Homework 2. Solutions.

1. (a) \[ EX = 1 \cdot \theta + 2 \cdot (1 - \theta) = 2 \theta - m_1, \quad \theta = 2 - m_1, \quad \hat{\theta} = 2 - \bar{X}. \]

In case of the sample \(1, 2, 2\) we have \(\hat{\theta} = 2 - (1 + 2 + 2)/3 = 1/3\).

(b) \(\text{lik}(\theta) = \theta^3(1 - \theta)^2 = \theta - 2\theta^2 + \theta^3\).

(c) Set \(\text{lik}'(\theta) = 1 - 4\theta + 3\theta^2 = 0 \iff \theta = 1/3\) or 1. Since the domain for \(\theta\) is \([0,1]\), and \(\text{lik}(\theta)\) is increasing on \((0, 1/3)\) and decreasing on \((1/3, 1)\), the point of maximum is \(1/3\). Thus the MLE is \(\hat{\theta} = 1/3\).

2. (a) \[ EX = \int_{-\infty}^{\infty} x f(x; \sigma) dx = \int_{-\infty}^{\infty} x e^{-|x|/2\sigma} dx = 0, \]

because we integrate an odd function. The first moment cannot be used for method of moments estimate. Consider

\[ EX^2 = \int_{-\infty}^{\infty} x^2 f(x; \sigma) dx = \int_{-\infty}^{\infty} x^2 e^{-|x|/2\sigma} dx = \int_{0}^{\infty} 2x^2 e^{-x/\sigma} dx = \sigma^2 \int_{0}^{\infty} ye^{-y} dy \]

\[ = \sigma^2[-y^2 e^{-y}]_{0}^{\infty} + \int_{0}^{\infty} 2ye^{-y} dy = \sigma^2[-2ye^{-y}]_{0}^{\infty} + 2 \int_{0}^{\infty} e^{-y} dy = 2\sigma^2, \]

where we used substitution \(x = y\sigma\) and the chain rule twice. Thus, \(EX^2 = m_2 = 2\sigma^2\) and \(\sigma = \sqrt{m_2/2}\). So the method of moments estimate of \(\sigma\) is \(\hat{\sigma} = \sqrt{\frac{2}{2n} \sum_{i=1}^{n} X_i^2}\).

(b) Since \(\ln f(x; \sigma) = -\ln 2 - \ln \sigma - \frac{|x|}{\sigma}\), the log-likelihood function is

\[ L(\sigma) = \sum_{i=1}^{n} \left[ -\ln 2 - \ln \sigma - \frac{|X_i|}{\sigma} \right] = -n \ln 2 - n \ln \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} |X_i|. \]

Set

\[ L'(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} |X_i| = 0 \iff \hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |X_i|. \]

Since

\[ L''(\sigma) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} \sum_{i=1}^{n} |X_i|, \quad L''(\hat{\sigma}) = -\frac{n}{\sigma^2} < 0, \]

the following point is the point of maximum, the MLE of \(\sigma\) is \(\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |X_i|\).

(c) The asymptotic variance of the MLE of \(\sigma\) is \(\frac{1}{nI(\sigma)}\). So we need to find Fisher information first.

\[ I(\sigma) = -E \frac{\partial^2 \ln f(X; \sigma)}{\partial \sigma^2} = -E \left( \frac{1}{\sigma^2} - \frac{2}{\sigma^3} |X| \right) = -\frac{1}{\sigma^2} + \frac{2}{\sigma^3} E|X| = \frac{1}{\sigma^2}. \]

Since

\[ E|X| = \int_{-\infty}^{\infty} \frac{|x|}{2\sigma} e^{-|x|/2\sigma} dx = 2 \int_{0}^{\infty} \frac{x}{2\sigma} e^{-x/\sigma} dx = \sigma \int_{0}^{\infty} ye^{-y} dy = \sigma [y(-e^{-y})]_{0}^{\infty} + \int_{0}^{\infty} e^{-y} dy = \sigma. \]

Thus the asymptotic variance of the MLE of \(\sigma\) is \(\frac{\sigma^2}{n}\).

