & (\nu \text{ even}) 
\end{cases}
\] (5.2)
where \( [\cdot] \) denotes the greatest integer function. Similarly, define \( \hat{f}_j \) to be the vector of Fourier coefficients for the innovations estimates \( \hat{\psi}_t(j) \), defined by replacing \( \psi_t(j) \) by \( \hat{\psi}_t(j) \), \( c_r(j) \) by \( \hat{c}_r(j) \), and \( s_r(j) \) by \( \hat{s}_r(j) \) in (5.1) and (5.2). We wish to describe the asymptotic distributional properties of these Fourier coefficients, to determine those that are statistically significantly different from zero. These are the coefficients that will be included in our model.

In order to compute the asymptotic distribution of the Fourier coefficients, it is convenient to work with the complex discrete Fourier transform (DFT) and its inverse
\[
\hat{\psi}_r(j) = \nu^{-1/2} \sum_{t=0}^{\nu-1} \exp \left( \frac{-2i\pi rt}{\nu} \right) \psi_t(j)
\] (5.3)
\[
\psi_t(j) = \nu^{-1/2} \sum_{r=0}^{\nu-1} \exp \left( \frac{2i\pi rt}{\nu} \right) \hat{\psi}_r^*(j)
\]
and similarly \( \hat{\psi}_r^*(j) \) is the complex DFT of \( \hat{\psi}_m(j) \). The complex DFT can also be written in matrix form. Recall from (3.30) the definition
\[
\hat{\psi}(\ell) = [\hat{\psi}_0(\ell), \hat{\psi}_1(\ell), \ldots, \hat{\psi}_{\nu-1}(\ell)]^T
\]
\[
\psi(\ell) = [\psi_0(\ell), \psi_1(\ell), \ldots, \psi_{\nu-1}(\ell)]^T
\]
and similarly define

\begin{align}
\hat{\psi}^*(j) &= \begin{bmatrix} \hat{\psi}_0^*(j), \hat{\psi}_1^*(j), \ldots, \hat{\psi}_{\nu-1}^*(j) \end{bmatrix}^T \\
\psi^*(j) &= \begin{bmatrix} \psi_0^*(j), \psi_1^*(j), \ldots, \psi_{\nu-1}(j) \end{bmatrix}^T
\end{align}

(5.4)

noting that these are all \(\nu\)-dimensional vectors. Define a \(\nu \times \nu\) matrix \(U\) with complex entries

\[ U = \nu^{-1/2} \left( e^{-\frac{2\pi ir\ell}{\nu}} \right)_{r,\ell=0,1,\ldots,\nu-1} \]

(5.5)

so that \(\psi^*(j) = U\psi(j)\) and \(\hat{\psi}^*(j) = U\hat{\psi}(j)\). This matrix form is useful because it is easy to invert. Obviously \(\psi^*(j) = U\psi(j)\) is equivalent to \(\psi(j) = U^{-1}\psi^*(j)\) since the matrix \(U\) is invertible. This is what guarantees that there exists a unique vector of complex DFT coefficients \(\psi^*(j)\) corresponding to any vector \(\psi(j)\) of moving average parameters. But in this case, the matrix \(U\) is also unitary (i.e., \(UU^T = I\)) which means that \(U^{-1} = U^T\), and the latter is easy to compute. Then we also have \(\psi(j) = U^T\psi^*(j)\) which is the matrix form of the second relation in (5.3).

Next we convert from complex to real DFT, and it is advantageous to do this in a way that also involves a unitary matrix. Define

\begin{align}
a_r(j) &= 2^{-1/2} \{ \psi_r^*(j) + \psi_{\nu-r}^*(j) \} \quad (r = 1, 2, \ldots, [(\nu - 1)/2]) \\
a_0(j) &= \psi_0^*(j) \quad (r = 0 \text{ or } \nu/2) \\
b_r(j) &= i2^{-1/2} \{ \psi_r^*(j) - \psi_{\nu-r}^*(j) \} \quad (r = 1, 2, \ldots, [(\nu - 1)/2])
\end{align}

(5.6)

and let

\[ e(j) = \begin{cases} 
[a_0(j), a_1(j), b_1(j), \ldots, a_{(\nu-1)/2}(j), b_{(\nu-1)/2}(j)]^T & \text{if } \nu \text{ odd} \\
[a_0(j), a_1(j), b_1(j), \ldots, b_{(\nu/2-1)}(j), a_{(\nu/2)}(j)]^T & \text{if } \nu \text{ even}
\end{cases} \]

(5.7)

and likewise for the coefficients of \(\hat{\psi}_t(j)\). These relations (5.6) define another \(\nu \times \nu\) matrix \(P\) with complex entries such that

\[ e(j) = PU\psi(j) = P\psi^*(j), \]

(5.8)
and it is not hard to check that $P$ is also unitary, so that $\psi^*(j) = P^{-1}e(j) = \tilde{P}^T e(j)$, the latter form being most useful for computations. The DFT coefficients $a_r(j)$ and $b_r(j)$ are not the same as the coefficients $c_r(j)$ and $s_r(j)$ in (5.1) but they are closely related. Substitute the first line of (5.3) into (5.6) and simplify to obtain

$$a_r(j) = \nu^{-1/2} \sum_{t=0}^{\nu-1} \cos \left( \frac{2\pi rt}{\nu} \right) \psi_t(j)$$  

$$b_r(j) = \sqrt{\frac{2}{\nu}} \sum_{t=0}^{\nu-1} \sin \left( \frac{2\pi rt}{\nu} \right) \psi_t(j)$$  

for $r = 1, 2, \ldots, [(\nu-1)/2]$. Inverting the relations (5.6) or, equivalently, using the matrix equation $\psi^*(j) = \tilde{P}^T e(j)$, we obtain $\psi^*_r(j) = a_r(j)$ for $r = 0$ or $\nu/2$ and

$$\psi^*_r(j) = 2^{-1/2} \{a_r(j) - ib_r(j)\} \quad \text{and} \quad \psi^*_{\nu-r}(j) = \tilde{\psi}^*_r(j) = 2^{-1/2} \{a_r(j) + ib_r(j)\}$$

for $r = 1, \ldots, k = [(\nu-1)/2]$. Substitute these relations into the second expression in (5.3) and simplify to obtain

$$\psi_t(j) = \nu^{-1/2} a_0(j) + \sqrt{\frac{2}{\nu}} \sum_{r=1}^{k} \left\{a_r(j) \cos \left( \frac{2\pi rt}{\nu} \right) + b_r(j) \sin \left( \frac{2\pi rt}{\nu} \right) \right\}$$

for $\nu$ odd and

$$\psi_t(j) = \nu^{-1/2} (a_0(j) + a_k(j)) + \sqrt{\frac{2}{\nu}} \sum_{r=1}^{k-1} \left\{a_r(j) \cos \left( \frac{2\pi rt}{\nu} \right) + b_r(j) \sin \left( \frac{2\pi rt}{\nu} \right) \right\}$$

for $\nu$ even, where $k$ is the total number of harmonics which is equal to $\nu/2$ or $(\nu - 1)/2$ depending on whether $\nu$ is even or odd, respectively. Comparison with (5.1) reveals that

$$c_r = \sqrt{\frac{2}{\nu}} a_r$$  

$$c_r = \nu^{-1/2} a_r$$  

$$s_r = \sqrt{\frac{2}{\nu}} b_r$$  

for $r = 1, 2, \ldots, [(\nu-1)/2]$.
Substituting into (5.9) yields

\[ c_r(j) = \nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \psi_m(j) \quad (r = 0 \text{ or } \nu/2) \]
\[ c_r(j) = 2\nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \psi_m(j) \quad (r = 1, 2, \ldots, [(\nu - 1)/2]) \]
\[ s_r(j) = 2\nu^{-1} \sum_{m=0}^{\nu-1} \sin \left( \frac{2\pi rm}{\nu} \right) \psi_m(j) \quad (r = 1, 2, \ldots, [(\nu - 1)/2]) \]

and likewise for the Fourier coefficients of \( \hat{\psi}_m(j) \). Define the \( \nu \times \nu \) diagonal matrix

\[
L = \begin{cases} 
\text{diag}(\nu^{-1/2}, \sqrt{2/\nu}, \ldots, \sqrt{2/\nu}) & (\nu \text{ odd}) \\
\text{diag}(\nu^{-1/2}, \sqrt{2/\nu}, \ldots, \sqrt{2/\nu}, \nu^{-1/2}) & (\nu \text{ even})
\end{cases}
\]

so that in view of (5.10) we have \( f(j) = Le(j) \) and \( \hat{f}(j) = \hat{L}\hat{e}(j) \). Substituting into (5.8) we obtain

\[
f(j) = LPU\psi(j) \quad \text{and} \quad \hat{f}(j) = LPU\hat{\psi}(j).
\]

**Theorem 5.1** For any positive integer \( j \)

\[
N_y^{1/2} \left[ \hat{f}(j) - f(j) \right] \Rightarrow \mathcal{N}(0, R_V)
\]

where

\[
R_V = LPUV_{jj} \hat{U}^T \hat{P}^T L^T.
\]

**PROOF.** From Theorem 3.7 we have

\[
N_y^{1/2} \left[ \hat{\psi}(j) - \psi(j) \right] \Rightarrow \mathcal{N}(0, V_{jj})
\]

where \( V_{ik} \) is from (3.31). Define \( B = LPU \) so that \( f(j) = B\psi(j) \) and \( \hat{f}(j) = B\hat{\psi}(j) \) using (5.13). Apply Proposition 3.8 with \( X_n = \hat{\psi}(j) \) and \( g(X) = BX \), so that \( \mu = \psi(j) \), \( g(\mu) = B\psi(j) = f(j) \), and \( D = [\partial g_i/\partial x_j] = B \). Then \( N_y^{1/2}[B\hat{\psi}(j) - B\psi(j)] \Rightarrow \mathcal{N}(0, BV_{jj}B^T) \) or in other words \( N_y^{1/2}[\hat{f}(j) - Bf(j)] \Rightarrow \mathcal{N}(0, BV_{jj}B^T) \). Although \( P \) and \( U \) are complex matrices, the product \( B = LPU \) is a real matrix, and therefore \( B^T = \hat{B}^T = \hat{U}^T \hat{P}^T L^T \). Then (5.14) and (5.15) follow, which finishes the proof.
Theorem 5.2 Let $X_t = \tilde{X}_t - \mu_t$, where $X_t$ is the periodic moving average process (3.1) and $\mu_t$ is a periodic mean function with period $\nu$. Then, under the null hypothesis that the mean-standardized process is stationary with $\psi_t(h) = \psi(h)$ and $\sigma_t = \sigma$, the elements of (5.2) are asymptotically independent with

$$N^{1/2}_y \{ \hat{c}_m(h) - \mu_m(h) \} \Rightarrow \mathcal{N}(0, \nu^{-1} \eta_V(h)) \quad (m = 0 \text{ or } \nu/2)$$

$$N^{1/2}_y \{ \hat{c}_m(h) - \mu_m(h) \} \Rightarrow \mathcal{N}(0, 2\nu^{-1} \eta_V(h)) \quad (m = 1, 2, \ldots, [(\nu - 1)/2]) \quad (5.17)$$

$$N^{1/2}_y \{ \hat{s}_m(h) - \mu_m(h) \} \Rightarrow \mathcal{N}(0, 2\nu^{-1} \eta_V(h)) \quad (m = 1, 2, \ldots, [(\nu - 1)/2])$$

for all $h \geq 1$, where

$$\mu_m(h) = \begin{cases} \psi(h) & (m = 0) \\ 0 & (m > 0) \end{cases} \quad (5.18)$$

$$\eta_V(h) = \sum_{n=0}^{h-1} \psi^2(n). \quad (5.19)$$

PROOF. Under the null hypothesis, $\psi_t(h) = \psi(h)$ and $\sigma_t = \sigma$, is constant in $t$ for each $h$ and hence the $F_n$ and $B_n$ matrices in (3.32) become, respectively, a scalar multiple of the identity matrix: $F_n = \psi(n)I$, and an identity matrix: $B_n = I$. Then, from (3.31),

$$V_{hh} = \sum_{n=0}^{h-1} \psi(n)\Pi^{(\nu-1)(h-1-n)}I{\psi(n)\Pi^{(\nu-1)(h-1-n)}}^T = \sum_{n=0}^{h-1} \psi^2(n)I = \eta_V(h)I$$

is also a scalar multiple of the identity matrix. Hence, since scalar multiples of the identity matrix commute in multiplication with any other matrix, we have from (5.15) that

$$PUV_{hh} \tilde{U}^T \tilde{P}^T = V_{hh} PU \tilde{U}^T \tilde{P}^T = V_{hh}$$

since $P$ and $U$ are unitary matrices (i.e, $\tilde{P}^T P = I$ and $U \tilde{U}^T = I$). Then in Theorem 5.1 we have

$$N^{1/2}_y \left[ \hat{f}(h) - f(h) \right] \Rightarrow \mathcal{N}(0, R_V) \quad (5.20)$$
where $R_V = LPUV_{hh} \tilde{U}^T \tilde{P}^T L^T = V_{hh} LL^T$, so that

$$R_V = \begin{cases} 
\eta_V(h) \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, 2\nu^{-1}) & (\nu \text{ odd}) \\
\eta_V(h) \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, \nu^{-1}) & (\nu \text{ even})
\end{cases}$$

Under the null hypothesis, $f(h) = [\psi(h), 0, \ldots, 0]^T$ and then the theorem follows by considering the individual elements of the vector convergence (5.20).

Under the hypothesis that $c_m(h)$ and $s_m(h)$ are zero for all $m \geq 1$ and $h \neq 0$, Theorem 5.2 tells us that, for example, if $\nu$ is odd then $\{\hat{c}_1(j), \hat{s}_1(h), \ldots, \hat{c}_{(\nu-1)/2}(h), \hat{s}_{(\nu-1)/2}(h)\}$ form $\nu - 1$ independent and normally distributed random variables with mean zero and standard error $(2\nu^{-1} \hat{\eta}_V(h)/N_y)^{1/2}$. The Bonferroni$^{14}$ $\alpha$-level test statistic rejects the null hypothesis that $c_m(h)$ and $s_m(h)$ are zero for all $m \geq 1$ if $|Z_c| > z_{\alpha'/2}$ and $|Z_s| > z_{\alpha'/2}$, respectively, and

$$Z_c = \frac{\hat{c}_m(h)}{(\hat{\lambda} \hat{\eta}_V(h)/N_y)^{1/2}} , \quad Z_s = \frac{\hat{s}_m(h)}{(\hat{\lambda} \hat{\eta}_V(h)/N_y)^{1/2}}$$

(5.21)

where

$$\hat{\lambda} = \begin{cases} 
\nu^{-1} & (m = \nu/2) \\
2\nu^{-1} & (m = 1, 2, \ldots, [(\nu - 1)/2])
\end{cases}$$

and $Z \sim N(0, 1)$, $\alpha' = \alpha/(\nu - 1)$. When $\alpha = 5\%$ and $\nu = 12$, $\alpha' = 0.05/11 = 0.0045$, $z_{\alpha'/2} = z_{0.0045} = 2.84$, and the null hypothesis is rejected when any $|Z_{c,s}| > 2.84$, indicating that the corresponding Fourier coefficients is statistically significantly different from zero.

$^{14}$See Section A-6, Appendix A.
5.3 Estimation of Fourier Coefficients of Model Parameters

For large $\nu$, it is anticipated that PARMA model parameters $\phi_t(\ell), \theta_t(\ell)$ and $\sigma_t$, will vary smoothly with respect to $t$, and can therefore be explained by a few of their non-zero Fourier coefficients. For a PARMA$(p, q)$ model, the Fourier series representation of the $\phi_t(\ell), \theta_t(\ell)$ and $\sigma_t$, can be obtained by

$$\theta_t(\ell) = c_{a0}(\ell) + \sum_{r=1}^{k} \left\{ c_{ar}(\ell) \cos \left( \frac{2\pi rt}{\nu} \right) + s_{ar}(\ell) \sin \left( \frac{2\pi rt}{\nu} \right) \right\}$$

$$\phi_t(\ell) = c_{b0}(\ell) + \sum_{r=1}^{k} \left\{ c_{br}(\ell) \cos \left( \frac{2\pi rt}{\nu} \right) + s_{br}(\ell) \sin \left( \frac{2\pi rt}{\nu} \right) \right\}$$

$$\sigma_t = c_{d0} + \sum_{r=1}^{k} \left\{ c_{dr} \cos \left( \frac{2\pi rt}{\nu} \right) + s_{dr} \sin \left( \frac{2\pi rt}{\nu} \right) \right\}$$

(5.22)

c_{ar,br,dr} and $s_{ar,br,dr}$ are the Fourier coefficients, $r$ is the harmonic and $k$ is the total number of harmonics (as in (5.1)). For instance, for monthly series where $\nu = 12$, we have $k = 6$; for weekly series with $\nu = 52$, $k = 26$ and for daily series with $\nu = 365$, $k = 182$. In practice, a small number of harmonic $k^* < k$ is used.

For general Fourier analysis of PARMA$\nu(p, q)$ model, it is desirable to represent (3.38) in terms of the Fourier coefficients of the model parameters and the Fourier transform of $\psi_t(j)$ (see Appendix B for a detailed discussion). However, we develop asymptotic distributions of the discrete Fourier coefficients of PARMA$\nu(1, 1)$ model parameters by using the relationship in (3.44) and (3.45).

Consider again the PARMA$\nu(1, 1)$ model given in (3.42). To simplify notation, we will express the model parameters, along with their Fourier coefficients, in terms of vector notation. Let $\theta = [\theta_0, \theta_1, \cdots, \theta_{\nu-1}]^T$, $\phi = [\phi_0, \phi_1, \cdots, \phi_{\nu-1}]^T$ and $\sigma = [\sigma_0, \sigma_1, \cdots, \sigma_{\nu-1}]^T$ be the vector of PARMA$\nu(1, 1)$ model parameters. These model parameters may be defined in terms of their complex finite Fourier coefficients $\theta_t^*, \phi_t^*$ and $\sigma^*$ as follows
\[ \theta^*(\ell) = U\theta(\ell) \quad \text{and} \quad \theta(\ell) = \tilde{U}^T\theta^*(\ell) \]
\[ \phi^*(\ell) = U\phi(\ell) \quad \text{and} \quad \phi(\ell) = \tilde{U}^T\phi^*(\ell) \]
(5.23)
\[ \sigma^* = U\sigma \quad \text{and} \quad \sigma = \tilde{U}^T\sigma^* \]

where \( \sim \) denotes the complex conjugate, \( U \) is the \( \nu \times \nu \) Fourier transform matrix defined in (5.5) and

\[ \theta^* = \left[ \theta^*_0, \theta^*_1, \ldots, \theta^*_{\nu-1} \right]^T \]
\[ \phi^* = \left[ \phi^*_0, \phi^*_1, \ldots, \phi^*_{\nu-1} \right]^T \]
\[ \sigma^* = \left[ \sigma^*_0, \sigma^*_1, \ldots, \sigma^*_{\nu-1} \right]^T \]

As in Theorem 5.1 let the vector form for transformed \( \theta \) and \( \phi \) be given by

\[ f_\theta = LP\theta^* = LPU\theta \]
\[ f_\phi = LP\phi^* = LPU\phi \]
(5.24)

where

\[ f_\theta = \begin{cases} \left[ c_{a0}, c_{a1}, s_{a1}, \ldots, c_{a(\nu-1)/2}, s_{a(\nu-1)/2} \right]^T & (\nu \ odd) \\ \left[ c_{a0}, c_{a1}, s_{a1}, \ldots, s_{a(\nu-1)/2}, c_{a(\nu-1)/2} \right]^T & (\nu \ even) \end{cases} \]
(5.25)
\[ f_\phi = \begin{cases} \left[ c_{b0}, c_{b1}, s_{b1}, \ldots, c_{b(\nu-1)/2}, s_{b(\nu-1)/2} \right]^T & (\nu \ odd) \\ \left[ c_{b0}, c_{b1}, s_{b1}, \ldots, s_{b(\nu-1)/2}, c_{b(\nu-1)/2} \right]^T & (\nu \ even) \end{cases} \]
(5.26)

\[ c_{ar} = \nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \theta_m \quad (r = 0 \ or \ \nu/2) \]
\[ c_{ar} = 2\nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \theta_m \quad (r = 1, 2, \ldots, [(\nu-1)/2]) \]
(5.27)
\[ s_{ar} = 2\nu^{-1} \sum_{m=0}^{\nu-1} \sin \left( \frac{2\pi rm}{\nu} \right) \theta_m \quad (r = 1, 2, \ldots, [(\nu-1)/2]) \]
\[ c_{br} = \nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \phi_m \quad (r = 0 \text{ or } \nu/2) \]
\[ c_{br} = 2\nu^{-1} \sum_{m=0}^{\nu-1} \cos \left( \frac{2\pi rm}{\nu} \right) \phi_m \quad (r = 1, 2, \ldots, [(\nu - 1)/2]) \]  
(5.28)
\[ s_{br} = 2\nu^{-1} \sum_{m=0}^{\nu-1} \sin \left( \frac{2\pi rm}{\nu} \right) \phi_m \quad (r = 1, 2, \ldots, [(\nu - 1)/2]) \]

and likewise for the Fourier coefficients of \( \hat{\theta}_m \) and \( \hat{\phi}_m \). Note that \([ \cdot ]\) denotes the greatest integer function. We wish to describe the asymptotic distributional properties of the elements of (5.25) and (5.26).

