Quiz 1

1. Let $X$ be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{3}{2}x^2, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$ 

(a) Find the mean $\mu$ and variance $\sigma^2$ of $X$.

(b) If $S = X_1 + X_2 + \cdots + X_{60}$, where $X_1, \ldots, X_{60}$ are i.i.d. random variables with pdf $f(x)$, what is the approximate value of $P(S > 15)$?

(a) We have

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{-1}^{1} x \cdot \frac{3}{2} x^2 \, dx = \left( \frac{3}{8} x^4 \right) \bigg|_{-1}^{1} = 0.$$  

In fact, we can immediately conclude $\mu = 0$ without calculation since the pdf is symmetric about 0.

Furthermore,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{-1}^{1} x^2 \cdot \frac{3}{2} x^2 \, dx = \left( \frac{3}{10} x^5 \right) \bigg|_{-1}^{1} = \frac{3}{5}.$$  

Thus

$$\sigma^2 = Var[X] = E[X^2] - E[X]^2 = \frac{3}{5}.$$  

(A general result is that $Var[X] = E[X^2]$ if $E[X] = 0$.)

(b) The central limit theorem tells us that

$$P(S > 15) = P \left( \frac{S - n\mu}{\sigma \sqrt{n}} > \frac{15 - n\mu}{\sigma \sqrt{n}} \right) = P \left( \frac{S - 60 \cdot 0}{\sqrt{0.6 \sqrt{60}}} > \frac{15 - 60 \cdot 0}{\sqrt{0.6 \sqrt{60}}} \right) \approx P(Z > 2.5) = 1 - 0.9938 = 0.0062.$$