Name: ID:

Homework for 2/24 Due 3/12

1. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$, where μ is the unknown parameter and σ^2 is assumed to be known. The hypotheses are

$$H_0 : \mu = \mu_0 \qquad \text{v.s.} \qquad H_A : \mu > \mu_0$$

Consider the test that reject H_0 if $\overline{X} > c$, where $\overline{X}$ is the sample mean.

a. Show that the significance level of this test is $1 - \Phi\left(\frac{c - \mu_0}{\sigma/\sqrt{n}}\right)$, where Φ is the cdf of the standard normal random variable, that is, $\Phi(z) = \mathbb{P}(Z \leq z)$ with $Z \sim N(0, 1)$.

b. Show that if the test is required to have significance level α, then $c = z_{1-\alpha}\frac{\sigma}{\sqrt{n}} + \mu_0$. Here $z_{1-\alpha}$ satisfies $\mathbb{P}(Z \leq z_{1-\alpha}) = 1 - \alpha$.

c. Show that if the sample gives $\overline{X} = \bar{x}$, then the p-value in this case is $1 - \Phi\left(\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right)$

d. Show that if in fact $\mu = \mu_1$ with $\mu_1 > \mu_0$, then the power of the test is $1 - \Phi\left(\frac{c - \mu_1}{\sigma/\sqrt{n}}\right)$. In particular, if the significance level is α, then the power is $1 - \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right)$.

2. The drying time of a certain type of paint under specified test conditions is known to be normally distributed with mean value 75 min and standard deviation 9 min. Chemists have proposed a new additive designed to decrease average drying time. It is believed that drying times with this additive will remain normally distributed with $\sigma = 9$. Because of the expense associated with the additive, evidence should strongly suggest an improvement in average drying time before such a conclusion is adopted. Let μ denote the true average drying time when the additive is used. The appropriate hypotheses are $H_0 : \mu = 75$ versus $H_A : \mu < 75$. Only if H_0 can be rejected will the additive be declared successful and then be used. Assume a sample of size $n = 25$ is obtained.

   a. If $\bar{x} = 72.3$, what is the conclusion using $\alpha = 0.01$?

   b. What is α for the test procedure that rejects H_0 when $\overline{X} < 69.816$?

   c. For the test procedure of part (b), what is $\beta(70)$?

   d. If a level .01 test is used with $n = 100$, what is the probability of a type I error when $\mu = 76$?

3. Let μ denote the true average tread life of a certain type of tire. Consider testing$H_0 : \mu = 30,000$ versus $H_A : \mu > 30,000$ based on a sample of size $n = 16$ from a normal population distribution with $\sigma = 1500$.

   a. If $\bar{x} = 30,960$ and a level $\alpha = 0.01$ test is used, what is the decision?

   b. If a level .01 test is used, what is $\beta(30,500)$?

   c. If $\bar{x} = 30,960$, what is the smallest α at which H_0 can be rejected (based on $n = 16$)?

4. The melting point of each of 16 samples of a certain brand of hydrogenated vegetable oil was determined, resulting in $\bar{x} = 94.32$. Assume that the distribution of the melting point is normal with $\sigma = 1.20$. c. What value of n is necessary to ensure that when ?

   a. Test $H_0 : \mu = 95$ versus $H_A : \mu \neq 95$ using a two-tailed level .01 test.

   b. If a level .01 test is used, what is $\beta(94)$, the probability of a type II error when $\mu = 94$?

Appendix

[1'.] Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$, where μ is the unknown parameter and σ^2 is assumed to be known. The hypotheses are

$$H_0 : \mu = \mu_0 \qquad \text{v.s.} \qquad H_A : \mu < \mu_0$$

Consider the test that reject H_0 if $\overline{X} < c$, where $\overline{X}$ is the sample mean.

a. The significance level of this test is $\Phi\left(\frac{c-\mu_0}{\sigma/\sqrt{n}}\right)$, where Φ is the cdf of the standard normal random variable, that is, $\Phi(z) = \mathbb{P}(Z \le z)$ with $Z \sim N(0,1)$.

b. If the test is required to have significance level α, then $c = z_\alpha \frac{\sigma}{\sqrt{n}} + \mu_0$. Here z_α satisfies $\mathbb{P}(Z \le z_\alpha) = \alpha$.

c. If the sample gives $\overline{X} = \bar{x}$, then the p-value in this case is $\Phi\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$

d. If in fact $\mu = \mu_1$ with $\mu_1 < \mu_0$, then the power of the test is $\Phi\left(\frac{c-\mu_1}{\sigma/\sqrt{n}}\right)$. In particular, if the significance level is α, then the power is $\Phi\left(z_\alpha + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right)$.

[1".] Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$, where μ is the unknown parameter and σ^2 is assumed to be known. The hypotheses are

$$H_0 : \mu = \mu_0 \qquad \text{v.s.} \qquad H_A : \mu \neq \mu_0$$

Consider the test that reject H_0 if $|\overline{X} - \mu_0| > c$, where $\overline{X}$ is the sample mean.

a. The significance level of this test is $2\left(1 - \Phi\left(\frac{c-\mu_0}{\sigma/\sqrt{n}}\right)\right)$.

b. If the test is required to have significance level α, then $c = z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$.

c. If the sample gives $\overline{X} = \bar{x}$, then the p-value in this case is $2\left(1 - \Phi\left(\frac{|\bar{x}-\mu_0|}{\sigma/\sqrt{n}}\right)\right)$.

d. If in fact $\mu = \mu_1$, then the power of the test is $1 + \Phi\left(\frac{\mu_0-\mu_1-c}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\mu_0-\mu_1+c}{\sigma/\sqrt{n}}\right)$. In particular, if the significance level is α, then the power is $1 - \Phi\left(z_{1-\frac{\alpha}{2}} + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right) + \Phi\left(-z_{1-\frac{\alpha}{2}} + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right)$.