**Theorem 5.3** Regarding the Fourier transformed \( \phi \) and \( \theta \), we have

\[
N^{1/2} \left[ \hat{f}_\theta - f_\theta \right] \Rightarrow \mathcal{N}(0, R_S)
\]
\[
N^{1/2} \left[ \hat{f}_\phi - f_\phi \right] \Rightarrow \mathcal{N}(0, R_Q)
\]

\[
f_\theta = LP\theta^* = LP\theta
\]
\[
R_S = LPUS\hat{U}^T \hat{P}^T L^T
\]
\[
f_\phi = LP\phi^* = LP\phi
\]
\[
R_Q = LPUQ\hat{U}^T \hat{P}^T L^T
\]  
(5.30)

where “ ~ ” denotes the complex conjugate, \( Q \) is given by (3.48) and \( S \) is given by (3.56).

**PROOF.** Same procedure as in proof of Theorem 5.1 by applying Proposition 3.8 along with Theorem 3.9 and Theorem 3.11.

**Theorem 5.4** Let \( X_t = \tilde{X}_t - \mu_t \), where \( X_t \) is the PARMA\( \nu \)(1, 1) process (3.42) and \( \mu_t \) is a periodic mean function with period \( \nu \). Then, under null hypothesis that the mean-standardized process is stationary with \( \phi_t = \phi, \theta_t = \theta \) and \( \sigma_t = \sigma \), the elements of \( \hat{f}_\phi \), defined by (5.26)

with \( c_{br} \) replaced by \( \hat{c}_{br} \) and \( s_{br} \) replaced by \( \hat{s}_{br} \), are asymptotically independent with
\[ \mathcal{N}_{y}^{1/2} \{ \hat{c}_{bm} - \mu_{bm} \} \Rightarrow \mathcal{N}(0, \nu^{-1}\eta_{Q}) \quad (m = 0 \text{ or } \nu/2) \]

\[ \mathcal{N}_{y}^{1/2} \{ \hat{c}_{bm} - \mu_{bm} \} \Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_{Q}) \quad (m = 1, 2, \ldots, [(\nu - 1)/2]) \quad (5.31) \]

\[ \mathcal{N}_{y}^{1/2} \{ \hat{s}_{bm} - \mu_{bm} \} \Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_{Q}) \quad (m = 1, 2, \ldots, [(\nu - 1)/2]) \]

where

\[ \mu_{bm} = \begin{cases} \phi & (m = 0) \\ 0 & (m > 0) \end{cases} \quad (5.32) \]

\[ \eta_{Q} = \psi^{-4}(1) \left\{ \psi^{2}(2) \left( 1 - \frac{2\psi^{2}(1)}{\psi(2)} \right) + \psi^{2}(1) \sum_{n=0}^{1} \psi^{2}(n) \right\} \quad (5.33) \]

\[ \psi(1) = \phi - \theta, \text{ and } \psi(2) = \phi\psi(1). \]

**PROOF.** The proof follows along the same lines as Theorem 3.9 and hence we adopt the same notation. As in proof of Theorem 5.2, we have \( B_{n} = I \) and \( F_{n} = \psi(n)I \) in (3.32) and so (3.31) implies \((x = \text{Min}(h, j))\)

\[ V_{jh} = \sum_{n=0}^{x-1} \psi(j - 1 - n)\Pi^{(\nu-1)(j-1-n)}\Pi^{(\nu-1)(h-1-n)}T \psi(h - 1 - n) \]

\[ = \sum_{n=0}^{x-1} \psi^{2}(j - 1 - n)I \quad \text{if } j = h \]

since then \( \Pi^{(\nu-1)(j-1-n)}\Pi^{(\nu-1)(h-1-n)}T = I \) so

\[ V_{11} = \psi^{2}(0)I = I \]

\[ V_{22} = [\psi^{2}(1) + \psi^{2}(0)]I = [\psi^{2}(1) + 1]I \]

Note that, for \( h \neq j, \Pi^{(\nu-1)(j-1-n)}\Pi^{(\nu-1)(h-1-n)}T = \Pi^{(\nu-1)(j-1-n)}\Pi^{-(\nu-1)(h-1-n)} = \Pi^{(\nu-1)(j-h)} \)

Then

\[ V_{12} = \psi(0)\Pi^{(\nu-1)(1-h)}\psi(1) = \psi(1)\Pi \]

\[ V_{21} = \psi(1)\Pi^{-1}\psi(0) = \psi(1)\Pi^{-1} = \psi(1)\Pi^{T} \]
and 

\[ Q = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} H_1^T \\ H_2^T \end{pmatrix} \]

where \( V_{21} = V_{12}^T \Rightarrow Q \) is symmetric.

Since \( X_t \) is stationary, every \( \psi_\ell(t) = \psi(t) \) in (3.1) and so

\[ H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & -\frac{\psi(2)}{\psi^2(1)} \\ -\frac{\psi(2)}{\psi^2(1)} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\psi(2)}{\psi^2(1)} & 0 \end{pmatrix} = -\frac{\psi(2)}{\psi^2(1)} \Pi^{-1} \]

and

\[ H_2 = \frac{1}{\psi(1)} I \]

\[ V_{11} = B_1 = I \]

\[ V_{22} = B_2 + F_1 \Pi^{-1} B_1 \Pi F_1 = I + \psi^2(1) \Pi^{-1} \Pi = \left[ \psi^2(1) + 1 \right] I \]

\[ V_{12} = B_1 \Pi F_1 = \Pi F_1 = \psi(1) \Pi \]

\[ V_{21} = V_{12}^T = \psi(1) \Pi^{-1} = \psi(1) \Pi^T \]

\[ Q = H_1 V_{11} H_1^T + H_2 V_{21} H_1^T + H_1 V_{12} H_2^T + H_2 V_{22} H_2^T \]

\[ = -\frac{\psi(2)}{\psi^2(1)} \Pi^{-1} I \left( -\frac{\psi(2)}{\psi^2(1)} \Pi \right) + \frac{\psi(2)}{\psi^2(1)} \Pi^{-1} \psi(1) \Pi \frac{1}{\psi(1)} \]

\[ + \frac{1}{\psi(1)} I \psi(1) \Pi^{-1} \left( -\frac{\psi(2)}{\psi^2(1)} \Pi \right) + \frac{1}{\psi(1)} I \left[ \psi^2(1) + 1 \right] I \frac{1}{\psi(1)} I \]

\[ Q = \left\{ \frac{\psi^2(2) - 2 \psi^2(1) \psi(2) + \psi^2(1) + 1}{\psi^2(1)} \right\} I \]

So \( Q = \eta Q I \) is actually a scalar multiple of the identity matrix \( I \). Then \( PU \tilde{Q} \tilde{U}^T \tilde{P}^T = QPU \tilde{U}^T \tilde{P}^T = Q \) and hence \( R_Q = LQL^T = QLL^T \) or in other words

\[ R_Q = \begin{cases} 
\eta_Q \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, 2\nu^{-1}) & (\nu \text{ odd}) \\
\eta_Q \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, \nu^{-1}) & (\nu \text{ even}).
\end{cases} \]
Under the null hypothesis, \( f_\phi = [\phi, 0, \ldots, 0]^T \) and then the theorem follows by considering the individual elements of the vector convergence from the second line of (5.29).

Under the hypothesis that \( c_{bm} \) and \( s_{bm} \) are zero for all \( m \geq 1 \) and \( h \neq 0 \), Theorem 5.4 tells us that, for example, if \( \nu \) is odd then \( \{\hat{c}_{b1}, \hat{s}_{b1}, \ldots, \hat{c}_{b(\nu-1)/2}, \hat{s}_{b(\nu-1)/2}\} \) form \( \nu - 1 \) independent and normally distributed random variables with mean zero and standard error \((2\nu^{-1}\eta_S/N_y)^{1/2}\). The Bonferroni \( \alpha \)-level test statistic rejects the null hypothesis that \( c_{bm} \) and \( s_{bm} \) are zero for all \( m \geq 1 \) and \( h \neq 0 \) if \( |Z_c| > z_{\alpha'/2} \) and \( |Z_s| > z_{\alpha'/2} \), respectively. Here, \( Z \sim N(0, 1) \), \( \alpha' = \alpha/(\nu - 1) \), and

\[
Z_c = \frac{\hat{c}_{bm}(h)}{(\lambda \hat{\eta}_Q/N_y)^{1/2}}, \quad Z_s = \frac{\hat{s}_{bm}(h)}{(\lambda \hat{\eta}_Q/N_y)^{1/2}}
\]

where

\[
\lambda = \begin{cases} 
\nu^{-1} & (m = \nu/2) \\
2\nu^{-1} & (m = 1, 2, \ldots, [(\nu - 1)/2])
\end{cases}
\]

\[
\hat{\eta}_Q = \hat{\psi}^{-4}(1) \left( \hat{\psi}^2(2) \left( 1 - \frac{2\hat{\psi}^2(1)}{\hat{\psi}(2)} \right) + \hat{\psi}^2(1) \sum_{n=0}^{1} \hat{\psi}^2(n) \right).
\]

**Theorem 5.5** Let \( X_t = \tilde{X}_t - \mu_t \), where \( X_t \) is the PARMA\( \nu(1, 1) \) process (3.42) and \( \mu_t \) a periodic mean function with period \( \nu \). Then, under null hypothesis that the mean-standardized process is stationary with \( \theta_t = \theta \), \( \phi_t = \phi \) and \( \sigma_t = \sigma \), the elements of \( \hat{f}_\theta \), defined by (5.25) with \( c_{br} \) replaced by \( \hat{c}_{br} \) and \( s_{br} \) replaced by \( \hat{s}_{br} \), are asymptotically independent with

\[
N_y^{1/2} \{\hat{c}_{am} - \mu_{am}\} \Rightarrow \mathcal{N}(0, \nu^{-1}\eta_S) \quad (m = 0 \text{ or } \nu/2)
\]

\[
N_y^{1/2} \{\hat{c}_{am} - \mu_{am}\} \Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_S) \quad (m = 1, 2, \ldots, [(\nu - 1)/2])
\]

\[
N_y^{1/2} \{\hat{s}_{am} - \mu_{am}\} \Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_S) \quad (m = 1, 2, \ldots, [(\nu - 1)/2])
\]

where
\[ \mu_{am} = \begin{cases} \theta & (m = 0) \\ 0 & (m > 0) \end{cases} \]  

(5.36)

\[ \eta_S = \psi^{-4}(1) \left\{ \psi^2(2) \left( 1 - \frac{2\psi^2(1)}{\psi(2)} \right) + \sum_{j=1}^{2} \psi^{A/2}(1) \sum_{n=0}^{j-1} \psi^2(n) \right\} \]  

(5.37)

\[ \psi(1) = \phi - \theta, \text{ and } \psi(2) = \phi \psi(1). \]

**PROOF.** The proof follows along the same lines as Theorem 3.11 and hence we adopt the same notation. Note that \[ S = Q + \tilde{S} \]

so as in Theorem 5.4 proof we get \[ R_S = LPUSU^T \tilde{P}^T L^T = SLP\tilde{U}U^T \tilde{P}^T L^T = SLL^T \]

so that

\[ R_S = \begin{cases} 
\eta_S \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, 2\nu^{-1}) & (\nu \text{ odd}) \\
\eta_S \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, \nu^{-1}) & (\nu \text{ even}). 
\end{cases} \]

Under the null hypothesis, \( f_\theta = [\theta, 0, \ldots, 0]^T \) and then the theorem follows by considering the individual elements of the vector convergence from the first line of (5.29).

Under the hypothesis that \( c_{am} \) and \( s_{am} \) are zero for all \( m \geq 1 \) and \( h \neq 0 \), Theorem 5.5 tells us that, for example, if \( \nu \) is odd then \( \{\hat{c}_{a1}, \hat{s}_{a1}, \ldots, \hat{c}_{a(\nu-1)/2}, \hat{s}_{a(\nu-1)/2}\} \) form \( \nu - 1 \) independent and normally distributed random variables with mean zero and standard errors obtained from \( (\lambda \hat{\eta}_S / N_g)^{1/2} \) where \( \lambda \) is defined below. The Bonferroni \( \alpha \)-level test statistic rejects the null hypothesis that \( c_{am} \) and \( s_{am} \) are zero for all \( m \geq 1 \) and \( h \neq 0 \) if \( |Z_c| > z_{\alpha'}/2 \) and \( |Z_s| > z_{\alpha'}/2 \), respectively. Here \( Z \sim N(0, 1), \alpha' = \alpha/\sqrt{\nu - 1}, \text{ and} \)
\[ Z_c = \frac{\hat{c}_{am}(h)}{(\lambda \hat{\eta}/N_y)^{1/2}} \quad Z_s = \frac{\hat{s}_{am}(h)}{(\lambda \hat{\eta}/N_y)^{1/2}} \] (5.38)

where

\[ \lambda = \begin{cases} \nu^{-1} & (m = \nu/2) \\ 2\nu^{-1} & (m = 1, 2, \ldots, [(\nu - 1)/2]) \end{cases} \]

\[ \hat{\eta}_S = \hat{\psi}^{-4}(1) \left\{ \psi^2(2) \left( 1 - \frac{2\hat{\psi}^2(1)}{\psi(2)} \right) + \sum_{j=1}^{2} \psi^{4/j}(1) \sum_{n=0}^{j-1} \hat{\psi}^2(n) \right\} \]

5.4 Simulation Study

A detailed simulation study was conducted to demonstrate the effectiveness of the results of the previous section using simulated data from different PARMA_{\nu}(p, q) models with finite fourth moment. For each model, individual realizations of \( N_y = 50, 100, 300, \) and 500 years of data (i.e., sample size of \( N = N_y \nu \)) were simulated and the innovations algorithm was used to obtain parameter estimates for each realization. In each case, estimates were obtained for \( k = 15 \) iterations. Then discrete Fourier transformed innovation estimates and model parameters were obtained from (5.11), (5.27) and (5.28). The test statistics for the Fourier coefficients were computed using (5.21), (5.34) and (5.38) to identify those estimates that were statistically significant. A FORTRAN program was used to simulate the PARMA samples as well as to make all the necessary calculations.

As an example, we summarize here the results of two particular cases of a PARMA_{12}(p, q) model and one case of a PARMA_{52}(p, q) model. We first consider the following PARMA_{12}(0, 1) model

\[ X_{k\nu+i} = \varepsilon_{k\nu+i} + \theta_i \varepsilon_{k\nu+i-1} \] (5.39)
where

\[ \theta_i = \sigma_0 + \sum_{r=1}^{6} \left\{ c_{ar} \cos \left( \frac{2\pi ri}{\nu} \right) + s_{ar} \sin \left( \frac{2\pi ri}{\nu} \right) \right\} \]

\[ \sigma_i = \sigma_0 + \sum_{r=1}^{6} \left\{ c_{dr} \cos \left( \frac{2\pi ri}{\nu} \right) + s_{dr} \sin \left( \frac{2\pi ri}{\nu} \right) \right\} \]

The periodic notation \( X_{kv+i} \) refers to the mean zero simulated data during the \( i^{th} \) period of cycle \( k \). The innovations \( \{ \delta_t = \sigma_i^{-1} \varepsilon_{kv+i} \} \) form an iid normal sequence of random variables with mean zero and standard deviation one.

For example, suppose we wish to test the hypothesis that \( \theta_i \) is dominated by two harmonics of period 12,

\[ \theta_i = \sigma_0 + \sum_{r=1}^{2} \left\{ c_{ar} \cos \left( \frac{2\pi ki}{12} \right) + s_{ar} \sin \left( \frac{2\pi ri}{12} \right) \right\} \] (5.40)

For \( c_0 = 0.45, c_1 = 0.25, s_1 = 0.75, c_2 = 0.80, \) and \( s_2 = 0.50 \), the values of \( \theta_i \) are calculated from (5.40), and assume \( \sigma_i = 2.00 + 0.15 \cos \frac{2\pi i}{12} + 0.90 \sin \frac{2\pi i}{12} \). Then using (5.39), a single realization with \( N_y = 500 \) years of data (i.e., sample size of \( N = 6000 \)) was generated. Table 5.2 shows the results after \( k = 15 \) iterations of the innovations algorithm.

| Table 5.1: Model parameters and estimates for simulated PARMA_{12}(0,1) data. |
|--------------------------|--------|--------|--------|--------|--------|
| True Value              | \( c_{a0} \) | \( c_{a1} \) | \( s_{a1} \) | \( c_{a2} \) | \( s_{a2} \) |
| Estimated               | 0.463  | 0.282  | 0.725  | 0.806  | 0.519  |

Fourier coefficients with test statistics \( z > 3.32 \) (for \( \alpha = 1\% \)) are considered to be significant. It seems that the \( \psi \)-weights are dominated by a few of Fourier coefficients; especially \( c_m(j) \) and \( s_m(j) \) where \( m = 1, 2 \). Besides, the Fourier coefficients are statistically insignificant after lag 1, which indicate that the simulated data are from PARMA_{12}(0,1). The estimated standard deviation \( (\hat{\sigma}_i \approx 1.92 + 0.15 \cos \frac{2\pi i}{12} + 0.88 \sin \frac{2\pi i}{12} ) \) is close to the true
values. The results of the residuals analysis also indicate that they look independent and normally distributed with mean zero and variance one. Those significant Fourier coefficients and the true values are summarized in Table 5.1 (neglecting the smaller coefficients). Again, the results show the closeness of the estimates.

