**Fortune's Formula** The final part of HW due 9-6-13. I’ve chosen to title this section of the course notes after the book by William Poundstone who has helped publicize the Kelly Principle of investment and how it fared in the hands of mathematician Edward O. Thorp (author of Beat the Dealer, a card counting system for winning at blackjack).

Here is an interesting review of Poundstone’s book by David Pogue:

A quote from Pogue:
“Fortune’s Formula” may be the world’s first history book, gambling primer, mathematics text, economics manual, personal finance guide and joke book in a single volume. Poundstone comes across like the best college professor you ever had, someone who can turn almost any technical topic into an entertaining and zesty lecture. But every now and then, you can’t help wishing there were some teaching assistants on hand to help.

Our real goal lies far beyond the casinos or the stock market, in the blizzard of transactions carried out in the electronic marketplace with its pop ups and other marketing devices. Poundstones’ book overlooks machine learning methods that come close to duplicating Kelly performance in random markets but without knowing the probabilities! Moreover the machine learning methods don’t require that there be any probabilities, introducing their own probabilities as random behavior in some cases, and can self guide themselves, for example to home in on the best revenue producing pop up ads while keeping revenue up.

I plan to take you through the idea behind John Kelly’s principle, which is to: 

allocate resources among investment opportunities according to which allocation achieves the largest expected value \( E \log(\text{return on a dollar}) \).
It is motivated by the simple fact that your fortune $F_n$ after $n$ investment periods starting with $F_0 = 1$ may be written

$$F_n = \frac{F_1}{F_0} \frac{F_2}{F_1} \cdots \frac{F_n}{F_{n-1}} = \prod_{1 \leq i \leq n} X_i,$$

where the $i$-th *price relative* is defined $X_i = \frac{F_i}{F_{i-1}}$ for $i \geq 1$. Your fortune after $n$ investments can then be described as the product of price relatives and so too

$$F_n = \exp^{\sum_{i=1}^n \log X_i}.$$

Under rather general conditions, to be described later, the exponent grows roughly like the sum of *conditional expectations*

$$\sum_{i=1}^n E(\log X_i \mid \text{information available before period } i).$$

In other words, follow the Kelly Principle to keep revenue up.

To generate some ideas let’s look at simple eye-opening examples.

**Making money in a go-nowhere market.**

Two stocks A, B. At each investment period one of A, B doubles in value but the other halves in value. The one to double is selected by a fresh toss of a fair coin.

Q1. Can we make money?

Q2. Calculate the return on one period if you invest $1 by splitting it equally on A, B. Have you made money?

Q3. Calculate the return over 20 periods if you always divide your money equally on A, B as was done for one period in Q2. Is this making money?
Are you understanding these exercises? email me with any questions.

Q4. Calculate E log(return on $1 in one period) for the betting scheme that invests p in A and 1-p in B, as a function of p between 0 and 1. Graph the function in R. What is the Kelly strategy?

doit <- function(p) { .5*log((2*p)+(.5*(1-p))) + .5*log((2*(1-p))+(.5*p)) }

x <- 0.01*(0:100)
plot(x, doit(x))

Q5. Do you understand the code? If not, ask.

Q6. In the definition of doit in the R code above note my heavy use of parentheses. Was it necessary?

**Wild ride.**

We’ll see what Kelly needs from the markets in order to succeed and how Kelly’s accumulating fortune can actually be used to price other investments (provided we can sell short, which means sell what we do not own with a guarantee that we will pay it back when the transaction has concluded). Selling short can be looked upon as an invention that makes understanding markets easier. You will learn that Kelly’s method is geared towards growing your fortune through earnings reinvestment in new ventures. It tends to diversify its investments but can be a wild ride as it gains strength and places ever larger amounts of money at risk in quest of increased returns. This is not unlike some multimillionaires who make a fortune, crash, then come back again. For p = 0.5 (in Q4) the returns are not random at all and it is a moneymaker.
Wheel.

A spinning game wheel (think of roulette) has probabilities 1/3 for each of three sectors to come up the (sole) winner. The sectors pay 2:1, 4:1, 8:1 respectively. You begin with $1. On your first play you can retain an amount b0, bet some fraction b1 of $1 on 2:1 payout sector, some amount b2 on 4:1, and some amount b3 on 8:1. Money bet on a sector that is not the winner pays nothing.

Q7. Choose some bets b0, b1, b2, b3, nonnegative, adding to one, and evaluate your expected log return. For example, if I hold back b0 = .2 and bet the remaining .8 on 8:1 my expected log return on the $1 is:

$$1/3 \log(.2 + 0) + 1/3 \log(.2 + 0) + 1/3 \log(.2 + 8*0.8)$$

Q8. Which choice of b0, b1, b2, b3 do you think is the best way to repeatedly play to grow your money?

Q9. Use R to hunt for the Kelly choice b0, b1, b2, b3. Are you surprised? Use b to denote a four entry vector.

```r
wheel <- function(b) {
  b0 <- b[1]
  b1 <- b[2]
  b2 <- b[3]
  b3 <- b[4]
  (1/3)*(log(b0 + 2*b1) + log(b0 + 4*b2) + log(b0 + 8*b3))
}
wheel(c(.1,.3,.2,.4))
```

Are you getting these questions? email if you have questions.
Horse Race.

A horse pays 2:1 to win. The horse wins with probability $p = 0.8$. You have $1. How much do you bet? Denote it by $b$. Kelly consults the expected log return on $1$ which is

```r
horse <- function(b) {.8*log(1-b + (b*2)) + .2*log(1-b)}
plot(.01*(0:100), horse(.01*(0:100)))
```

Was your intuition of help?

What’s next?

In future exercises, which will be part of your second assignment due later than 9-6-2013, we will run simulations of wheel and horse race to see how Kelly’s earnings do. We will also program exact solutions obtained by Kelly in 1946 for wheel like investments and horse race like investments.

Following the above, we will describe machine learning algorithms that make such investments (play such games) without knowing the particulars of the probabilities or payouts but only observing them in repeated trials. We will then compare their performance to that of the Kelly strategy when the games (investments) are actually governed by probabilities.