(d) An approximate 100(1 - \(\alpha\))% confidence interval for \(\sigma\) is

\[ \hat{\sigma} \pm \frac{z_{1-\alpha/2}}{\sqrt{nI(\hat{\sigma})}} \iff \hat{\sigma} \pm \frac{z_{1-\alpha/2}}{\sqrt{n}}. \]
Here for $\alpha = 0.08$, $n = 64$, and $\bar{\sigma} = 2$ we have $z_{0.96} = 1.75$ and the interval $2 \pm 0.4375$ or $(1.5625, 2.4375)$. We are 92% confident that the parameter $\sigma$ is between 1.5625 and 2.4375. It means that if the sampling of 64 observations, calculating $\tilde{\sigma}$ is repeated many times, approximately 92% of the resulting intervals would cover the parameter $\sigma$.

3. (a) Use substitution $y = x - \theta$ to get

$$EX = \int_{\theta}^{\infty} xe^{-(x-\theta)}dx = \int_{0}^{\infty} (y + \theta)e^{-y}dy = 1 + \theta \iff \theta = m_1 - 1.$$ 

Thus the method of moments estimator of $\theta$ is $\tilde{\theta} = \bar{X} - 1 = \frac{1}{n}\sum_{i=1}^{n} X_i - 1$.

(b) The likelihood function is

$$lik(\theta) = \begin{cases} e^{-(X_1-\theta)\cdots e^{-(X_n-\theta)}} & \text{if } X_1, \ldots, X_n \geq \theta, \\ 0 & \text{elsewhere} \end{cases}$$

This function is positive and increasing on its domain $\theta \leq \min_{1\leq i\leq n} X_i$ and it is 0 elsewhere. Thus it reaches its maximum at the endpoint $\hat{\theta} = \min_{1\leq i\leq n} X_i$, which is the MLE of $\theta$.

4. The small sample $100(1 - \alpha)\%$ confidence intervals for $\mu$ and $\sigma^2$ are

$$\bar{X} \pm t_{1-\alpha/2} \frac{s}{\sqrt{n}} \quad \text{and} \quad \left(\frac{(n - 1)s^2}{\chi^2_{1-\alpha/2}}, \frac{(n - 1)s^2}{\chi^2_{\alpha/2}}\right)$$

Since $n = 15$, $\bar{X} = 10$, $s^2 = 25$, $\alpha = 0.1$, $t_{0.95} = 1.761$, $\chi^2_{0.95} = 6.57$, $\chi^2_{0.05} = 23.68$ we have the 90% confidence intervals $(7.7266, 12.2734)$ and $(14.7804, 53.2725)$.

5. (a) Let $C$ be a $\chi^2_{n-1}$ random variable, then $EC = n - 1, VarC = 2(n - 1)$. Thus $Es^2 = \frac{\sigma^2}{(n-1)}EC = \sigma^2$ and $E\hat{\sigma}^2 = \frac{\sigma^2}{n}EC = \sigma^2\frac{n-1}{n}$. So $s^2$ is unbiased estimate of $\sigma^2$.

(b) Recall $MSE = Var + bias^2$. Then

$$MSE(s^2) = Var(s^2) + bias(s^2)^2 = Var(s^2) = Var\left(\frac{\sigma^2}{(n-1)}C\right) = \frac{\sigma^4}{(n-1)^2}VarC$$

$$= \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1}$$

and

$$MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) + bias(\hat{\sigma}^2)^2 = Var\left(\frac{\sigma^2}{n}C\right) + (\frac{\sigma^2}{n} - \sigma^2)^2 = \frac{\sigma^4}{n^2}2(n-1) + \frac{\sigma^4}{n^2} = \frac{\sigma^4(2n-1)}{n^2}$$

Since $\frac{2n-1}{n^2} < \frac{2}{(n-1)} \iff 1 < 3n$, $\hat{\sigma}^2$ has the smaller MSE.

(c)

$$MSE\left(\rho \sum_{i=1}^{n} (X_i - \bar{X})^2\right) = Var(\rho \sigma^2 C) + (E(\rho \sigma^2 C) - \sigma^2)^2 = \rho^2 \sigma^4 VarC + \sigma^4(\rho EC - 1)^2$$

$$= \sigma^4[\rho^2 2(n-1) + (\rho(n-1) - 1)^2] = \sigma^4[(n^2 - 1)\rho^2 - 2(n-1)\rho + 1] = g(\rho).$$

Set

$$g'(\rho) = 0 \iff 2(n^2 - 1)\rho - 2(n-1) = 0 \iff \rho = 1/(n + 1).$$

It is the point of minimum, since $g$ is a parabola with branches going up.