The second model for which simulation results will be presented is a PARMA\(_{12}(1, 1)\) model

$$X_{kv+i} = \phi_i X_{kv+i-1} + \varepsilon_{kv+i} + \theta_i \varepsilon_{kv+i-1}$$  \hfill (5.41)

For example, suppose we wish to test the hypothesis that \(\theta_i\) and \(\phi_i\) are dominated by two harmonics of period 12,

$$\phi_i = c_{i0} + \sum_{k=1}^{2} \left\{ c_{bk} \cos \left( \frac{2 \pi k i}{12} \right) + s_{bk} \sin \left( \frac{2 \pi k i}{12} \right) \right\}$$  \hfill (5.42)

For coefficients given in Table 5.5, the values of \(\theta_i\) and \(\phi_i\) are calculated from (5.41), and (5.42), respectively. Here we assume constant innovation variance (\(\sigma_i^2 = 1\) for all i). From the above model, a single realization with \(N_y = 500\) years of quarterly data (sample size of \(N = 6000\)) was generated. Table 5.3 shows the results after \(k = 15\) iterations of the innovations algorithm. Since the discrete Fourier transform of \(\hat{\psi}_i\) weights do not generally cut-off to (statistically) zero at a certain lag, we choose a parsimonious mixed model that captures the periodic behavior as well as the exponential decay evidenced in the autocorrelation function.

After making all the diagnostics, we found that a PARMA\(_{12}(1, 1)\) model is adequate. The discrete Fourier transform of parameter estimates for this model, obtained using equations (5.27) and (5.28), are summarized in Table 5.4. It seems that estimated model parameters \((\hat{\theta}_i, \hat{\phi}_i)\) of PARMA\(_{12}(1, 1)\) model are approximately dominated by two harmonics, and these Fourier coefficients as well as true values are summarized in Table 5.5. The result shows the
closeness of the estimates.

The last model used to illustrate the Fourier analysis technique is PARMA\textsubscript{52}(1, 1). The same Fourier coefficients as for PARMA\textsubscript{12}(1, 1) model are used for calculating the PARMA\textsubscript{52}(1, 1) model parameters, and then a single realization with \( N_y = 500 \) years of weekly data (sample size of \( N = 26000 \)) was generated. The Fourier transformed values of the estimated PARMA\textsubscript{52}(1, 1) model parameters, the standard errors and the test statistics are summarized in Table 5.6 and Table 5.7. Fourier coefficients with test statistics \( z > 3.72 \) (for \( \alpha = 1\% \)) are considered to be significant. They are roughly dominated by two harmonics, and these Fourier coefficients as well as true values are summarized in Table 5.8. The result shows again the closeness of the estimates.

5.5 Application to Modeling of River Flows

First, we present a detailed Fourier analysis of monthly Fraser river flow series. We found that PARMA\textsubscript{12}(1, 1) is a reasonable fit for monthly flows (see Section 4.5). The monthly variation of the parameters (see Figure 5.1) is fairly smooth indicating that the Fourier transform of \( \hat{\theta}_t \) and \( \hat{\phi}_t \) may be dominated by a few components. The Fourier transformed values of these model parameters, the standard errors and the test statistics are summarized in Table 5.9. Based on Theorem 5.4 and Theorem 5.5, Fourier coefficients with test statistics \( z > 3.32 \) (for \( \alpha = 1\% \)) are considered to be significant. It seems that the PARMA\textsubscript{12}(1, 1) model parameters are dominated by few Fourier coefficients (see Table 5.9). Those coefficient with a test statistic close to 3.32 \((s_{a2}, s_{a4}s_{b4})\) can also be considered statistically insignificant so as to achieve model parsimony (see the effect in Figure 5.1)

Using those significant Fourier coefficients \((c_{a0} = 0.304, s_{a1} = -0.426, c_{a3} = 0.665, c_{b0} = 0.337, s_{b1} = 0.466, c_{b2} = 0.408, s_{b2} = 0.355 \) and \(c_{b3} = -0.649\), see Table 5.9), the
residuals are calculated again using (3.62). The ACF and PACF (not shown) indicate that there is no serial dependence. Statistical summary and lognormal probability plot of these residuals are shown in Figure 5.2, indicating a mixture of lognormal with an appropriate tail distribution (compare with Section 4.5).

Figure 5.2: Statistical summary (a) and lognormal probability plot (b) for model residuals, Fourier model parameters, monthly Fraser river data. Compare Figure 3.7 (a).

Next we model the 72 years of weekly river flow data from October 1912 to September 1984, and use as a second example for Fourier analysis. The weekly river flow series are obtained by averaging daily measurements over 7 days (8 or 9 days at the end of a each water year) starting from October of each year. In this analysis, $\nu = 0$ corresponds to the
first 7 days of October and \( \nu = 51 \) corresponds to the last 7 to 9 days of September. The weekly statistics such as the mean, standard deviation, and lag-1 and lag-2 serial correlation coefficient are displayed in Figure 5.3. Following similar reasoning as in Section 4.5, we fit \( \text{PARMA}_{32}(1, 1) \) model to the data. The parameter estimates for this model, obtained using equations (3.44) and (3.45), are shown in Figure 5.6. It must be noted that \( \theta_t \) in (3.44) must be replaced with \(-\theta_t\) so as to be consistent with the representation in (2.1). Model residuals were estimated using equation (3.62). Figure 5.4 shows the ACF and PACF of the model residuals. Although a few values lie slightly outside of the 95% confidence bands, there is no apparent pattern, providing some evidence that the \( \text{PARMA}_{32}(1, 1) \) model is adequate.

![Graphs of weekly river flow data](image.png)

Figure 5.3: Sample Statistics (a) mean (b) standard deviation and autocorrelations at lag 1 (c) and lag 2 (d) of weekly river flow data for the Fraser River at Hope, BC.

Statistical summary and lognormal probability plot of model residuals are also shown
Figure 5.4: ACF and PACF for model residuals, showing the bounds ±1.96/√N, indicate no serial dependence.

Figure 5.5: Statistical summary (left) and lognormal probability plot (right) for model residuals, weekly Fraser river data. Compare Figure 3.7
Figure 5.6: Plot of PARMA$_{52}(1,1)$ model parameters (with their Fourier fit) of average weekly flow series for the Fraser River at Hope, BC.

Coefficients: $c_{a0} = 0.255$, $c_{a1} = -0.197$, $c_{a2} = 0.121$ and $c_{b0} = 0.732$) indicate that there is no serial dependence (see Chapter 6).
Table 5.2: Discrete Fourier transform of moving average parameter estimates \( \hat{\psi}_i(\ell) \) at season \( i \) and lag \( \ell = 1, \ldots, 4 \), and standard errors (SE), after \( k = 15 \) iterations of the innovations algorithm applied to simulated \( N_y = 500 \) years of simulated PARMA\( _{12}(0, 1) \) data. Note that the value in (.) is the test statistic (5.21).

<table>
<thead>
<tr>
<th>lag</th>
<th>Coefficient</th>
<th>harmonic 0</th>
<th>harmonic 1</th>
<th>harmonic 2</th>
<th>harmonic 3</th>
<th>harmonic 4</th>
<th>harmonic 5</th>
<th>harmonic 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \hat{c}_m(1) )</td>
<td>0.463</td>
<td>0.282*</td>
<td>0.806*</td>
<td>-0.013</td>
<td>-0.008</td>
<td>-0.000</td>
<td>0.029</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(15.434)</td>
<td>(44.143)</td>
<td>(-0.700)</td>
<td>(-0.436)</td>
<td>(-0.016)</td>
<td>(2.230)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \hat{s}_m(1) )</td>
<td>0.725*</td>
<td>0.519*</td>
<td>0.005</td>
<td>-0.028</td>
<td>-0.013</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(39.711)</td>
<td>(28.441)</td>
<td>(0.288)</td>
<td>(-1.544)</td>
<td>(-0.719)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.018</td>
<td>0.018</td>
<td>0.018</td>
<td>0.018</td>
<td>0.018</td>
<td>0.013</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \hat{c}_m(2) )</td>
<td>0.015</td>
<td>0.053</td>
<td>0.027</td>
<td>0.049</td>
<td>0.001</td>
<td>-0.011</td>
<td>-0.015</td>
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<tr>
<td></td>
<td></td>
<td>(2.678)</td>
<td>(1.354)</td>
<td>(2.466)</td>
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<td>(-0.536)</td>
<td>(-1.073)</td>
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<tr>
<td></td>
<td>( \hat{s}_m(2) )</td>
<td>0.001</td>
<td>-0.004</td>
<td>-0.037</td>
<td>-0.008</td>
<td>0.006</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.058)</td>
<td>(-0.210)</td>
<td>(-1.887)</td>
<td>(-0.456)</td>
<td>(0.299)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.014</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \hat{c}_m(3) )</td>
<td>0.010</td>
<td>0.035</td>
<td>0.091*</td>
<td>0.049</td>
<td>0.038</td>
<td>0.006</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.750)</td>
<td>(4.604)</td>
<td>(2.495)</td>
<td>(1.939)</td>
<td>(0.319)</td>
<td>(1.219)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \hat{s}_m(3) )</td>
<td>-0.024</td>
<td>-0.004</td>
<td>0.031</td>
<td>-0.015</td>
<td>-0.007</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.211)</td>
<td>(-0.220)</td>
<td>(1.579)</td>
<td>(-0.741)</td>
<td>(-0.370)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.014</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \hat{c}_m(4) )</td>
<td>0.007</td>
<td>0.027</td>
<td>0.045</td>
<td>-0.007</td>
<td>-0.008</td>
<td>-0.018</td>
<td>-0.021</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.349)</td>
<td>(2.270)</td>
<td>(-0.352)</td>
<td>(-0.408)</td>
<td>(-0.917)</td>
<td>(-1.490)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \hat{s}_m(4) )</td>
<td>0.016</td>
<td>0.067*</td>
<td>0.022</td>
<td>0.036</td>
<td>0.010</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.798)</td>
<td>(3.421)</td>
<td>(1.091)</td>
<td>(1.815)</td>
<td>(0.523)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.014</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic \( \geq 3.32 \)
Table 5.3: Discrete Fourier transform of moving average parameter estimates $\hat{\psi}_i(\ell)$ at season $i$ and lag $\ell = 1, \ldots, 4$, and standard errors (SE), after $k = 15$ iterations of the innovations algorithm applied to simulated $N_y = 500$ years of simulated PARMA$_{12}(1, 1)$ data. Note that the value in (.) is the test statistic (5.21).

<table>
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<tr>
<th>lag</th>
<th>Coefficient</th>
<th>harmonic $m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\hat{c}_m(1)$</td>
<td>0.698</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(22.452)</td>
</tr>
<tr>
<td></td>
<td>$\hat{s}_m(1)$</td>
<td>0.762*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(41.727)</td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.018</td>
</tr>
<tr>
<td>2</td>
<td>$\hat{c}_m(2)$</td>
<td>0.378</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10.464)</td>
</tr>
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<td></td>
<td>$\hat{s}_m(2)$</td>
<td>0.378*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(17.897)</td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.022</td>
</tr>
<tr>
<td>3</td>
<td>$\hat{c}_m(3)$</td>
<td>0.157</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.530)</td>
</tr>
<tr>
<td></td>
<td>$\hat{s}_m(3)$</td>
<td>0.225*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10.329)</td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.022</td>
</tr>
<tr>
<td>4</td>
<td>$\hat{c}_m(4)$</td>
<td>0.069</td>
</tr>
<tr>
<td></td>
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<td>(-3.406)</td>
</tr>
<tr>
<td></td>
<td>$\hat{s}_m(4)$</td>
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<tr>
<td></td>
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<td>(5.771)</td>
</tr>
<tr>
<td></td>
<td>SE</td>
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*Fourier coefficients with test statistic $\geq 3.32$
Table 5.4: Discrete Fourier transform of model parameters estimates and standard errors (SE) for simulated PARMA_{12}(1,1) data. Note that the value in (.) is the test statistic (5.34) and (5.38).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Statistic</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\theta}_m$</td>
<td>0.361</td>
<td>0.181*</td>
<td>0.228*</td>
<td>-0.023</td>
<td>-0.048</td>
<td>0.009</td>
<td>-0.020</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.711)</td>
<td>(7.204)</td>
<td>(-0.715)</td>
<td>(-1.529)</td>
<td>(0.289)</td>
<td>(-0.895)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\hat{s}_m$</td>
<td>0.418*</td>
<td>0.314*</td>
<td>0.021</td>
<td>-0.021</td>
<td>-0.047</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(13.234)</td>
<td>(9.916)</td>
<td>(0.668)</td>
<td>(-0.652)</td>
<td>(-1.473)</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>SE</td>
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<td>0.032</td>
<td>0.032</td>
<td>0.032</td>
<td>0.032</td>
<td>0.032</td>
<td>0.022</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\phi}_m$</td>
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<td>0.229*</td>
<td>0.455*</td>
<td>-0.015</td>
<td>0.028</td>
<td>0.009</td>
<td>0.019</td>
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<tr>
<td></td>
<td></td>
<td>(6.280)</td>
<td>(12.471)</td>
<td>(-0.406)</td>
<td>(0.760)</td>
<td>(0.244)</td>
<td>(0.733)</td>
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</tr>
<tr>
<td></td>
<td>$\hat{s}_m$</td>
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<td>-0.133*</td>
<td>-0.006</td>
<td>-0.004</td>
<td>0.034</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(9.404)</td>
<td>(-3.652)</td>
<td>(-0.162)</td>
<td>(-0.104)</td>
<td>(0.943)</td>
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<td>0.022</td>
<td>0.022</td>
<td>0.022</td>
<td>0.022</td>
<td>0.015</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.32$

Table 5.5: Significant discrete Fourier transform coefficients for simulated PARMA_{12}(1,1) model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$c_{a0}$</th>
<th>$c_{a1}$</th>
<th>$s_{a1}$</th>
<th>$c_{a2}$</th>
<th>$s_{a2}$</th>
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</thead>
<tbody>
<tr>
<td>$\theta_t$</td>
<td>True Value</td>
<td>0.35</td>
<td>0.15</td>
<td>0.40</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>Estimated</td>
<td>0.361</td>
<td>0.181</td>
<td>0.418</td>
<td>0.228</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$c_{b0}$</th>
<th>$c_{b1}$</th>
<th>$s_{b1}$</th>
<th>$c_{b2}$</th>
<th>$s_{b2}$</th>
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</thead>
<tbody>
<tr>
<td>$\phi_t$</td>
<td>True Value</td>
<td>0.35</td>
<td>0.25</td>
<td>0.35</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>Estimated</td>
<td>0.336</td>
<td>0.229</td>
<td>0.343</td>
<td>0.455</td>
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</table>
Table 5.6: Discrete Fourier transform of moving average parameter estimate, \( \hat{\theta}_t \) (with standard error, \( SE = 0.009 \) for \( m = 26 \) and \( SE = 0.013 \) for \( m \neq 26 \)) for the simulated PARMA\(_{32}(1,1)\) data. Note that the value in (.) is the test statistic (5.38).

<table>
<thead>
<tr>
<th>harmonic m</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</thead>
<tbody>
<tr>
<td>( \hat{c}_{am} )</td>
<td>0.301</td>
<td>0.148*</td>
<td>0.310*</td>
<td>-0.046*</td>
<td>0.063*</td>
<td>0.007</td>
<td>-0.049*</td>
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<td>-0.033</td>
</tr>
<tr>
<td></td>
<td>(11.836)</td>
<td>(24.675)</td>
<td>(-3.691)</td>
<td>(5.014)</td>
<td>(0.559)</td>
<td>(-3.918)</td>
<td>(4.723)</td>
<td>(-2.611)</td>
<td></td>
</tr>
<tr>
<td>( \hat{s}_{am} )</td>
<td>0.452*</td>
<td>0.315*</td>
<td>0.003</td>
<td>0.028</td>
<td>-0.070*</td>
<td>0.019</td>
<td>0.009</td>
<td>-0.076*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(36.031)</td>
<td>(25.097)</td>
<td>(0.259)</td>
<td>(2.235)</td>
<td>(-5.508)</td>
<td>(1.480)</td>
<td>(0.712)</td>
<td>(-6.036)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>( \hat{c}_{am} )</td>
<td>-0.031</td>
<td>0.054*</td>
<td>-0.068*</td>
<td>-0.008</td>
<td>0.036</td>
<td>-0.033</td>
<td>0.028</td>
<td>0.060*</td>
<td>-0.049*</td>
</tr>
<tr>
<td></td>
<td>(-2.495)</td>
<td>(4.339)</td>
<td>(-5.388)</td>
<td>(-0.626)</td>
<td>(2.901)</td>
<td>(-2.603)</td>
<td>(2.207)</td>
<td>(4.816)</td>
<td>(-3.910)</td>
</tr>
<tr>
<td>( \hat{s}_{am} )</td>
<td>0.056*</td>
<td>-0.018</td>
<td>-0.037</td>
<td>0.066*</td>
<td>-0.019</td>
<td>-0.025</td>
<td>0.079*</td>
<td>-0.049*</td>
<td>-0.027</td>
</tr>
<tr>
<td></td>
<td>(4.451)</td>
<td>(-1.422)</td>
<td>(-2.974)</td>
<td>(5.257)</td>
<td>(-1.544)</td>
<td>(-1.972)</td>
<td>(6.301)</td>
<td>(-3.920)</td>
<td>(-2.174)</td>
</tr>
<tr>
<td></td>
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<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
</tr>
<tr>
<td>( \hat{c}_{am} )</td>
<td>-0.018</td>
<td>0.056*</td>
<td>-0.039</td>
<td>0.025</td>
<td>0.061*</td>
<td>-0.056*</td>
<td>0.025</td>
<td>0.002</td>
<td>-0.017</td>
</tr>
<tr>
<td></td>
<td>(-1.432)</td>
<td>(4.428)</td>
<td>(-3.117)</td>
<td>(2.024)</td>
<td>(4.895)</td>
<td>(-4.433)</td>
<td>(2.000)</td>
<td>(0.162)</td>
<td>(-1.868)</td>
</tr>
<tr>
<td>( \hat{s}_{am} )</td>
<td>0.032</td>
<td>-0.041</td>
<td>0.021</td>
<td>0.069*</td>
<td>-0.074*</td>
<td>-0.015</td>
<td>0.028</td>
<td>-0.070*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.599)</td>
<td>(-3.244)</td>
<td>(1.687)</td>
<td>(5.475)</td>
<td>(-5.865)</td>
<td>(-1.218)</td>
<td>(2.242)</td>
<td>(-5.544)</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic \( \geq 3.72 \)
Table 5.7: Discrete Fourier transform of autoregressive parameter estimate, $\hat{\phi}_t$ (with standard error, $SE = 0.011$ for $m = 26$ and $SE = 0.015$ for $m \neq 26$) for the simulated PARMA$_{32}(1,1)$ data. Note that the value in (.) is the test statistic (5.34).

