PROBABILITY

- Sample space and Events
- Rules of Probability
- Conditional Probability and Independence
- Combinatorial Concepts
Toss a coin Twice

\[ P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2} \]

Tacitly assumed ‘fair’ coin. Two possible outcomes. Both equally likely.

\[ S = \{ H, T \} \] - Sample space
Throw a six faced die twice

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Assume all outcomes have equal probability

So the probability of any outcome is \( \frac{1}{36} \)
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Probability that both the throws give even numbers = $\frac{9}{36}$

Probability that the sum of both throws is 8 = $\frac{5}{36}$

Probability that the at least one throw is even = $\frac{27}{36}$
Formal definitions

- **SAMPLE SPACE**
  - Sample space $S$ is the set of all possible outcomes

- **EVENT**
  - Any collection of outcomes is called an Event. The event occurs if the outcome belongs to the subset.
Let $S = \{s_1, s_2, s_3, \ldots, s_n\}$

Specify prob of each outcome $P(s_i) = p_i$

for $i = 1, 2, \ldots, n$

$p_i$ satisfy

\[ p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1 \]

For any event $A$,

\[ P(A) = \text{sum of prob of all outcomes in } A = \sum_{s_i \text{ in } A} p_i \]
Problem 3.2
Let $A =$ first throw is a “6“, $P(A) = \frac{6}{36}$

Let $B =$ Second throw is a “6“, $P(B) = \frac{6}{36}$

“$A$ and $B$“: first throw is “6“ AND ‘ second throw is“6“

‘ all outcomes that are both in $A$ and $B ‘$

This called ‘ $A$ intersection $B ‘$, written as $A \cap B$. 

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Let A = first throw is a “6“, $P(A) = \frac{6}{36}$

Let B = Second throw is a “6“, $P(B) = \frac{6}{36}$

“A or B“ : first throw is “6“ or ‘ second throw is“6“

‘ all outcomes that are at least in one of A or B ‘

This called ‘ A Union B ‘, written as $A \cup B$. 
\[
P(A \cup B) = \frac{11}{36}
\]

\[
P(A) = \frac{6}{36} \quad P(B) = \frac{6}{36}
\]

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
Let $A =$ first throw is a “6“, $P(A) = \frac{6}{36}$

C = first throw is a “4“, $P(C) = \frac{6}{36}$

$P(A \cup C) = \frac{12}{36} = P(A) + P(C)$

A and C have no outcome in common.

A, C are said to be ‘Disjoint‘ or ‘Mutually Exclusive‘
Let $A =$ first throw is a “6“, $P(A) = \frac{6}{36}$

Probability ‘ the first throw is NOT a “6 “ $= \frac{30}{36}$

$A^c$ – A Complement : All outcomes not in A

$P(A^c) = 1 - P(A)$
We have mentioned

- event
- complement of an event
- union of two events
- intersection of two events

Next we look at these abstractly
Formal definitions

- **SAMPLE SPACE**
  
  Sample space $S$ is the set of all possible outcomes.

- **EVENT**
  
  An event is a subset of the sample space. The event occurs if the outcome belongs to the subset.
Formal definitions

- $A$ is an event.

$\bar{A}$ (A- COMPLEMENT) stands for “Not A”. A does not
Sometimes I would denote A- complement by $\bar{A}$ occur

- $A, B$ events. $A \cup B$ (read A UNION B) At least one of A, B occurs. A or B . All outcomes in at least one of A,B

- $A \cap B$ (read A INTERSECTION B) Both A and B occur. All outcomes which are both in A and in B
$\overline{A}$
A-Complement

\[ \bar{A} \]
A-Union B
A-Union B
A-Union B

$A \cup B$
A-intersection B
A-intersection B
A-intersection B
Mutually Exclusive events

A and B are MUTUALLY EXCLUSIVE OR DISJOINT if
\[ A \cap B = \emptyset \]

i.e., A and B have no outcomes in common

Both A and B cannot occur at the same time
Mutually Exclusive
Toss a coin three times. Let $H_i$ be the event of getting a head and $T_i$ of getting a tail at ith time for $i=1,2,3$. Which of the following events is the event of getting at least one head in the three tosses.

a. $H_1 \cap H_2 \cap H_3$

b. $T_1 \cup T_2 \cup T_3$

c. $T_1 \cup T_2 \cup T_3$

d. $H_1 \cup H_2 \cup H_3$
Toss a coin three times. Let $H_i$ be the event of getting a head and $T_i$ of getting a tail at ith time for $i=1,2,3$.

The complement of getting at least one head is

a) Getting all tails

b) Getting all heads

c) $T_1 \cup T_2 \cup T_3$

d) Getting at least one tail
Convince yourselves that

\[ A \cup B = \overline{A} \cap \overline{B} \]

\[ A \cap B = \overline{A} \cup \overline{B} \]
Probability: Formal Properties

Probability is an assignment of numbers $P(A)$ to every event $A$, such that

1. $0 \leq P(A) \leq 1$

2. $P(S) = 1$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4. If $A$ and $B$ are mutually exclusive, since $P(A \cap B) = 0$,
   \[ P(A \cup B) = P(A) + P(B) \]

5. (i) and (ii) imply that $P(\overline{A}) = 1 - P(A)$