<table>
<thead>
<tr>
<th>harmonics m</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{c}_{am}$</td>
<td>0.385</td>
<td>0.240</td>
<td>0.417*</td>
<td>0.053*</td>
<td>-0.055*</td>
<td>-0.007</td>
<td>0.061</td>
<td>-0.058</td>
<td>0.027</td>
</tr>
<tr>
<td></td>
<td>(15.661)</td>
<td>(27.233)</td>
<td>(3.460)</td>
<td>(-3.576)</td>
<td>(-0.485)</td>
<td>(4.014)</td>
<td>(-3.767)</td>
<td>(1.739)</td>
<td></td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>0.280*</td>
<td>-0.127</td>
<td>-0.004</td>
<td>-0.021</td>
<td>0.064*</td>
<td>-0.038</td>
<td>-0.009</td>
<td>0.077*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(18.260)</td>
<td>(-8.308)</td>
<td>(-1.355)</td>
<td>(4.183)</td>
<td>(-2.512)</td>
<td>(-0.559)</td>
<td>(5.029)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>$\hat{c}_{am}$</td>
<td>0.046</td>
<td>-0.051*</td>
<td>0.057*</td>
<td>0.016</td>
<td>-0.041</td>
<td>0.040</td>
<td>-0.027</td>
<td>-0.071*</td>
<td>0.062*</td>
</tr>
<tr>
<td></td>
<td>(3.014)</td>
<td>(-3.307)</td>
<td>(3.698)</td>
<td>(1.064)</td>
<td>(-2.697)</td>
<td>(2.609)</td>
<td>(-1.764)</td>
<td>(-4.624)</td>
<td>(4.024)</td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>-0.060*</td>
<td>0.020</td>
<td>0.028</td>
<td>-0.068*</td>
<td>0.009</td>
<td>0.024</td>
<td>-0.070*</td>
<td>0.061*</td>
<td>0.029</td>
</tr>
<tr>
<td></td>
<td>(-3.920)</td>
<td>(1.308)</td>
<td>(1.813)</td>
<td>(-4.475)</td>
<td>(0.593)</td>
<td>(1.556)</td>
<td>(-4.563)</td>
<td>(3.955)</td>
<td>(1.910)</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
</tr>
<tr>
<td>$\hat{c}_{am}$</td>
<td>0.022</td>
<td>-0.046</td>
<td>0.048</td>
<td>-0.022</td>
<td>-0.047</td>
<td>0.044</td>
<td>-0.015</td>
<td>-0.011</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>(1.436)</td>
<td>(-3.012)</td>
<td>(3.154)</td>
<td>(-1.446)</td>
<td>(-3.071)</td>
<td>(2.855)</td>
<td>(-0.976)</td>
<td>(-0.694)</td>
<td>(1.643)</td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>-0.049</td>
<td>0.041</td>
<td>-0.020</td>
<td>-0.073*</td>
<td>0.060*</td>
<td>0.010</td>
<td>-0.047</td>
<td>0.064*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-3.178)</td>
<td>(2.691)</td>
<td>(-1.307)</td>
<td>(-4.750)</td>
<td>(3.902)</td>
<td>(0.676)</td>
<td>(-3.053)</td>
<td>(4.166)</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.72$
Table 5.8: Significant discrete Fourier transform coefficients for simulated PARMA_{52}(1,1) model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True Value</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_t$</td>
<td>$c_{a0}$ = 0.35, $c_{a1}$ = 0.15, $s_{a1}$ = 0.40, $c_{a2}$ = 0.25, $s_{a2}$ = 0.35</td>
<td>$c_{a0}$ = 0.300, $c_{a1}$ = 0.148, $s_{a1}$ = 0.452, $c_{a2}$ = 0.310, $s_{a2}$ = 0.314</td>
</tr>
<tr>
<td>$\phi_t$</td>
<td>$c_{b0}$ = 0.35, $c_{b1}$ = 0.25, $s_{b1}$ = 0.35, $c_{b2}$ = 0.45, $s_{b2}$ = -0.15</td>
<td>$c_{b0}$ = 0.385, $c_{b1}$ = 0.240, $s_{b1}$ = 0.279, $c_{b2}$ = 0.417, $s_{b2}$ = -0.127</td>
</tr>
</tbody>
</table>

Table 5.9: Discrete Fourier transform of model parameters estimates and standard errors (SE) of average monthly flow series for the Fraser River at Hope, BC. Note that the value in (.) is the test statistic (5.34) and (5.38).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Statistic</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{c}_{am}$ = 0.090</td>
<td>0 1 2 3 4 5 6</td>
</tr>
<tr>
<td></td>
<td>0.304</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>(0.128)</td>
<td>(-2.816)</td>
</tr>
<tr>
<td></td>
<td>-0.426*</td>
<td>0.301*</td>
</tr>
<tr>
<td></td>
<td>(-4.741)</td>
<td>(3.351)</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{c}_{bm}$ = 0.102</td>
<td>0.337</td>
</tr>
<tr>
<td></td>
<td>(-0.355)</td>
<td>(3.998)</td>
</tr>
<tr>
<td></td>
<td>0.466*</td>
<td>-0.355*</td>
</tr>
<tr>
<td></td>
<td>(4.570)</td>
<td>(-3.487)</td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.32$
Table 5.10: Discrete Fourier transform of moving average parameter estimate, $\hat{\theta}_t$ (with standard error, SE = 0.017 for $m = 26$ and SE = 0.024 for $m \neq 26$) for PARMA$_{32}(1,1)$ model of average weekly flow series for Fraser River at Hope, BC. Note that the value in (.) is the test statistic (5.38).

| harmonics $m$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\hat{c}_{am}$ | 0.255 | -0.197* | 0.121* | -0.040 | 0.081 | -0.022 | -0.031 | -0.013 | -0.078 | 0.044 | 0.052 | 0.023 | -0.071 | 0.009 | 0.021 | 0.139* | -0.053 | 0.017 | 0.031 | -0.007 | 0.113* | 0.019 | 0.075 | -0.016 | 0.018 | -0.065 | 0.019 |
|               | (-8.25) | (5.062) | (-1.670) | (3.388) | (-0.941) | (-1.300) | (-0.531) | (-3.283) |               | (1.833) | (2.169) | (0.957) | (-2.996) | (0.392) | (0.880) | (5.836) | (-2.240) | (0.696) |               | (0.753) | (1.118) | (-2.471) | (-1.436) | (-2.154) | (-2.993) | (1.714) | (3.078) | (-0.396) |
| $\hat{s}_{am}$ | -0.0198 | 0.081 | -0.117* | 0.015 | -0.044 | -0.041 | -0.020 | -0.016 | (-0.828) | (3.392) | (-4.89) | (0.633) | (-1.823) | (-1.731) | (-0.832) | (-0.680) |               |               |               |               |               |               |               |               |               |               |
|               | (-0.828) | (3.392) | (-4.89) | (0.633) | (-1.823) | (-1.731) | (-0.832) | (-0.680) |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |

*Fourier coefficients with test statistic $\geq 3.72$
Table 5.11: Discrete Fourier transform of autoregressive parameter estimate, $\hat{\phi}_t$ (with standard error, SE = 0.023 for $m = 26$ and SE = 0.033 for $m \neq 26$) for PARMA$_{52}(1,1)$ model of average weekly flow series for the Fraser River at Hope, BC. Note that the value in (.) is the test statistic (5.34).

<table>
<thead>
<tr>
<th>harmonics $m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{c}_{am}$</td>
<td>0.732</td>
<td>-0.021</td>
<td>0.051</td>
<td>-0.076</td>
<td>-0.005</td>
<td>-0.045</td>
<td>0.052</td>
<td>0.002</td>
<td>0.082</td>
</tr>
<tr>
<td></td>
<td>(-0.621)</td>
<td>(1.536)</td>
<td>(-2.302)</td>
<td>(-0.174)</td>
<td>(-1.362)</td>
<td>(1.557)</td>
<td>(0.058)</td>
<td>(2.483)</td>
<td></td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>-0.065</td>
<td>-0.049</td>
<td>0.063</td>
<td>0.019</td>
<td>-0.009</td>
<td>0.020</td>
<td>0.043</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.957)</td>
<td>(-1.470)</td>
<td>(1.902)</td>
<td>(0.583)</td>
<td>(-0.275)</td>
<td>(0.588)</td>
<td>(1.305)</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td>9 10 11 12 13 14 15 16 17</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{c}_{am}$</td>
<td>0.009</td>
<td>-0.043</td>
<td>-0.018</td>
<td>0.032</td>
<td>0.024</td>
<td>0.021</td>
<td>-0.055</td>
<td>0.044</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>(0.275)</td>
<td>(-1.298)</td>
<td>(-0.539)</td>
<td>(0.959)</td>
<td>(0.711)</td>
<td>(0.648)</td>
<td>(-1.667)</td>
<td>(1.319)</td>
<td>(0.287)</td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>0.005</td>
<td>-0.011</td>
<td>0.024</td>
<td>-0.041</td>
<td>0.040</td>
<td>0.016</td>
<td>-0.064</td>
<td>-0.026</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>(0.150)</td>
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<td>(0.714)</td>
<td>(-1.226)</td>
<td>(1.194)</td>
<td>(0.493)</td>
<td>(-1.931)</td>
<td>(-0.797)</td>
<td>(1.279)</td>
</tr>
<tr>
<td>18 19 20 21 22 23 24 25 26</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{c}_{am}$</td>
<td>-0.019</td>
<td>0.036</td>
<td>-0.014</td>
<td>0.023</td>
<td>-0.003</td>
<td>0.000</td>
<td>-0.025</td>
<td>0.037</td>
<td>-0.000</td>
</tr>
<tr>
<td></td>
<td>(-0.580)</td>
<td>(1.071)</td>
<td>(-0.418)</td>
<td>(0.695)</td>
<td>(-0.103)</td>
<td>(0.000)</td>
<td>(-0.756)</td>
<td>(1.105)</td>
<td>(-0.007)</td>
</tr>
<tr>
<td>$\hat{s}_{am}$</td>
<td>0.035</td>
<td>-0.001</td>
<td>0.066</td>
<td>0.004</td>
<td>0.071</td>
<td>0.043</td>
<td>-0.010</td>
<td>-0.009</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.054)</td>
<td>(-0.019)</td>
<td>(1.999)</td>
<td>(0.119)</td>
<td>(2.124)</td>
<td>(1.283)</td>
<td>(-0.304)</td>
<td>(-0.282)</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.72$
6 Application of PARMA Models: Hydrological Drought Analysis

6.1 General

Obtaining drought properties is important for planning and management of water resources system. For example, the design of water supply capacity of a given city may be based on meeting water demands during a critical drought that may occur in a specified planning horizon (Frick et al., 1990). Moreover, the estimation of return periods associated to severe droughts can provide useful information in order to improve water systems management under drought condition.

It is difficult to give a universal definition since there exists widely diverse views about the interpretation of droughts among the scientists of different disciplines. In a broader sense, a drought is defined as a deficit of water in time. It can be divided into three categories—meteorological, hydrological, and agricultural drought—which may occur as a result of precipitation, river flow, and soil moisture deficits, respectively (see for example, Dracup et al. 1980; Sen 1980, etc). In this study the emphasis is on hydrological drought, which is defined as uninterrupted sequences of river flows below a threshold level. It has three components—duration, intensity, and severity. Since river flow sequences are stochastic variables, the corresponding drought properties are random and must be described in probabilistic terms.

Yevjevich (1967) proposed the theory of runs as a major tool to use in objectively defining droughts and studying their statistical properties. Many studies of drought properties followed Yevjevich’s definition of droughts. For example, probabilistic behavior of drought properties has been derived analytically, assuming a given stochastic structure of the underlying hydrological series (e.g., Downer et al., 1967; Llamas and Siddiqui, 1969;
Another subject of much interest when analyzing extreme events such as drought is the determination of the return period. The return period can be defined in different ways for different applications. Some authors (e.g., Lloyd, 1970; Loaiciga and Mariño, 1991; Fernandez and Salas, 1999a,b; Shiau and Shen, 2001) have assumed the return period as average elapsed time between occurrences of specified events (i.e., droughts exceeding a critical value). An alternative definition of return period is the average number of trials (or time periods) required to the first occurrence of a specified drought event (e.g. Vogel, 1987; Bras, 1990; Fernandez and Salas,1999a,b; Douglas et al., 2002).

Hydrologic droughts can be determined from the historical record alone by using non-parametric methods but, because the number of drought events that can be drawn from the historical sample is generally small, the “historical” drought properties have a large degree of uncertainty. Other alternatives for finding drought properties include using stochastic models that can represent the underlying river flows, simulating long records of such river flows, and then deriving droughts properties from the simulated samples based on the theory of runs. In this chapter, we use PARMA models to model river flows and generate a large sequence of synthetic flows, and then drought properties are derived from the simulated samples based on the theory of runs. The applicability of these methods is demonstrated by using weekly river flow data for the Fraser River near Hope in British Colombia.

6.2 Definitions

An objective definition (as suggested by Yevjevich, 1967) will be made on the basis of a given sequence of river flows \( \{X_t\} \) and a preselected level of reference \( X_{0t} \), see Figure
6.1. A hydrologic drought event is defined as uninterrupted sequences of river flows below the preselected level $X_{0t}$ (also called negative run). This preselected reference level can correspond to various hydrological quantities (see, for example, Sen, 1976; Dracup et al. 1980a). In case of water supply, $X_{0t}$ may be considered equal to demand, which may be constant or variable. Thus the drought duration $L$ (negative run length) is the number of consecutive time intervals in which $X_t < X_{0t}$, preceded and followed by (at least one interval where) $X_t \geq X_{0t}$. Likewise, a deficit at time $t$ has a magnitude $S_t = X_{0t} - X_t$ so that the accumulated deficit (also called severity) $D$ is the sum of deficit $S_t$ over the drought duration $L$. The drought intensity is the average accumulated deficit, i.e., $I = D/L$. Periods of deficit and surplus are sometimes referred to as dry and wet periods, respectively. Such an objective and simple drought definition may enable one to characterize droughts using stochastic approaches.

Another subject of much interest when analyzing extreme events such as drought is the determination of the return period. In this analysis, return period of a hydrologic drought is defined as the mean interarrival time of hydrologic droughts with magnitude that are equal or greater than a certain arbitrary value $M_0$ (usually critical drought).

6.3 Methodology

Drought properties of various return periods are needed to assess the degree to which a water supply system will be able to cope with future droughts and accordingly, to plan alternate water supply strategies. They can be determined from the historical record alone by using non-parametric methods but, because the number of drought events that can be drawn from the historical sample is generally small, the “historical” drought properties have a large degree of uncertainty. In such cases, drought properties can be derived by synthetically
Estimation Drought properties

Both drought duration and severity are random in character and, therefore, probabilistic distributions are useful to describe them. In general, several droughts result in a time series of given demand level and sample size N. Assume that m droughts with durations $L_1, \ldots, L_m$ and severities $D_1, \ldots, D_m$ occur. The means($\bar{L}, \bar{D}$), standard deviation($\bar{s}_L, \bar{s}_D$) and the maximum ($L^*, D^*$) of drought duration and severity are important characteristics describing a given time series.
\[
\bar{L} = m^{-1} \sum_{j=1}^{m} L_j \quad \text{and} \quad \bar{D} = m^{-1} \sum_{j=1}^{m} D_j
\]

\[
\bar{s}_L = \left[ \frac{1}{m-1} \sum_{j=1}^{m} (L_j - \bar{L})^2 \right]^{1/2} \quad \text{and} \quad \bar{s}_D = \left[ \frac{1}{m-1} \sum_{j=1}^{m} (D_j - \bar{D})^2 \right]^{1/2}
\]

(6.1)

\[
L^* = \max[L_1, \ldots, L_m] \quad \text{and} \quad D^* = \max[D_1, \ldots, D_m]
\]

It must be noted that for a given \(N\) and demand level \(X_0t\), the above characteristics are random variables. In this analysis, the hydrologic drought properties are derived from the times series model (PARMA model) fitted to the historical river flows series by Monte Carlo simulation. That is, first a PARMA model is fitted to the observed river flow data, and a synthetic river flow series is generated from the PARMA model. Then, drought properties (e.g. maximum drought duration and severity) are determined by the runs analysis. This process is repeated several times to find the array of drought properties (e.g. maximum drought duration) whose average is an estimate of that particular property.

**Estimation Return period**

The return period of hydrologic droughts, based on the concept of stochastic processes, is derived in the following manner. Drought events \(M\) (severity \(D\) or duration \(L\)) are assumed to be independent and identically distributed. The occurrence of hydrologic drought event is observed beginning at time 0, one arbitrary point on the time axis, and illustrated in Figure 6.1. The interarrival time of drought events is defined as the period of time between two successive drought events \(L_i\), namely, the time elapsing from the initiation of a drought to the beginning of the next drought. In Figure 6.1, hydrologic drought events with magnitude equal to or greater than any arbitrary value (usually critical drought) \(M_0\), \(M \geq M_0\), are denoted by \(\otimes\). It is assumed that there occurs a hydrologic event with \(M \geq M_0\) at any time.
Once a hydrologic drought event with $M \geq M_0$ occurs, let $N$ be the number of hydrologic drought events until the occurrence of the next hydrologic drought event with $M \geq M_0$. Let $W$ denote the interarrival time between two successive hydrologic drought events with $M \geq M_0$. Then

$$W = \sum_{i=1}^{N} L_i$$

(6.2)

where $L_i$ is the drought interarrival time between any two successive drought events.

According to the above definition, the return period $T$ of a hydrologic drought event with $M \geq M_0$ is the expected value of $W$. If the drought interarrival times, $L_i$ are assumed to have the same distribution, then

$$T = EW = E \left( \sum_{i=1}^{N} L_i \right) = E(N)E(L_i).$$

(6.3)

In this analysis, the return period can be determined by Monte Carlo simulation. Recall that the return period is defined as the expected time between successive occurrences of a drought greater or equal to a specified critical drought, $M_0$. Start the simulation and continue until two successive droughts occur with magnitude greater or equal to the critical drought. Denote the time between these two droughts (interarrival time) by $W_1$. Repeat the simulation $n$ times, to obtain a random sequence $W_1, W_2, \ldots, W_n$. Then, the estimate of the mean return period is $T = n^{-1} \sum W_i$.

6.4 Example: Hydrologic Drought Analysis for the Fraser River

6.4.1 Modeling of PARMA$_{52}(1,1)$ Model Residuals

We present a detailed analysis of weekly river flow series for drought analysis. As observed in Chapter 5, the PARMA$_{52}(1,1)$ model parameters are dominated by few Fourier coefficients, and are obtained from
\[ \hat{\theta}_i = 0.255 - 0.197\cos \left( \frac{\pi i}{26} \right) + 0.121\cos \left( \frac{2\pi i}{26} \right) \]
\[ \hat{\phi}_i = 0.732 \]

and see Table 6.1 for other parameters such as \( \hat{\sigma}_t \) and \( \hat{\mu}_t \). The standardized residuals for \( \text{PARMA}_v(1, 1) \) are computed using (3.62). Inspection of the ACF and PACF of these residuals indicate that they are iid (see Figure 6.2).

Figure 6.2: (a) ACF and (b) PACF for \( \text{PARMA}_{52}(1, 1) \) model residuals of the Fraser River at Hope, BC.

A summary (histogram) of the residuals as well as probability plot for a three parameter lognormal distribution are shown in Figure 6.3, indicating that lognormal (scale = 0.094, location = 2.561 and threshold = −13.01) gives adequate fit except at the tails. This suggest for the use of a mixture of three parameter lognormal with appropriate tail distributions.

Hill’s estimator (4.12) gives \( \hat{\alpha} = 2.512 \) and \( \hat{C} = 0.379 \) for the upper tail \( (r = 215 \text{ residuals}) \), and \( \hat{\alpha} = 3.518 \) and \( \hat{C} = 0.252 \) for the lower tail \( (r = 207 \text{ residuals}) \). Using (3.70), the p-values are 0.001 and 0.588, respectively. The small p-value as well as the visual evidence of the probability plot in Figure 6.4 suggests that the Pareto model is not a good fit for the upper tail of the estimated residuals. Therefore, we decided to fit a truncated Pareto to roughly the upper 5% of the residuals. Since \( p > 0.05 \), Pareto is a good fit for the
Figure 6.3: Statistical summary (left) and lognormal probability plot (right) for \( \text{PARMA}_{52}(1,1) \) model residuals of the weekly Fraser River data, after applying significant Fourier coefficients. Compare Figure 5.5 (right).

lower tail. However, we chose a truncated Pareto for the lower 10% of the residuals because river flows are bounded from below. Then we computed \(-1.537 = \text{10th percentile and} \) 2.115 = the 95th percentile of the three parameter lognormal distribution we fit to the body of the residuals. Next we determined that \( r = 215 \) residuals exceed the 95th percentile, and \( r = 207 \) residuals fall below the 10th percentile. Then the MLE from Theorem 3.13 was used to estimate the parameters \( (\tilde{\beta} = 8.374, \tilde{\gamma} = 0.547, \tilde{\alpha} = 2.068) \) of the best fitting truncated Pareto distribution, and the theoretical distribution tail \( P(R > r) \) was plotted over the 215 largest positive residuals in Figure 6.4 (left). In Figure 6.4 (right), we used the same method to fit a truncated Pareto \( (\tilde{\beta} = 8.386, \tilde{\gamma} = 0.667, \tilde{\alpha} = 3.459) \) to the 207 largest negative residuals, after a change of sign. Both of the plots in Figures 6.4 indicate an adequate fit.

A mixture distribution with lognormal body and truncated Pareto tails was used to simulate the innovations. The mixture has cumulative distribution function (cdf)

\[
P(\delta \leq r) = \begin{cases} 
F_{-}(r) & \text{if } r < -1.537 \\
F_{0}(r) & \text{if } -1.537 \leq r \leq 2.115 \\
F_{+}(r) & \text{if } r > 2.115
\end{cases}
\]  

(6.5)
where $F_0$ is the cdf of the lognormal, and $F_+, F_-$ are truncated Pareto cdfs of the positive and negative tails, respectively. The truncated Pareto distributions were slightly shifted (by $s = -0.141$ on the positive tail and $s = 0.240$ on the negative tail) to make the mixture cdf continuous. Now innovations could be simulated by the inverse cumulative distribution function method $\delta = F^{-1}(U)$ where $U$ is a pseudorandom number uniformly distributed on the unit interval $(0, 1)$. However, this is impractical in the present case since the lognormal cdf is not analytically invertible. Instead, we used the Box-Muller method to generate standard normal random variates $Z$ (see Gentle, 2003). Then lognormal random variates were calculated using $\delta = 2.561 + \exp(2.561 + 0.094Z)$. If $R > 2.115$, the 95th percentile of the lognormal, we generated another uniform $(0, 1)$ random variate $U$ and substituted $\delta = F_+^{-1}(0.95 + 0.05U)$. If $R < -1.537$, the 10th percentile of the lognormal, we substituted $\delta = F_-^{-1}(0.1U)$. This gives simulated innovations $\delta$ with the mixture distribution (6.5).

Figure 6.5 shows a probability plot for $N = N_y \nu$ simulated innovations (for $\nu = 52$ weeks and $N_y = 100$ years) from the mixture distribution (6.5). Comparison with Figure 6.3 (right) shows that the simulated innovations are statistically identical to the computed model residuals in terms of distribution. Substituting the simulated innovations into the
model (3.63) generates $N_y$ years of simulated river flow. It is advantageous to simulate several extra years of river flows and throw out the initial years (100 years in this case), since we did not simulate $X_t$ for $t < 0$. This ensures that the simulated series is periodically stationary (see Figure 6.6). Figure 6.7 shows the main statistical characteristics (mean, standard deviation and autocorrelations) of a typical synthetic river flow time series obtained by this method, as well as the same statistical measures for the observed time series. It is apparent that PARMA$_{52}(1, 1)$ model closely reproduces the main statistical characteristics, indicating that the model can used for generating synthetic river flows in drought analysis of the Fraser River.

### 6.4.2 Drought Statistics Estimation

The main purpose of this section is to illustrate the proposed methodology for estimating drought properties based on simulated weekly river flow series of the Fraser River at Hope for a given water demand. Just as there are criteria for defining flood levels and event, criteria for defining drought years must also be developed. A more useful approach would be to define a drought year by a minimum flow required to meet (a) all water right priorities or
Figure 6.6: Plot of (a) observed and (b) simulated weekly river flows for the Fraser River at Hope, BC, indicating similarity.

(b) just municipal water requirements, and by how many years would some portion of that flow would not be met. In general, water demand is high during summer months and low in winter months. Therefore, we assumed a hypothetical sinusoidal demand curve given by

\[ D_i = 34300 + 8000\cos \left( \frac{2\pi i}{52} \right) - 191000\sin \left( \frac{2\pi i}{52} \right) \]  

(6.6)

Using PARMA_{52}(1,1) model, 1000 years and 30 realizations of weekly synthetic flow sequences for the Fraser River at Hope are generated. These synthetic flows were analyzed to ensure that the historical statistics were reproduced, and then equations (6.1) and (6.3) were applied to derive the necessary drought statistics (for example, maximum duration, severity and return periods for different values of \( L_0 \) and \( M_0 \)) for the assumed water demand (6.6). The resulting droughts (a total of 30 for each drought property) were averaged and summarized in Table 6.2. The historical data were also analyzed to determine the drought statistics (see Table 6.2), indicating few drought events (which are not useful in obtaining drought events of large return periods, say 100 years). However, we were able to determine the return periods of drought duration and severity up to 385 years and 333 years, respectively, from the generated data set.
Figure 6.7: Comparison of mean, standard deviation, and autocorrelations for simulated vs. observed weekly river flow data for the Fraser River at Hope, BC.

It is observed that in 22 realizations a drought with the largest severity is different from the drought with the longest drought duration. However, in the remaining 8 realization, a drought with the worst severity is the same as the drought with the longest duration. It must be noted the result depends on the assumed demand curve and number of years of simulation.

In summary, a reliable analysis of the drought statistics based on historical data series can not be properly carried out due to the limited number of drought events in historical data series. It is essential for the water resource planner to use stochastic models such as PARMA models to generate long records and obtain more drought events for statistical analysis of droughts.
Table 6.1: Other parameter estimates for PARMA$_{32}(1, 1)$ model of average weekly flow series for the Fraser River at Hope, BC.

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<th>Parameter</th>
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Table 6.2: Drought statistics estimation of the historical and simulated data for the Fraser River at Hope, BC. Note that the drought duration (L), severity (D) and return period (T) are in weeks, in cms and in years, respectively.

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<th>Statistics</th>
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<th>Historical</th>
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<td>Max. Duration</td>
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<td>Max. Severity*</td>
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<th>$D_0^*$</th>
<th>$T$</th>
<th>$L_0$</th>
<th>$T$</th>
<th>$D_0^*$</th>
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* The values can be multiplied by 604800 sec. so as to obtain Severity in $(m^3)$
7 Summary and Conclusions

Generation of synthetic river flow data is important in planning, design and operation of water resources systems. River flow series usually exhibit both heavy tails and periodical stationarity; that is, their mean and covariance functions are periodic with respect to time. The common procedure in modeling such periodic river flow series is first to standardize or filter the series and then fit an appropriate stationary stochastic model to the reduced series. However, standardizing or filtering most river flow series may not yield stationary residuals due to periodic autocorrelations. Periodic autoregressive moving average (PARMA) models provide a powerful tool for the modeling of periodic hydrologic series in general and river flow series in particular. PARMA models are extensions of commonly used ARMA models that allow parameters to depend on season. The PARMA modeling procedure involves iterative steps of model identification, parameter estimation, model diagnosis and fitting the residuals (noise) with a probability distribution function (pdf). The opposite process to step-by-step modeling is the use of models to generate (simulate) new samples or a long sample of the process. One starts with the random noise and its pdf by generating its sample(s). Then generate the corresponding data samples by using the fitted PARMA model.

The main purposes of this dissertation were to fit a PARMA model to represent a given river flow data, estimate parameters, check for goodness of fit to the data, model the residuals, and to use the fitted model for generating synthetic river flows and apply them to extreme analysis such as droughts. The innovations algorithm is used to obtain estimates of the PARMA model parameters. Asymptotic distributions for the innovations estimates and model parameters (which are developed here) provide us with a general technique for identifying PARMA models with the minimum number of parameters.

First the innovations algorithm estimation procedure, as well as model identification
using simulated data from different Gaussian PARMA$\nu(p, q)$ models, is discussed in detail. A simulation study demonstrates that the innovations algorithm and asymptotic distributions are an efficient and reliable technique for parameter estimation and model identification of PARMA models, respectively. However, such techniques often do not adequately identify the order of the model when the model is mixed in the sense that both the autoregressive (AR) and moving average (MA) components are present. In this case, the model identification technique can be supplemented by modeler experience or the Akaike’s information criterion (AIC). We used monthly river flow data for the Fraser River in British Columbia in order to illustrate the model identification procedure and residual modeling procedure, and to prove the ability to generate realistic synthetic river flows. This example illustrates that the innovations algorithm is useful for modeling river flow time series. For monthly average river flow data for the Fraser River, a first order periodic autoregressive moving average model is adequate to capture the essential features. A mixture of three parameter lognormal body and truncated Pareto tails fits the model residuals nicely. This mixture model is then applied with satisfactory results to generate synthetic monthly river flow records.

Secondly, we applied the same methods to monthly flow data for the Salt River in Arizona, which has heavy tails. In this case, we are able to generate realistic synthetic river flows by developing a mixture probability model (stable with truncated Pareto tails) for the residuals. The methodology presented herein provides a useful tool for river flow modeling and synthetic river flow data generation. The results allow practitioners and planners to explore realistic decision-making scenarios for a given water resource system.

Thirdly, for analysis and design of water resources systems, it is sometimes required to generate river flow data with high resolution (that is, weekly or daily values). Fitting the PARMA model to historical weekly or daily data, however, requires estimation of too
many parameters which violates the principle of parsimony (model with minimum number of parameters). In an effort to obtain a parsimonious model representing these periodically stationary series, we develop the asymptotic distribution of the discrete Fourier transform of the innovation estimates and then determine those statistically significant Fourier coefficients, which are directly applicable to the selection of significant Fourier coefficients of the parameters of pure $\text{PMA}_\nu(q)$ and $\text{PAR}_\nu(1)$ processes. For higher order PARMA models, it is more difficult to obtain an explicit representation for the Fourier transform of the model parameters, and hence the selection of significant Fourier coefficients is a complicated problem. However, we developed the asymptotic distribution of the Fourier coefficients of the model parameters for $\text{PARMA}_\nu(1,1)$. The simulation study demonstrates that these asymptotic results are efficient and reliable technique for identification of significant Fourier coefficients of PARMA model parameters. We demonstrated its application for modeling monthly and weekly Fraser River flow time series. It seems that even for monthly time series the parameters of $\text{PARMA}_{12}(1,1)$ indeed vary smoothly with respect to $t$ (dominated by a few Fourier coefficients). For weekly data series, we also found that $\text{PARMA}_{52}(1,1)$ is adequate to capture the essential features. It is true that the $\text{PARMA}_{52}(1,1)$ model parameters are dominated by few Fourier coefficients. This analysis can be easily extended to historical daily or weekly time series.

Finally, obtaining drought properties is important for planning and management of water resources system. For example, hydrologic drought properties (severity and duration) of various return periods are needed to assess the degree to which a water supply system will be able to cope with future droughts and, accordingly, to plan alternative water supply strategies. They can be determined from the historical record alone by using non-parametric methods but, because the number of drought events that can be drawn from the historical sample is
generally small, the “historical” drought properties have a large degree of uncertainty. Other alternatives for finding drought properties include using stochastic models (such as PARMA models) that can represent the underlying river flows, simulating long records of such river flows, and then deriving droughts properties from the simulated samples based on the theory of runs. In this study, we used lower order PARMA models such as PARMA$_{52}(1, 1)$ and illustrated the methodology for assessing drought properties. It seems that a PARMA$_{52}(1, 1)$ model is a good fit for the weekly Fraser River flows. Therefore, the PARMA$_{52}(1, 1)$ model was used to generate a number of synthetic flow sequences of 1000 years. These synthetic flows were analyzed to ensure that the historical statistics were reproduced, and then equations (6.1) and (6.3) were applied to derive the necessary drought statistics for the Fraser River at Hope. We were able to determine the return periods of drought duration and severity up to 385 years and 333 years, respectively, from the generated data set. We found that the methodology is useful for drought analysis in rivers where the high resolution data series (such as weekly) are described by low order PARMA models. In some cases, however, low order PARMA models may not be good fit to high resolution data series (e.g. weekly flows for the Truckee River at Farad) due to long term dependence of the weekly data series (long memory property of time series). Fractionally differenced low order PARMA models can be useful for river flows with such long memory property. This procedure can also be extended to flood frequency analysis.

The techniques presented in this research allow hydrologists, practitioners and planners to explore realistic decision-making scenarios for a given water resource system. As part of future work, comparison of this new method with the traditional approach in water resources will be made; a guideline on how (and when) to use the techniques effectively to water resources application, will be prepared.
Directions for future research include the development of asymptotic distributions for the parameters of higher order PARMA models, sensitivity analysis, more general discrete Fourier analysis for PARMA models and fractionally integrated PARMA models. Demonstration of the PARMA modeling procedure for other applications such as reservoir operation, flood analysis and river flow forecasting will also be part of future work.

As for drought analysis, the focus was on the illustration the proposed methodology for estimating drought properties based on simulated weekly river flow of the Fraser River. In the future, detailed investigation of the empirical probability distribution of the drought properties (duration and severity) both for the historical and simulated data, the goodness of fit as well as the sensitivity of the drought properties to the number of years and number of realizations will be carried out. As another case example, a detailed drought study of the Truckee River (which the main source of water supply in Truckee Meadows) will be conducted. It involves the determination of the prolonged drought characteristics for the river as well as the evaluation of drought impact on the water supply system for both present and future conditions.
References


A Basic Concepts

A-1 Definitions, Theorems, etc.

Time series is a set of observations $X_t$, each one being recorded at a specific time $t$. For example, $X_t$ can be river flow (daily, monthly, etc.) quantities at time $t$. A time series model for the observed data $\{X_t\}$ is a specification of the joint distributions (or possible only the means and covariances) of a sequence of random variables $\{X_t\}$ of which $\{x_t\}$ is postulated to be a realization. A mathematical model representing a stochastic process is called “Stochastic model” or “Time Series model”. It has a certain mathematical form or structure and a set of parameters.

A complete probabilistic time series model for the sequences of random variables $\{X_1, X_2, ...\}$ would specify all of the joint distributions of the random vectors $(X_1, X_2, ..., X_n)'$, $n = 1, 2, ...$ or equivalently all of the probabilities

$$F(x_1, x_2, ..., x_n) = P\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\} \quad -\infty < x_1, ..., x_n < \infty, \ n = 1, 2, ..$$

(A-1)

Such specification is rarely used in time series analysis (unless the data are generated by some well-understood simple mechanism), since in general it will contain far too many parameters to be estimated from the available data. Instead we specify on the first- and second-order moments of the joint distributions, i.e., the expected values $E X_t$ and the expected products $E X_{t+h}X_t$, $t = 1, 2, ..., h = 1, 2, ...$, focusing on the properties of the sequence $\{X_t\}$ which depends only on these. Such properties of $\{X_t\}$ are referred to as second-order properties. In the particular case when all the joint distributions are multivariate normal, the second-order properties of $\{X_t\}$ completely determine the joint distributions and hence give a complete probabilistic characterization of the sequence. In general we shall lose a certain amount of
information by looking at the time series “through second order spectacles”; however, the
theory of minimum mean squared error linear prediction depends only on the second order
properties, thus providing further justification for the use of the second order characterization
of time series models.

If the joint distribution can be factored into the product of the marginal distributions as
\[ F(x_1, x_2, ..., x_n) = P\{X_1 \leq x_1\}P\{X_2 \leq x_2\} \ldots P\{X_n \leq x_n\} \]
\[ = F(x_1)F(x_2)\ldots F(x_n) \] (A-2)
then process becomes an independent stochastic process and the series is an independent
series. Otherwise there is certain type of serial dependence among the variables and the pro-
cess is called a serially dependent stochastic process and correspondingly a serially dependent
time series.

Definition A.1 (The Autocovariance Function). If \{X_t, t \in T\} is a process such that
\[ \text{Var}(X_t) < \infty \text{ for each } t \in T, \] then the autocovariance function \( \gamma_X(\cdot, \cdot) \) of \{X_t\} is defined by
\[ \gamma_X(r, s) = \text{Cov}(X_r, X_s) = E[(X_r - EX_r)(X_s - EX_s)], \quad r, s \in T \] (A-3)

The autocovariance measures the linear dependence between two points on the same series
observed at different times. Very smooth series exhibit autocovariance functions that stay
large even when the \( r \) and \( s \) are far apart, whereas choppy series tend to have autocovariance
functions that are nearly zero for large separations. Recall from classical statistics that if
\( \gamma_X(r, s) = 0 \), \( X_r \) and \( X_s \) are not linearly related, but there still may be some dependence
structure between them. If, however, \( X_r \) and \( X_s \) are bivariate normal, \( \gamma_X(r, s) = 0 \) ensure
their independence.

Definition A.2 (Stationarity). The time series \{X_t, t \in Z\}, with index set \( Z = \{0, \pm 1, \pm 2, \ldots\} \),
is said to be stationary if
1. \( E|X_t|^2 < \infty \) for all \( t \in \mathbb{Z} \)

2. \( E X_t = \mu \) for all \( t \in \mathbb{Z} \) and

3. \( \gamma_X(r, s) = \gamma_X(r + t, s + t) \) for all \( r, s, t \in \mathbb{Z} \)

Remark A.3 Stationarity as just defined is frequently referred to in the literature as weak stationarity, covariance stationarity, stationarity in a wider sense or second-order stationarity.

Remark A.4 If \( \{X_t, t \in \mathbb{Z}\} \) is stationary then \( \gamma_X(r, s) = \gamma_X(r - s, 0) \) for all \( r, s \in \mathbb{Z} \).

It is therefore convenient to redefine the autocovariance function of a stationary process as function of just one variable,

\[
\gamma_X(h) \equiv \gamma_X(h, 0) = \text{Cov}(X_{t+h}, X_t), \quad \text{for all } t, h \in \mathbb{Z} \quad (A-4)
\]

The function \( \gamma_X(\cdot) \) is referred to as the autocovariance function of \( \{X_t\} \) and \( \gamma_X(h) \) as its value at “lag” \( h \). The autocorrelation function (ACF) of \( \{X_t\} \) is defined analogously as the function whose value at lag \( h \) is

\[
\rho_X(h) \equiv \gamma_X(h)/\gamma_X(0) = \text{Corr}(X_{t+h}, X_t), \quad \text{for all } t, h \in \mathbb{Z} \quad (A-5)
\]

Remark A.5 It will be noticed that we have defined stationarity only in the case when \( T = \mathbb{Z} \). It is not difficult to define stationarity using a more general index set, but for our purposes this will not be necessary. If we wish to model a set of data \( \{x_t, t \in T \subset \mathbb{Z}\} \) as a realization of a stationary process, we can always consider it to be part of the realization of a stationary process \( \{X_t, t \in \mathbb{Z}\} \).

The autocorrelation function measure the linear predictability of the series at time \( t \), say \( X_t \), using the only values \( X_s \). We can easily show that \(-1 \leq \rho(s = t + h, t) \leq 1 \) using
the Cauchy-Schwarz inequality. If we can predict \( X_t \) perfectly from \( X_s \) through a linear relationship, \( X_t = \beta_0 + \beta_1 X_s \), then the correlation will be 1 when \( \beta_1 > 0 \), -1 when \( \beta_1 < 0 \), and 0 when \( \beta_1 = 0 \). Hence, we have a rough measure of the ability to forecast the series at time \( t \) from the value at time \( s \).

**Partial autocorrelation Function (PACF)**

To define the PACF for mean-zero stationary time series, let \( X_{h-1}^h \) denote the best linear predictor of \( X_h \) based on the \( \{X_1, X_2, \ldots, X_{h-1}\} \). We note \( X_{h-1}^h \) has the form:

\[
X_{h-1}^h = \beta_1 X_{h-1} + \beta_2 X_{h-2} + \ldots + \beta_{h-1} X_1,
\]

(A-6)

where the \( \beta' \)s are chosen to minimize the mean square linear prediction error, \( E (X_h - X_{h-1}^h)^2 \).  

In addition, let \( X_{0}^{h-1} \) denote the minimum mean square linear predictor of \( X_0 \) based on \( \{X_1, X_2, \ldots, X_{h-1}\} \). \( X_{0}^{h-1} \) can be written as

\[
X_{0}^{h-1} = \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_{h-1} X_{h-1}.
\]

(A-7)

Equation (A-6) can be thought of as the linear regression of \( X_h \) on the past, \( X_{h-1}, X_{h-2}, \ldots, X_1 \), and (A-7) can be thought of as the linear regression of \( X_0 \) on the past, \( X_1, X_2, \ldots, X_{h-1} \). The coefficients, \( X_h \) on the past, \( \beta_1, \beta_2, \ldots, \beta_{h-1} \) are the same as in (A-6) and (A-7), which means that, for stationary process, linear prediction backward in time is equivalent to linear prediction forward in time.

Formally, for a stationary time series, \( X_t \), we define the partial autocorrelation function (PACF), \( \phi_{hh}, h = 1, 2, \ldots \), by

\[
\phi_{11} = \text{Corr}(X_1, X_0)
\]

(A-8)

and

\[
\phi_{hh} = \text{Corr}(X_h - X_{h-1}^h, X_0 - X_{0}^{h-1}), \quad h \geq 2
\]

(A-9)
Both \((X_h - X_{h-1})\) and \((X_0 - X_{h-1})\) are uncorrelated with \(\{X_1, X_2, \ldots, X_{h-1}\}\). By stationarity, the PACF, \(\phi_{hh}\), is the correlation between \(X_t\) and \(X_{t-h}\) with the linear effect of \(\{X_{t-1}, X_{t-2}, \ldots, X_{t-(h-1)}\}\), on each, removed. If the process \(X_t\) is Gaussian, then

\[
\phi_{hh} = \text{Corr}(X_t, X_{t-h}|X_{t-1}, X_{t-2}, \ldots, X_{t-(h-1)}).
\]

That is, \(\phi_{hh}\) is the correlation coefficient between \(X_t\) and \(X_{t-h}\) in the bivariate distribution of \((X_t, X_{t-h})\) conditional on \(\{X_{t-1}, X_{t-2}, \ldots, X_{t-(h-1)}\}\).

**Definition A.6 (Strict Stationarity).** The time series \(\{X_t, t \in \mathbb{Z}\}\) is said to be strictly stationary if the joint distribution of \((X_{t_1}, \ldots, X_{t_k})'\) and \((X_{t_1+h}, \ldots, X_{t_k+h})'\) are the same for all positive integers \(k\) and for all \(t_1, \ldots, t_k, h \in \mathbb{Z}\).

**Remark A.7** Definition A.6 is equivalent to the statement that \((X_1, \ldots, X_k)'\) and \((X_{1+h}, \ldots, X_{k+h})'\) have the same joint distributions for all integers \(h\) and \(k > 0\).

If \(\{X_t\}\) is strictly stationary it immediately follows, on taking \(k = 1\) in Definition A.6, that \(X_t\) has the same distribution for each \(t \in \mathbb{Z}\). If \(E|X_t|^2 < \infty\) this implies in particular that \(E X_t\) and \(\text{Var}(X_t)\) are both constant. Moreover, taking \(k = 2\) in Definition A.6, we find that \(X_{t+h}\) and \(X_t\) have the same joint distribution and hence the same covariance for all \(h \in \mathbb{Z}\).

Thus a strictly stationary process with finite second moments is stationary.

The converse of the previous statement is not true. For example if \(X_t\) is a sequence of independent random variables such that \(X_t\) is exponentially distributed with mean one when \(t\) is odd and normally distributed with mean one and variance one when \(t\) is even, then \(X_t\) is stationary with \(\gamma_X(0) = 1\) and \(\gamma_X(h) = 0\) for \(h \neq 0\). However since \(X_1\) and \(X_2\) have different distributions, and \(X_t\) can not be strictly stationary.

There is one important case however in which stationarity does imply strict stationarity.
Definition A.8 (Gaussian Time Series). The process $X_t$ is a Gaussian time series if and only if the distribution functions of $X_t$ are all multivariate normal.

If $\{X_t, t \in \mathbb{Z}\}$ is a stationary Gaussian process then $\{X_t\}$ is strictly stationary, since for all $n \in \{1, 2, \ldots\}$ and for all $h, t_1, t_1, \ldots \in \mathbb{Z}$, the random vectors $(X_{t_1}, \ldots, X_{t_n})'$ and $(X_{t_1+h}, \ldots, X_{t_n+h})'$ have the same mean and covariance matrix, and hence the same distribution.

Definition A.9 The time series $X_t$ is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all $t$, where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Note that $WN(0, \sigma^2)$ is white noise (sequence of uncorrelated random variables) with mean 0 and variance $\sigma^2$.

A-2 Statistical Properties of Time Series

A stationary process $X_t$ is characterized, at least from second-order point of view, by its mean $\mu$ and its autocovariance function $\gamma(.)$. The estimation of $\mu, \gamma(.)$ and the autocorrelation function $\rho(.) = \gamma(.)/\gamma(0)$ from observations $X_1, \ldots, X_n$ therefore plays a crucial role in problem of inference and in particular in the problem of constructing an appropriate model for the data.

For a set of observations $X_1, \ldots, X_n$, the sample mean, autocovariance and autocorrelation functions are defined by
\[\hat{\mu} = \bar{X}_n = n^{-1} \sum_{t=1}^{n} X_t\]
\[\hat{\gamma}(h) = (n)^{-1} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \bar{X}_n)(X_t - \bar{X}_n)\]
\[\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}\]  

Definition A.10 Let \(\{X_n\}\) be a sequence of random variables, and \(X\) a random variable. Then we say that \(X_n\) converges in probability to \(X\) if

\[\forall \epsilon > 0 \quad P(|X_n - X| > \epsilon) \to 0\]  
\[\text{as } n \to \infty.\]  

We denote convergence in probability by \(\overset{\text{p}}{\to}\).

Definition A.11 Let \(\{X_n\}\) be a sequence of random variables and \(X\) an random variable. Then we say that \(X_n\) converges in distribution to \(X\) if

\[F_n(x) \to F(x)\]  
\[\text{for all } x \text{ such that } F(x) \text{ is continuous, as } n \to \infty\]  

where \(F_n(x)\) is the distribution of \(X_n\), and \(F(x)\) is the distribution of \(X\). We denote convergence in distribution by \(\Rightarrow\).

Definition A.12 A sequence of normal random variables \(\{X_n\}\) is said to be asymptotical normal with “mean” \(\mu_n\) and “standard deviations” \(\sigma_n\) if \(\sigma_n > 0\) for \(n\) sufficiently large and

\[\sigma_n^{-1}(X_n - \mu_n) \Rightarrow Z,\]  
\[\text{where } Z \sim \mathcal{N}(0,1).\]  

We shall abbreviate this as

\[X_n \sim \text{AN}(\mu_n, \sigma_n^2)\]  

where \(\sim\) will denote “is distributed as”.
Theorem A.13 If $X_t$ is a stationary linear process of the form (A-10) and $\sum_j \psi_j \neq 0$, then

$$\bar{X}_n \sim AN(\mu_X, n^{-1}V),$$  \hspace{1cm} (A-16)

where

$$V = \sum_{h=-\infty}^{\infty} \gamma(h) = \sigma^2 \left( \sum_{j=-\infty}^{\infty} \psi_j \right)^2$$  \hspace{1cm} (A-17)

and $\gamma(.)$ is the autocovariance function of $X_t$

PROOF. See Theorem 1.5 of Shumway and Stoffer (2000) for a proof.

Theorem A.14 If $X_t$ is a stationary linear process of the form (A-10) satisfying the fourth moment condition (i.e., $E Z_t^4 = \eta \sigma^4 < \infty$ where $\eta$ is some constant), then

$$\begin{pmatrix} \hat{\gamma}(0) \\ \hat{\gamma}(1) \\ \vdots \\ \hat{\gamma}(K) \end{pmatrix} \sim AN \begin{pmatrix} \gamma(0) \\ \gamma(1) \\ \vdots \\ \gamma(K) \end{pmatrix}, n^{-1}V$$  \hspace{1cm} (A-18)

where $V$ is the matrix with elements given by

$$v_{pq} = (\eta - 3)\sigma^4 \gamma(p)\gamma(q) + \sum_{u=-\infty}^{\infty} [\gamma(u)\gamma(u-p+q) + \gamma(u+q)\gamma(u-p)]$$  \hspace{1cm} (A-19)

PROOF. See Theorem 1.6 of Shumway and Stoffer (2000) for a proof.

Theorem A.15 If $X_t$ is a stationary linear process of the form (A-10) satisfying the fourth moment condition (i.e., $E Z_t^4 = \eta \sigma^4 < \infty$ where $\eta$ is some constant), then

$$\begin{pmatrix} \hat{\rho}(0) \\ \hat{\rho}(1) \\ \vdots \\ \hat{\rho}(K) \end{pmatrix} \sim AN \begin{pmatrix} \rho(0) \\ \rho(1) \\ \vdots \\ \rho(K) \end{pmatrix}, n^{-1}W$$  \hspace{1cm} (A-20)
where $W$ is the matrix with elements given by

$$w_{pq} = \sum_{u=1}^{\infty} [\rho(u+p) + \rho(u-p) - 2\rho(p)\rho(u)] \times [\rho(u+q) + \rho(u-q) - 2\rho(q)\rho(u)]$$

(A-21)

PROOF. See Theorem 1.7 of Shumway and Stoffer (2000) for a proof.

### A-3 Spectral Representation of a Stationary Process

Suppose that $\{X_t\}$ is a non-zero stationary time series with autocovariance function $\gamma(.)$ satisfying that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. The spectral density of $\{X_t\}$ is the function $f(.)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \text{Exp}(-ih\lambda)\gamma(h), \quad -\infty < \lambda < \infty$$

(A-22)

where $\text{Exp}(-i\lambda) = \cos(\lambda) + i\sin(\lambda)$ and $i = \sqrt{-1}$.

**Basic properties of $f$**

1. $f$ is even, i.e., $f(\lambda) = f(-\lambda)$

2. $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$ and

3. $\gamma(k) = \int_{-\pi}^{\pi} \text{Exp}(ik\lambda)f(\lambda)d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda)f(\lambda)d\lambda$.

**Definition A.16** A function $f$ is the spectral density of a stationary time series $\{X_t\}$ with ACVF $\gamma(.)$ if

1. $f(\lambda) \geq 0$ for all $\lambda \in (0, \pi]$

2. $\gamma(h) = \int_{-\pi}^{\pi} \text{Exp}(ih\lambda)f(\lambda)d\lambda$ for all integers $h$

**Remark A.17** Spectral densities are essentially unique. That is, if $f$ and $g$ are two spectral densities corresponding the autocovariance function $\gamma(.)$, i.e., $\gamma(h) = \int_{-\pi}^{\pi} \text{Exp}(ih\lambda)f(\lambda)d\lambda =$
\[ \int_{-\pi}^{\pi} \exp(ih\lambda)g(\lambda)d\lambda \] for all integers \( h \), then \( f \) and \( g \) have the same Fourier coefficients and hence are equal.

**Proposition A.18** A real-valued function \( f \) defined on \((-\pi, \pi]\) is the spectral density of a stationary process if and only if

1. \( f(\lambda) = f(-\lambda) \)

2. \( f(\lambda) \geq 0 \) and

3. \( \int_{-\pi}^{\pi} f(\lambda)d\lambda < \infty \)

**PROOF.** See Proposition 4.1.1 of Brockwell and Davis (1996) for a proof.

**Corollary A.19** An absolutely summable function \( \gamma(.) \) is the autocovariance function of a stationary time series if and only if it is even and

\[
 f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-ih\lambda)\gamma(h), \quad -\infty < \lambda < \infty \tag{A-23}
\]

in which case \( f(.) \) is the spectral density of \( \gamma(.) \)

**PROOF.** See Corollary 4.1.1 of Brockwell and Davis (1996) for a proof.

**Theorem A.20** (Spectral representation of the ACVF) A function \( \gamma(.) \) defined on the integers is the AVCF of a stationary time series if and only if there exists a right-continuous, nondecreasing, bounded function \( F \) on \([-\pi, \pi]\) with \( F(-\pi) = 0 \) such that

\[
 \gamma(h) = \int_{(-\pi,\pi]} \exp(ih\lambda)dF(\lambda) \tag{A-24}
\]

for all integers \( h \). (For real-valued time series, \( F \) is symmetric in the sense that \( \int_{(a,b]} dF(x) = \int_{(-b,-a)} dF(x) \) for all \( a \) and \( b \) such that \( 0 < a < b \).)
PROOF. See Theorem 4.1.1 of Brockwell and Davis (1996) for a proof.

Remark A.21 The function $F$ is a generalized distribution function on $[-\pi, \pi]$ if $G(\lambda) = F(\lambda)/F(\pi)$ is a probability distribution function on $[-\pi, \pi]$. Note that since $F(\pi) = \gamma(0) = \text{Var}(X_1)$, the ACF of $\{X_t\}$ has spectral representation

$$\rho(h) = \int_{(-\pi,\pi]} \text{Exp}(ih\lambda)dG(\lambda).$$ (A-25)

The function $F$ in (A-25) is called the spectral distribution function of $\gamma(.)$. If $F(\lambda)$ can be expressed as $F(\lambda) = \int_{-\pi}^{\lambda} f(y)dy$ for all $\lambda \in [-\pi, \pi]$, then $f$ is the spectral density function and the time series is said to have a continuous spectrum. If $F$ is a discrete distribution (i.e., if $G$ is a discrete probability distribution), then the time series is said to have a discrete spectrum.

A-4 Simulation of Continuous Random Variates

Generating Uniform Random Variates

Consider uniform continuous-values deviates $X$ that lie in the range $[0, 1]$. The probability density function (pdf) of $X$ is

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$ (A-26)

The mean and standard deviation for this distribution are $\mu = 1/2, \sigma^2 = 1/12$

Because many statistical methods rely on random samples, applied statisticians often need a source of “random numbers”. Nowadays, a computer is used to generate “random” (psuedorandom) numbers directly. Psuedorandom numbers are meant to simulate random numbers.
A number of different methods for generating pseudo-random uniform numbers (Gentle, 2000). We describe here a linear congruential generators. The linear congruential method produces a sequence of integers $X_1, X_2, \ldots$ between zero and $m-1$ according to the following recursive relationship: $X_{i+1} = (aX_i + c) \mod m$, $i = 0, 1, 2, \ldots$. The initial value $X_0$ is called the seed, $a$ is called the constant multiplier, $c$ is the increment, and $m$ is the modulus. All values are integers. If $c \neq 0$, the form is called the mixed congruential method. If $c = 0$, the form is known as the multiplicative congruential method.

Each $X_i$ is scaled into a unit interval $(0, 1)$ by division by $m$, that is

$$u_i = \frac{X_i}{m}$$

(A-27)

If $a$ and $m$ are properly chosen, the $u_i$s will “look like” they are randomly and uniformly distributed between 0 and 1.

The selection of the values for $a$, $c$, $m$ and $X_0$ are of vital importance to the quality of the generator. If they are not chosen properly, the generator may not have the longest possible period, or generated numbers may not exhibit good randomness, or the generator may not be efficiently implementable. For example, the book by Press et al. (1996), suggests that $a = 16807$ and $m = 2^{31} - 1$.

**The Inverse Transformation Method**

A general method for simulating a random variable having a continuous distribution - called the inverse transformation method-is based on the following proposition.

**Proposition A.22** Let $U$ be a uniform $(0, 1)$ random variable. For any continuous distribution function $F$ if we define the random variable $X$ by

$$X = F^{-1}(U)$$

(A-28)
then the random variable \(X\) has the distribution function \(F\). \([F^{-1}(u)\) is defined to equal that value \(x\) for which \(F(x) = u]\)

PROOF. See Proposition 11.1 Ross (2000) for a proof.

**Generating Normal Random Variates** A simple method for generating standard normal random variates (i.e., with zero mean and unit standard deviation) is based on Box-Müller method arising from a polar transformation: If \(U_1\) and \(U_2\) are independently distributed as \(U(0,1)\), and

\[
X_1 = \sqrt{-2 \log(U_1)} \cos(2\pi U_2) \tag{A-29}
\]

\[
X_2 = \sqrt{-2 \log(U_1)} \sin(2\pi U_2)
\]

then \(X_1\) and \(X_2\) are independently distributed as \(\mathcal{N}(0,1)\).

**A-5 Akaike Information Criterion**

A mathematical formulation which considers the principle of parsimony in model building is the Akaike Information Criterion (AIC) proposed by Akaike (1974). For comparing among competing ARMA \((p, q)\) models he used

\[
AIC(p, q) = N \ln(\hat{\sigma}_e^2) + 2(p + q)
\]

where \(N\) is the sample size and \(\hat{\sigma}_e^2\) is the maximum likelihood estimate of the residual variance. Akaike suggested such criterion to select the correct model among competing ARMA models. Under this criterion the model which gives the minimum AIC is the one to be selected. In case of PARMA\(_\nu(p, q)\), the AIC is given within an additive constant\((C)\) by

\[
AIC = -2 \ln L(\phi(\ell), \theta(\ell), \sigma) + C
\]

where \(\phi(\ell) = [\phi_0(\ell), \phi_1(\ell), \ldots, \phi_{\nu-1}(\ell)]^T\), \(1 \leq \ell \leq p\); \(\theta(\ell) = [\theta_0(\ell), \theta_1(\ell), \ldots, \theta_{\nu-1}(\ell)]^T\), \(1 \leq \ell \leq q\); \(\sigma = [\sigma_0, \sigma_1, \ldots, \sigma_{\nu-1}]^T\) and \(L(\phi(\ell), \theta(\ell), \sigma)\) is the likelihood function. See, for example,
Lund and Basawa (2000) for the Gaussian likelihood of the PARMA model.

A-6 The Bonferroni Inequality and Simultaneous Confidence Intervals

**Proposition A.23** (The Bonferroni Inequality). If $A_1, \ldots, A_k$ are events (not necessarily independent) which have probability $1 - p$ of occurring, then the probability they all occur is at least $1 - kp$.

**Proof:** The probability that $A_1, \ldots, A_k$ all occur is

\[
P(A_1 \cap A_2 \cap \ldots \cap A_k) = 1 - P(A_1^c \cup A_2^c \cup \ldots \cup A_k^c) \tag{A-30}
\]

i.e., the probability they all occur is one minus the probability at least one does not occur. Now,

\[
P(A_1^c \cup A_2^c \cup \ldots \cup A_k^c) \leq P(A_1^c) + P(A_2^c) + \ldots + P(A_k^c) \tag{A-31}
\]

The right side of the above equal to $kp$ since $P[A_i^c] = 1 - (1 - p) = p$. Therefore,

\[
P(A_1 \cap A_2 \cap \ldots \cap A_k) = 1 - P(A_1^c \cup A_2^c \cup \ldots \cup A_k^c) \geq 1 - kp \tag{A-32}
\]

**Use with the confidence intervals**

A $1 - \alpha'$ confidence interval for a population parameter is constructed such that, before we collect the data, there is a probability $1 - \alpha$ that the confidence interval will contain the parameter. However, if we are going to construct $k$ confidence intervals, each with level $1 - \alpha$, for different parameters from the same data set, the probability that the confidence intervals all contain their respective parameters is less than $1 - \alpha$. One way to ensure that this probability is at least $1 - \alpha$ to make each confidence interval individually have level $1 - \alpha/k$. Then, by the Bonferroni inequality, the probability they will all contain their respective
parameters is at least \(1 - k(\alpha/k) = 1 - \alpha\). When we construct confidence intervals such that the confidence that they are all correct is \(1 - \alpha\), we call these ‘simultaneous \(1 - \alpha\) level confidence intervals’. For example, suppose we wish to construct 95% confidence intervals for the population means of 5 variables measured on the same sample. Then \(\alpha = 0.05\) so we need to make each confidence interval separately a \(1 - (0.05/5) = 0.99\) level confidence interval. We use the \(t\) procedure to construct a 99% confidence interval for each mean. We then call the set of 5 confidence intervals ‘simultaneous 99% confidence intervals for the population means’ or, to be more specific, ‘Benferroni simultaneous 99% confidence intervals’. You could also these 5 confidence intervals ‘individual 99% confidence intervals’. Including the word ‘individual’ indicates that the confidence level applies to each interval separately and not to the whole group simultaneously.
B Vector Difference Equation for Discrete Fourier Coefficients

We wish to relate the Fourier coefficients of $\psi(j)$ to the Fourier coefficients of the model parameters. Let $\phi(\ell) = [\phi_0(\ell), \phi_1(\ell), \ldots, \phi_{\nu-1}(\ell)]^T$, $1 \leq \ell \leq p$, be the vector of periodic autoregressive parameters at lag $\ell$, $\theta(\ell) = [\theta_0(\ell), \theta_1(\ell), \ldots, \theta_{\nu-1}(\ell)]^T$, $1 \leq \ell \leq q$, be the vector of periodic moving average parameters at lag $\ell$, and let $\sigma = [\sigma_0, \sigma_1, \ldots, \sigma_{\nu-1}]^T$ denote the periodic residual standard deviation parameters. These model parameters may be defined in terms of their finite Fourier coefficients $\phi^*_t(\ell), \theta^*_t(\ell)$ and $\sigma^*_t$ as follows

$$
\phi^*(\ell) = U\phi(\ell) \quad \text{and} \quad \phi(\ell) = \tilde{U}^T\phi^*(\ell)
\theta^*(\ell) = U\theta(\ell) \quad \text{and} \quad \theta(\ell) = \tilde{U}^T\theta^*(\ell)
$$

(B-1)

$$\sigma^* = U\sigma \quad \text{and} \quad \sigma = \tilde{U}^T\sigma^*$$

where "~" denotes the complex conjugate and $U$ is the $\nu \times \nu$ Fourier transform matrix defined in (5.5).

Using the Fourier transformed of $\psi^*(\ell) = U\psi(\ell)$ and $\psi(\ell) = \tilde{U}^T\psi^*(\ell)$, then (3.38) leads to

$$
\begin{cases}
\psi^*(j) - \sum_{k=1}^p UA_k\Pi^{-k}\tilde{U}^T\psi^*(j - k) = 0 & j \geq \text{Max}(p, q + 1) \\
\psi^*(j) - \sum_{k=1}^j UA_k\Pi^{-k}\tilde{U}^T\psi^*(j - k) = -\theta^*(j) & 0 \leq j \leq \text{Max}(p, q + 1)
\end{cases}
$$

(B-2)

Note that $\psi^*_t(0) = 1$.

Define

$$D_j = \nu^{-1/2}\text{diag}\{1, w^j, w^{2j}, \ldots, w^{(\nu-1)j}\}$$

(B-3)

where $w^q = e^{(\frac{2\pi q}{\nu})}$ and $w^{-q} = w^{\nu-q}$ with $e^{iy} = \cos y + isin y$ (the complex exponential function). By properties of the Fourier transform matrix $U$ (Brockwell and Davis, 1991), we
have

\[ D_{v+j} = D_j = \nu^{-1/2} U \Pi^{-j} \tilde{U}^T \]  

(B-4)

also note that \( U \) is the unitary matrix \( (U \tilde{U}^T = I) \).

**Theorem B.1** If we define \( D_j \) as in (B-3), then

\[ D_j = \Pi^m U \tilde{D}_m \Pi^{-j} \tilde{U}^T \]  

(B-5)

where \( \tilde{D}_j = \nu^{-1/2} \text{diag}\{1, w^{\nu-j}, w^{\nu-2j}, \ldots, w^{\nu-(\nu-1)j}\} \)


Writing

\[ A_\ell = \sum_{m=0}^{\nu-1} \phi_m^*(\ell) \tilde{D}_m \]  

(B-6)

where \( \phi_m^*(\ell) \) is the \( m^{th} \) element of \( \phi^*(\ell) \). Then (B-2) becomes

\[
\begin{align*}
\left\{ \begin{array}{l}
\psi^*(j) - \sum_{k=1}^p \left( \sum_{m=0}^{\nu-1} \phi_m^*(k) \Pi^{-m} \right) D_k \psi^*(j-k) = 0 & j \geq \text{Max}(p, q+1) \\
\psi^*(j) - \sum_{k=1}^j \left( \sum_{m=0}^{\nu-1} \phi_m^*(k) \Pi^{-m} \right) D_k \psi^*(j-k) = -\theta^*(j) & 0 \leq j \leq \text{Max}(p, q+1)
\end{array} \right.
\end{align*}
\]  

(B-7)

Thus equation (B-7) gives the desired difference equation relating the Fourier transformed \( \psi(j) \) to the Fourier coefficients of the model parameters. It may be further simplified by defining

\[ P_\ell = \left( \sum_{m=0}^{\nu-1} \phi_m^*(\ell) \Pi^{-m} \right)^T \]  

(B-8)

then observing that

\[ P_\ell = \text{circ}\{\phi_0^*(\ell), \phi_1^*(\ell), \ldots, \phi_{\nu-1}^*(\ell)\} \]  

(B-9)
Because the model parameters are real-valued, it follows that $\phi_k^*(j) = \tilde{\phi}_{r-k}^*(j)$ (Fuller 1996). This implies that $P_\ell$ is circulant as well as Hermitian (i.e., $P_\ell = \tilde{P}_\ell^T$). Using this fact along the substitution of (B-8) into (B-7) gives

\[
\begin{align*}
\psi^*(j) - \sum_{k=1}^p \tilde{P}_k D_k \psi^*(j - k) &= 0 & j \geq \text{Max}(p, q + 1) \\
\psi^*(j) - \sum_{k=1}^j \tilde{P}_k D_k \psi^*(j - k) &= -\theta^*(j) & 0 \leq j \leq \text{Max}(p, q + 1)
\end{align*}
\]  

(B-10)

(a) Periodic Moving Average Processes

The vector difference equation concerning the Fourier transformed innovation estimates of a PMA(q) process given by (B-10) is

\[
\begin{align*}
\psi^*(j) &= -\theta^*(j) & 0 \leq j \leq q \\
\psi^*(j) &= 0 & j > q
\end{align*}
\]  

(B-11)

(b) Periodic Autoregressive Processes

The vector difference equation concerning the Fourier transformed innovation estimates of a PAR(p) process given by (B-10) is

\[
\psi^*(j) = \sum_{k=1}^p \tilde{P}_k D_k \psi^*(j - k) & j \geq p
\]  

(B-12)
C Figures and Tables

Figure C-1: Normal probability plots of the residuals (for k=15) from PARMA$_4(0, 1)$, support the normality assumptions of residuals.

Figure C-2: Normal probability plots of the residuals (for k=15) from PARMA$_4(1, 1)$, support the normality assumptions of residuals.
Table C-1: Moving average parameter estimates and \( p \)-values after \( k = 20 \) iterations of the innovations algorithm applied to \( N_y = 50, 100, 300 \) years of simulated PARMA\(_4(0,1)\) data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
</tbody>
</table>

**Number of Years, \( N_y = 50 \)**

<table>
<thead>
<tr>
<th>Lag</th>
<th>( \hat{\psi}_0(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_1(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_2(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_3(\ell) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.253</td>
<td>0.002</td>
<td>0.425</td>
<td>0.178</td>
<td>0.889</td>
<td>0.000</td>
<td>0.621</td>
<td>0.212</td>
</tr>
<tr>
<td>2</td>
<td>0.247</td>
<td>0.430</td>
<td>-0.066</td>
<td>0.718</td>
<td>-0.279</td>
<td>0.332</td>
<td>-0.039</td>
<td>0.734</td>
</tr>
<tr>
<td>3</td>
<td>-0.036</td>
<td>0.610</td>
<td>1.215</td>
<td>0.058</td>
<td>-0.114</td>
<td>0.490</td>
<td>0.294</td>
<td>0.238</td>
</tr>
<tr>
<td>4</td>
<td>0.230</td>
<td>0.138</td>
<td>0.190</td>
<td>0.204</td>
<td>1.218</td>
<td>0.038</td>
<td>-0.355</td>
<td>0.014</td>
</tr>
<tr>
<td>( \hat{\sigma} )</td>
<td>0.630</td>
<td>1.399</td>
<td>0.313</td>
<td>1.099</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Number of Years, \( N_y = 100 \)**

<table>
<thead>
<tr>
<th>Lag</th>
<th>( \hat{\psi}_0(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_1(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_2(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_3(\ell) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.179</td>
<td>0.012</td>
<td>0.687</td>
<td>0.000</td>
<td>0.932</td>
<td>0.000</td>
<td>0.394</td>
<td>0.088</td>
</tr>
<tr>
<td>2</td>
<td>-0.020</td>
<td>0.904</td>
<td>0.004</td>
<td>0.976</td>
<td>0.051</td>
<td>0.794</td>
<td>0.081</td>
<td>0.250</td>
</tr>
<tr>
<td>3</td>
<td>-0.022</td>
<td>0.668</td>
<td>-0.223</td>
<td>0.522</td>
<td>-0.009</td>
<td>0.952</td>
<td>0.064</td>
<td>0.652</td>
</tr>
<tr>
<td>4</td>
<td>-0.039</td>
<td>0.704</td>
<td>0.077</td>
<td>0.466</td>
<td>-0.160</td>
<td>0.624</td>
<td>-0.038</td>
<td>0.712</td>
</tr>
<tr>
<td>( \hat{\sigma} )</td>
<td>0.824</td>
<td>1.661</td>
<td>0.503</td>
<td>1.162</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Number of Years, \( N_y = 300 \)**

<table>
<thead>
<tr>
<th>Lag</th>
<th>( \hat{\psi}_0(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_1(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_2(\ell) )</th>
<th>( p )</th>
<th>( \hat{\psi}_3(\ell) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.250</td>
<td>0.000</td>
<td>0.640</td>
<td>0.000</td>
<td>0.899</td>
<td>0.000</td>
<td>0.399</td>
<td>0.008</td>
</tr>
<tr>
<td>2</td>
<td>0.289</td>
<td>0.016</td>
<td>0.061</td>
<td>0.516</td>
<td>-0.009</td>
<td>0.936</td>
<td>0.064</td>
<td>0.092</td>
</tr>
<tr>
<td>3</td>
<td>0.041</td>
<td>0.174</td>
<td>0.054</td>
<td>0.826</td>
<td>0.044</td>
<td>0.604</td>
<td>-0.121</td>
<td>0.122</td>
</tr>
<tr>
<td>4</td>
<td>-0.049</td>
<td>0.424</td>
<td>0.078</td>
<td>0.198</td>
<td>-0.043</td>
<td>0.842</td>
<td>-0.042</td>
<td>0.472</td>
</tr>
<tr>
<td>( \hat{\sigma} )</td>
<td>0.844</td>
<td>1.736</td>
<td>0.428</td>
<td>1.123</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table C-2: Moving average parameter estimates and $p$-values after $k = 15$ iterations of the innovations algorithm applied to $N_y = 50, 100, 300$ years of simulated PARMA$_4(0,1)$ data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 50$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.217</td>
<td>0.012</td>
<td>0.692</td>
<td>0.032</td>
<td>0.879</td>
<td>0.000</td>
<td>0.703</td>
<td>0.106</td>
</tr>
<tr>
<td>2</td>
<td>0.171</td>
<td>0.536</td>
<td>-0.241</td>
<td>0.234</td>
<td>-0.099</td>
<td>0.734</td>
<td>-0.057</td>
<td>0.590</td>
</tr>
<tr>
<td>3</td>
<td>0.018</td>
<td>0.788</td>
<td>1.108</td>
<td>0.078</td>
<td>-0.090</td>
<td>0.610</td>
<td>0.279</td>
<td>0.250</td>
</tr>
<tr>
<td>4</td>
<td>0.228</td>
<td>0.132</td>
<td>0.190</td>
<td>0.218</td>
<td>0.747</td>
<td>0.168</td>
<td>-0.326</td>
<td>0.028</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.671</td>
<td>1.530</td>
<td>0.364</td>
<td>1.115</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 100$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.372</td>
<td>0.000</td>
<td>0.687</td>
<td>0.000</td>
<td>0.881</td>
<td>0.000</td>
<td>0.268</td>
<td>0.276</td>
</tr>
<tr>
<td>2</td>
<td>0.108</td>
<td>0.596</td>
<td>0.162</td>
<td>0.266</td>
<td>0.027</td>
<td>0.872</td>
<td>-0.114</td>
<td>0.124</td>
</tr>
<tr>
<td>3</td>
<td>0.014</td>
<td>0.818</td>
<td>0.380</td>
<td>0.294</td>
<td>0.061</td>
<td>0.632</td>
<td>0.105</td>
<td>0.448</td>
</tr>
<tr>
<td>4</td>
<td>0.046</td>
<td>0.682</td>
<td>0.011</td>
<td>0.920</td>
<td>0.209</td>
<td>0.502</td>
<td>-0.019</td>
<td>0.850</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.836</td>
<td>1.541</td>
<td>0.459</td>
<td>1.131</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 300$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.268</td>
<td>0.048</td>
<td>0.634</td>
<td>0.000</td>
<td>0.891</td>
<td>0.000</td>
<td>0.385</td>
<td>0.004</td>
</tr>
<tr>
<td>2</td>
<td>0.104</td>
<td>0.348</td>
<td>0.108</td>
<td>0.226</td>
<td>-0.033</td>
<td>0.748</td>
<td>0.005</td>
<td>0.904</td>
</tr>
<tr>
<td>3</td>
<td>0.004</td>
<td>0.904</td>
<td>-0.116</td>
<td>0.582</td>
<td>0.081</td>
<td>0.308</td>
<td>-0.074</td>
<td>0.332</td>
</tr>
<tr>
<td>4</td>
<td>-0.083</td>
<td>0.174</td>
<td>0.049</td>
<td>0.418</td>
<td>-0.083</td>
<td>0.660</td>
<td>-0.025</td>
<td>0.668</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.880</td>
<td>1.667</td>
<td>0.487</td>
<td>1.144</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table C-3: Moving average parameter estimates and $p$–values after $k = 10$ iterations of the innovations algorithm applied to $N_y = 50, 100, 300$ years of simulated PARMA$_4$(0,1) data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 50$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.187</td>
<td>0.026</td>
<td>0.457</td>
<td>0.184</td>
<td>0.879</td>
<td>0.000</td>
<td>0.394</td>
<td>0.378</td>
</tr>
<tr>
<td>2</td>
<td>0.215</td>
<td>0.442</td>
<td>-0.108</td>
<td>0.604</td>
<td>-0.299</td>
<td>0.338</td>
<td>-0.061</td>
<td>0.528</td>
</tr>
<tr>
<td>3</td>
<td>0.020</td>
<td>0.748</td>
<td>0.573</td>
<td>0.390</td>
<td>-0.143</td>
<td>0.448</td>
<td>0.279</td>
<td>0.242</td>
</tr>
<tr>
<td>4</td>
<td>0.343</td>
<td>0.022</td>
<td>0.131</td>
<td>0.368</td>
<td>0.486</td>
<td>0.418</td>
<td>-0.250</td>
<td>0.086</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.736</td>
<td>1.792</td>
<td>0.389</td>
<td>1.229</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 100$

<table>
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<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.369</td>
<td>0.000</td>
<td>0.690</td>
<td>0.000</td>
<td>0.888</td>
<td>0.000</td>
<td>0.279</td>
<td>0.250</td>
</tr>
<tr>
<td>2</td>
<td>0.087</td>
<td>0.674</td>
<td>0.143</td>
<td>0.332</td>
<td>0.070</td>
<td>0.682</td>
<td>-0.112</td>
<td>0.126</td>
</tr>
<tr>
<td>3</td>
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<td>0.904</td>
<td>0.460</td>
<td>0.200</td>
<td>0.045</td>
<td>0.726</td>
<td>0.103</td>
<td>0.436</td>
</tr>
<tr>
<td>4</td>
<td>0.057</td>
<td>0.604</td>
<td>0.031</td>
<td>0.772</td>
<td>0.315</td>
<td>0.318</td>
<td>-0.017</td>
<td>0.866</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.876</td>
<td>1.570</td>
<td>0.470</td>
<td>1.139</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 300$

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<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.267</td>
<td>0.000</td>
<td>0.630</td>
<td>0.000</td>
<td>0.892</td>
<td>0.000</td>
<td>0.389</td>
<td>0.004</td>
</tr>
<tr>
<td>2</td>
<td>0.094</td>
<td>0.396</td>
<td>0.099</td>
<td>0.272</td>
<td>-0.040</td>
<td>0.696</td>
<td>0.008</td>
<td>0.842</td>
</tr>
<tr>
<td>3</td>
<td>0.001</td>
<td>0.968</td>
<td>-0.119</td>
<td>0.568</td>
<td>0.067</td>
<td>0.400</td>
<td>-0.075</td>
<td>0.318</td>
</tr>
<tr>
<td>4</td>
<td>-0.085</td>
<td>0.164</td>
<td>0.049</td>
<td>0.424</td>
<td>-0.095</td>
<td>0.610</td>
<td>-0.028</td>
<td>0.632</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.880</td>
<td>1.667</td>
<td>0.487</td>
<td>1.144</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table C-4: Moving average parameter estimates and $p$–values after $k = 20$ iterations of the innovations algorithm applied to $N_y = 50, 100, 300$ years of simulated PARMA$_4(1, 1)$ data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-0.90</td>
<td>0.50</td>
<td>0.80</td>
<td>0.25</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
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</table>

Number of Years, $N_y = 50$

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<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.794</td>
<td>0.000</td>
<td>1.372</td>
<td>0.000</td>
<td>1.639</td>
<td>0.000</td>
<td>0.257</td>
<td>0.342</td>
</tr>
<tr>
<td>2</td>
<td>-0.099</td>
<td>0.780</td>
<td>-0.482</td>
<td>0.082</td>
<td>1.238</td>
<td>0.000</td>
<td>0.652</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.383</td>
<td>0.004</td>
<td>-0.621</td>
<td>0.250</td>
<td>-0.476</td>
<td>0.200</td>
<td>0.046</td>
<td>0.802</td>
</tr>
<tr>
<td>4</td>
<td>-0.058</td>
<td>0.764</td>
<td>-0.232</td>
<td>0.272</td>
<td>-1.366</td>
<td>0.056</td>
<td>-0.602</td>
<td>0.002</td>
</tr>
</tbody>
</table>

| $\hat{\theta}$ | 0.000 | 0.764 | 0.736 | -0.142 |
| $\hat{\phi}$   | 0.000 | 0.607 | 0.903 | 0.398  |
| $\hat{\sigma}$ | 0.000 | 1.084 | 0.420 | 0.795  |

Number of Years, $N_y = 100$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.669</td>
<td>0.000</td>
<td>1.320</td>
<td>0.000</td>
<td>1.685</td>
<td>0.000</td>
<td>0.330</td>
<td>0.180</td>
</tr>
<tr>
<td>2</td>
<td>-0.349</td>
<td>0.174</td>
<td>-0.306</td>
<td>0.114</td>
<td>1.217</td>
<td>0.000</td>
<td>0.478</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.331</td>
<td>0.000</td>
<td>-0.003</td>
<td>0.992</td>
<td>-0.183</td>
<td>0.528</td>
<td>0.292</td>
<td>0.064</td>
</tr>
<tr>
<td>4</td>
<td>-0.342</td>
<td>0.020</td>
<td>-0.029</td>
<td>0.810</td>
<td>0.198</td>
<td>0.277</td>
<td>-0.106</td>
<td>0.412</td>
</tr>
</tbody>
</table>

| $\hat{\theta}$ | 0.390 | 0.864 | 0.768 | 0.047  |
| $\hat{\phi}$   | 0.000 | 0.457 | 0.922 | 0.283  |
| $\hat{\sigma}$ | 0.000 | 1.738 | 0.440 | 1.081  |

Number of Years, $N_y = 300$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.620</td>
<td>0.000</td>
<td>1.226</td>
<td>0.000</td>
<td>1.698</td>
<td>0.000</td>
<td>0.584</td>
<td>0.000</td>
</tr>
<tr>
<td>2</td>
<td>-0.528</td>
<td>0.000</td>
<td>-0.410</td>
<td>0.000</td>
<td>1.043</td>
<td>0.000</td>
<td>0.365</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.300</td>
<td>0.000</td>
<td>-0.128</td>
<td>0.618</td>
<td>-0.422</td>
<td>0.006</td>
<td>0.330</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>-0.333</td>
<td>0.000</td>
<td>0.001</td>
<td>0.984</td>
<td>0.012</td>
<td>0.976</td>
<td>-0.078</td>
<td>0.254</td>
</tr>
</tbody>
</table>

| $\hat{\theta}$ | 0.000 | 0.565 | 0.847 | 0.369  |
| $\hat{\phi}$   | 0.000 | 0.661 | 0.850 | 0.215  |
| $\hat{\sigma}$ | 0.000 | 1.736 | 0.472 | 1.171  |
Table C-5: Moving average parameter estimates and $p$–values after $k = 15$ iterations of the innovations algorithm applied to $N_y = 50, 100, 300$ years of simulated PARMA4(1,1) data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-0.90</td>
<td>0.50</td>
<td>0.80</td>
<td>0.25</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 50$

<table>
<thead>
<tr>
<th>Lag</th>
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<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.706</td>
<td>0.000</td>
<td>1.260</td>
<td>0.000</td>
<td>1.651</td>
<td>0.000</td>
<td>0.296</td>
<td>0.304</td>
</tr>
<tr>
<td>2</td>
<td>-0.152</td>
<td>0.652</td>
<td>-0.560</td>
<td>0.028</td>
<td>1.304</td>
<td>0.000</td>
<td>0.648</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.429</td>
<td>0.000</td>
<td>-0.731</td>
<td>0.180</td>
<td>-0.405</td>
<td>0.272</td>
<td>0.236</td>
<td>0.234</td>
</tr>
<tr>
<td>4</td>
<td>-0.194</td>
<td>0.322</td>
<td>-0.036</td>
<td>0.858</td>
<td>-1.002</td>
<td>0.186</td>
<td>-0.458</td>
<td>0.016</td>
</tr>
</tbody>
</table>

$\hat{\theta} = -0.193$, $p = 0.466$, $0.616$, $-0.096$

$\hat{\phi} = -0.513$, $p = 0.794$, $1.035$, $0.393$

$\hat{\sigma} = 0.857$, $p = 1.218$, $0.442$, $0.904$

Number of Years, $N_y = 100$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.673</td>
<td>0.000</td>
<td>1.379</td>
<td>0.000</td>
<td>1.692</td>
<td>0.000</td>
<td>0.325</td>
<td>0.190</td>
</tr>
<tr>
<td>2</td>
<td>-0.287</td>
<td>0.266</td>
<td>-0.304</td>
<td>0.114</td>
<td>1.308</td>
<td>0.000</td>
<td>0.455</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.324</td>
<td>0.000</td>
<td>0.163</td>
<td>0.734</td>
<td>-0.171</td>
<td>0.556</td>
<td>0.343</td>
<td>0.028</td>
</tr>
<tr>
<td>4</td>
<td>-0.393</td>
<td>0.008</td>
<td>-0.001</td>
<td>0.992</td>
<td>0.401</td>
<td>0.576</td>
<td>-0.070</td>
<td>0.582</td>
</tr>
</tbody>
</table>

$\hat{\theta} = 0.211$, $p = 0.927$, $0.744$, $0.056$

$\hat{\phi} = -0.884$, $p = 0.452$, $0.949$, $0.269$

$\hat{\sigma} = 0.898$, $p = 1.789$, $0.456$, $1.133$

Number of Years, $N_y = 300$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.615</td>
<td>0.000</td>
<td>1.241</td>
<td>0.000</td>
<td>1.693</td>
<td>0.000</td>
<td>0.604</td>
<td>0.000</td>
</tr>
<tr>
<td>2</td>
<td>-0.546</td>
<td>0.000</td>
<td>-0.412</td>
<td>0.000</td>
<td>1.063</td>
<td>0.000</td>
<td>0.351</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.291</td>
<td>0.000</td>
<td>-0.156</td>
<td>0.542</td>
<td>-0.407</td>
<td>0.008</td>
<td>0.322</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>-0.330</td>
<td>0.000</td>
<td>0.027</td>
<td>0.696</td>
<td>-0.002</td>
<td>0.992</td>
<td>-0.070</td>
<td>0.304</td>
</tr>
</tbody>
</table>

$\hat{\theta} = 0.289$, $p = 0.571$, $0.836$, $0.396$

$\hat{\phi} = -0.905$, $p = 0.670$, $0.857$, $0.207$

$\hat{\sigma} = 0.861$, $p = 1.754$, $0.476$, $1.191$
Table C-6: Moving average parameter estimates and $p$–values after $k = 10$ iterations of the innovations algorithm applied to $N_y = 50, 100, 300$ years of simulated PARMA$_4(1,1)$ data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Season 0</th>
<th>Season 1</th>
<th>Season 2</th>
<th>Season 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.25</td>
<td>0.65</td>
<td>0.90</td>
<td>0.35</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-0.90</td>
<td>0.50</td>
<td>0.80</td>
<td>0.25</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.90</td>
<td>1.90</td>
<td>0.50</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 50$

<table>
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<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
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<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.649</td>
<td>0.000</td>
<td>1.412</td>
<td>0.000</td>
<td>1.653</td>
<td>0.000</td>
<td>0.467</td>
<td>0.186</td>
</tr>
<tr>
<td>2</td>
<td>-0.190</td>
<td>0.596</td>
<td>-0.473</td>
<td>0.066</td>
<td>1.325</td>
<td>0.002</td>
<td>0.468</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.310</td>
<td>0.002</td>
<td>-0.541</td>
<td>0.412</td>
<td>-0.527</td>
<td>0.156</td>
<td>0.340</td>
<td>0.126</td>
</tr>
<tr>
<td>4</td>
<td>-0.225</td>
<td>0.266</td>
<td>0.096</td>
<td>0.604</td>
<td>-0.913</td>
<td>0.338</td>
<td>-0.251</td>
<td>0.156</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>-0.242</td>
<td>0.683</td>
<td>0.714</td>
<td>0.184</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>-0.406</td>
<td>0.729</td>
<td>0.938</td>
<td>0.283</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.880</td>
<td>1.634</td>
<td>0.452</td>
<td>1.133</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Years, $N_y = 100$

<table>
<thead>
<tr>
<th>Lag</th>
<th>$\hat{\psi}_0(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.679</td>
<td>0.000</td>
<td>1.413</td>
<td>0.000</td>
<td>1.693</td>
<td>0.000</td>
<td>0.274</td>
<td>0.272</td>
</tr>
<tr>
<td>2</td>
<td>-0.247</td>
<td>0.338</td>
<td>-0.328</td>
<td>0.092</td>
<td>1.358</td>
<td>0.000</td>
<td>0.438</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>-0.320</td>
<td>0.000</td>
<td>0.234</td>
<td>0.632</td>
<td>-0.224</td>
<td>0.448</td>
<td>0.344</td>
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</tr>
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<td>0.929</td>
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Number of Years, $N_y = 300$

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<th>$p$</th>
<th>$\hat{\psi}_1(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_2(\ell)$</th>
<th>$p$</th>
<th>$\hat{\psi}_3(\ell)$</th>
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<td>0.574</td>
<td>0.000</td>
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<td>0.000</td>
<td>-0.402</td>
<td>0.000</td>
<td>1.081</td>
<td>0.000</td>
<td>0.345</td>
<td>0.000</td>
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Table C-7: Discrete Fourier transform of moving average parameter estimates $\hat{\psi}_i(\ell)$ at season $i$ and lag $\ell = 1, \ldots, 4$, and standard errors (SE), after $k = 15$ iterations of the innovations algorithm applied to simulated $N_y = 50$ years of simulated PARMA$_{12}(0,1)$ data. Note that the value in (.) is the test statistic (5.21).

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<th>6</th>
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<td>0.360</td>
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<td>0.851*</td>
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<td></td>
<td></td>
<td>(2.226)</td>
<td>(14.733)</td>
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<td>(-1.152)</td>
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<tr>
<td></td>
<td>$\hat{s}_m(1)$</td>
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<td>0.575*</td>
<td>0.461*</td>
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<td>0.050</td>
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<td></td>
<td></td>
<td>(9.960)</td>
<td>(7.976)</td>
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<td>(3.269)</td>
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</tr>
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<td>0.058</td>
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<td>0.058</td>
<td>0.058</td>
<td>0.058</td>
<td>0.041</td>
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<td>(-0.559)</td>
<td>(-2.742)</td>
<td>(-0.340)</td>
<td>(0.082)</td>
<td>(-0.081)</td>
<td>(0.967)</td>
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<tr>
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<td>$\hat{s}_m(2)$</td>
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<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.044</td>
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<tr>
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<td>(-0.200)</td>
<td>(2.221)</td>
<td>(2.901)</td>
<td>(1.796)</td>
<td>(2.224)</td>
<td>(2.075)</td>
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<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.044</td>
</tr>
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<td>4</td>
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<td>0.017</td>
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<td>-0.218*</td>
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<td>(1.526)</td>
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*Fourier coefficients with test statistic $\geq 3.32$
Table C-8: Discrete Fourier transform of moving average parameter estimates $\hat{\psi}_i(\ell)$ at season $i$ and lag $\ell = 1, \ldots, 4$, and standard errors (SE), after $k = 15$ iterations of the innovations algorithm applied to simulated $N_y = 100$ years of simulated PARMA$_{12}(0, 1)$ data. Note that the value in (.) is the test statistic (5.21).

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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td></td>
<td>(6.398)</td>
<td>(19.984)</td>
<td>(-0.742)</td>
<td>(0.004)</td>
<td>(0.303)</td>
<td>(0.308)</td>
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<td>0.014</td>
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<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
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<td>0.003</td>
<td>-0.031</td>
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<td>(0.061)</td>
<td>(-0.701)</td>
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<td>$\hat{s}_m(2)$</td>
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<tr>
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<td>0.045</td>
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*Fourier coefficients with test statistic $\geq 3.32$
Table C-9: Discrete Fourier transform of moving average parameter estimates \( \hat{\psi}_i(\ell) \) at season \( i \) and lag \( \ell = 1, \ldots, 4 \), and standard errors (SE), after \( k = 15 \) iterations of the innovations algorithm applied to simulated \( N_y = 300 \) years of simulated PARMA\(_{12}(0, 1) \) data. Note that the value in (.) is the test statistic (5.21).

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<th>4</th>
<th>5</th>
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<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
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<td>-0.050</td>
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<td>0.045</td>
<td>0.045</td>
<td>0.032</td>
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<td>0.047</td>
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<td>0.051</td>
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<td>-0.010</td>
<td>0.012</td>
<td>-0.048</td>
<td>-0.004</td>
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<td>(0.106)</td>
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<td>0.045</td>
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<td>0.045</td>
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<td>0.032</td>
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<td>( \hat{c}_n(4) )</td>
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<td>0.061</td>
<td>0.054</td>
<td>0.001</td>
<td>0.006</td>
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<td>(0.230)</td>
<td>(-0.567)</td>
<td>(-1.500)</td>
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<tr>
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<td>0.102*</td>
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<td>0.022</td>
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<tr>
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<td></td>
<td>(2.498)</td>
<td>(3.946)</td>
<td>(0.880)</td>
<td>(2.060)</td>
<td>(0.876)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SE</td>
<td>0.045</td>
<td>0.045</td>
<td>0.045</td>
<td>0.045</td>
<td>0.045</td>
<td>0.032</td>
<td></td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic ≥ 3.32
Table C-10: Discrete Fourier transform of model parameters estimates and standard errors (SE) for simulated PARMA$_{12}(1,1)$ data ($N_y = 100$ years). Note that the value in (.) is the test statistic (5.34) and (5.38).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Statistic</th>
<th>harmonic m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.438</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>(3.930)</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>0.256*</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>(3.554)</td>
</tr>
<tr>
<td>SE</td>
<td></td>
<td>0.072</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>0.245</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>(0.769)</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>0.466*</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>(5.640)</td>
</tr>
<tr>
<td>SE</td>
<td></td>
<td>0.083</td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.32$

Table C-11: Discrete Fourier transform of model parameters estimates and standard errors (SE) for simulated PARMA$_{12}(1,1)$ data ($N_y = 300$ years). Note that the value in (.) is the test statistic (5.34) and (5.38).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Statistic</th>
<th>harmonic m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.356</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>(3.388)</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>0.430*</td>
</tr>
<tr>
<td>$\hat{\theta}_t$</td>
<td>$\hat{\epsilon}_{am}$</td>
<td>(10.449)</td>
</tr>
<tr>
<td>SE</td>
<td></td>
<td>0.041</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>0.332</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>(5.283)</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>0.340*</td>
</tr>
<tr>
<td>$\hat{\phi}_t$</td>
<td>$\hat{\epsilon}_{bm}$</td>
<td>(7.163)</td>
</tr>
<tr>
<td>SE</td>
<td></td>
<td>0.083</td>
</tr>
</tbody>
</table>

*Fourier coefficients with test statistic $\geq 3.32$
<table>
<thead>
<tr>
<th>Distribution</th>
<th>pdf and/or cdf</th>
<th>Range</th>
<th>Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>( f_X(x) = \frac{1}{b-a} )</td>
<td>(-\infty &lt; a &lt; b &lt; \infty)</td>
<td>( \mu_x = \frac{a+b}{2}, \sigma_x^2 = \frac{(b-a)^2}{12} )</td>
</tr>
<tr>
<td>Normal</td>
<td>( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{(x-\mu)^2}{2\sigma^2} \right] )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
<td>( \mu_x, \sigma_x^2 )</td>
</tr>
<tr>
<td>Lognormal</td>
<td>( f_X(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right] )</td>
<td>( x &gt; 0 )</td>
<td>( \mu_y = \exp\left( \mu_y + \frac{\sigma_y^2}{2} \right), \sigma_y^2 = \mu_y^2[\exp\sigma_y^2 - 1] )</td>
</tr>
<tr>
<td>2-parameter Exponential</td>
<td>( f_X(x) = \beta \exp(-\beta(x - \xi)) )</td>
<td>( x \geq \xi, \beta &gt; 0 )</td>
<td>( \mu_x = \xi + \frac{1}{\beta}, \sigma_x^2 = \frac{1}{\beta^2} )</td>
</tr>
<tr>
<td></td>
<td>( F_X(x) = 1 - \exp(-\beta(x - \xi)) )</td>
<td></td>
<td>( CS_x = 3\text{CV}_x + 3\text{CV}_y^3 )</td>
</tr>
<tr>
<td>3-parameter Gamma</td>
<td>( f_X(x) = \frac{\alpha^\alpha (x-\xi)^{\alpha-1} \exp(-\beta(x-\xi))}{\Gamma(\alpha)} )</td>
<td>( x \geq \xi, \alpha, \beta &gt; 0 )</td>
<td>( \mu_x = \xi + \frac{\alpha}{\beta}, \sigma_x^2 = \frac{\alpha^2}{\beta^2} )</td>
</tr>
<tr>
<td>(Pearson Type 3)</td>
<td>( \Gamma(\alpha))-Gamma function</td>
<td></td>
<td>( \Gamma(n+1) = \alpha \Gamma(n), \Gamma(1/2) = \pi^{1/2} )</td>
</tr>
<tr>
<td></td>
<td>( \Gamma(n) = (n-1)! ) for any integer ( n \geq 1 )</td>
<td></td>
<td>( CS_x = \frac{2}{n\alpha} )</td>
</tr>
<tr>
<td>GEV</td>
<td>( F_X(x) = \exp\left{ -\left[ 1 - \left( \frac{x-\xi}{\alpha} \right) \right]^{1/\alpha} \right} )</td>
<td>( (\sigma_x^2 \text{ exists for } \alpha &gt; -0.5) )  ( \mu_x = \xi + \left( \frac{\alpha}{\beta} \right) [1 - \Gamma(1+\alpha)] )</td>
<td>( \text{whence } ) ( x &lt; \left( \xi + \frac{\alpha}{\beta} \right) \Rightarrow \sigma_x^2 = \left( \frac{\alpha}{\beta} \right)^2 \left[ \Gamma(1+2\alpha) - [\Gamma(1+\alpha)]^2 \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{whence } x &gt; \left( \xi + \frac{\alpha}{\beta} \right) \Rightarrow \sigma_x^2 = \left( \frac{\alpha}{\beta} \right)^2 \left[ \Gamma(1+2\alpha) - [\Gamma(1+\alpha)]^2 \right] )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \mu_x = \alpha \Gamma\left( 1 + \frac{1}{\alpha} \right) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \sigma_x^2 = \alpha^2 \left[ \Gamma\left( 1 + \frac{2}{\alpha} \right) - \Gamma\left( 1 + \frac{1}{\alpha} \right)^2 \right] )</td>
</tr>
<tr>
<td>Weibull</td>
<td>( f_X(x) = \left( \frac{b}{a} \right)^k \left( \frac{x}{a} \right)^{k-1} \exp\left{ -\left( \frac{x}{a} \right)^k \right} )</td>
<td>( x &gt; 0, \alpha, k &gt; 0 )</td>
<td>( \mu_x = \alpha \Gamma\left( 1 + \frac{1}{k} \right) )</td>
</tr>
<tr>
<td></td>
<td>( F_X(x) = 1 - \exp\left{ -(x/a)^k \right} )</td>
<td></td>
<td>( \sigma_x^2 = \alpha^2 \left[ \Gamma\left( 1 + \frac{2}{k} \right) - \Gamma\left( 1 + \frac{1}{k} \right)^2 \right] )</td>
</tr>
<tr>
<td>Generalized Pareto</td>
<td>( f_X(x) = \left( \frac{\mu}{\alpha} \right)^{\alpha} \left( 1 - \kappa \frac{(x-\xi)}{\alpha} \right)^{\alpha-1} )</td>
<td>( \kappa &lt; 0, \xi \leq x &lt; \infty )</td>
<td>( \mu_x = \xi + \frac{\alpha}{\beta} \left[ 1 + \left( \frac{1}{\alpha} \right)^{\alpha} \right] )</td>
</tr>
<tr>
<td></td>
<td>( F_X(x) = 1 - \left[ 1 - \kappa \frac{(x-\xi)}{\alpha} \right]^{1/\alpha} )</td>
<td>( \kappa &gt; 0, \xi \leq x \leq \xi + \frac{\alpha}{\beta} )</td>
<td>( \mu_x = \mu_x \left[ 1 + \frac{\alpha}{\beta} \left( 1 + 2\kappa \right) \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( (\text{CS}_x \text{ exists for } \kappa &gt; -0.33) ) ( \text{CS}_x = \frac{2\Gamma(1+2\kappa)\Gamma(1+2\kappa+1/\alpha)}{\Gamma(1+\alpha)} )</td>
</tr>
</tbody>
</table>

where Mean: \( \mu_x = E(X) \), Variance: \( \sigma_x^2 = E(X - \mu_x)^2 \), Coefficient of Variation: \( \text{CV}_x = \frac{\mu_x}{\sigma_x} \), and Coefficient of Skewness: \( \text{CS}_x = \frac{E(X - \mu_x)^3}{\sigma_x^3} \)