\( x_1, \ldots, x_n \sim N(\theta, \sigma^2), \sigma > 0 \)

\[ p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2 \right\} \]

By Factorization Th., \( T = (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2) \) is sufficient

This is exp. family with \( \eta = (\frac{1}{\sigma}, -\frac{1}{2\sigma^2}) \)

Consider \( \eta^{(0)} = (1, -\frac{1}{2}); \eta^{(1)} = (2, -2); \eta^{(2)} = (4, -8) \)

Construct matrix \( ||\eta^{(i)} - \eta^{(0)}|| \Rightarrow \begin{pmatrix} 1 - \frac{2}{3} \\ -\frac{2}{3} - \frac{1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 - \frac{2}{3} \\ 0 - 3 \end{pmatrix} \Rightarrow \text{rank} 2 \)

\( T \) is minimal by Cor 6.16 (ii)

(a) \( U(0, \theta), \theta > 0 \)

\[ p(\mathbf{x}) = \frac{1}{\theta^n} I(\mathbf{x}(n) \leq \theta) \]

By Factorization Th., \( T = \mathbf{x}(n) \) is sufficient

Consider \( \frac{p_1}{p_0} = \frac{\theta^n}{\theta^n} \frac{I(\mathbf{x}(n) \leq \theta)}{I(\mathbf{x}(n) \leq \theta_0)} \)

It is a f-n of \( T \)

\( T \) is minimal by Th. 6.12

(b) \( U(\theta_1, \theta_2), -\infty < \theta_1 < \theta_2 < \infty \)

\[ p(\mathbf{x}) = \frac{1}{(\theta_2 - \theta_1)^n} I(\theta_1 \leq x(1), x(n) \leq \theta_2) \]

By Factorization Th., \( T = (x(1), x(n)) \) is sufficient

Consider \( \frac{p_1}{p_0} = \frac{(\theta_2 - \theta_1)^n}{(\theta_2 - \theta_1)^n} \frac{I(\theta_1 \leq x(1), x(n) \leq \theta_2)}{I(\theta_1 \leq x(1), x(n) \leq \theta_2)} \)

It is a f-n of \( T \)

\( T \) is minimal by Th. 6.12

(c) \( U(\theta - \frac{1}{2}, \theta + \frac{1}{2}), -\infty < \theta < \infty \)

\[ p(\mathbf{x}) = I(\theta - \frac{1}{2} \leq x(1), x(n) \leq \theta + \frac{1}{2}) \]

By Factorization Th., \( T = (x(1), x(n)) \) is suff.

Again consider \( \frac{p_1}{p_0} = \frac{I(\theta - \frac{1}{2} \leq x(1), x(n) \leq \theta + \frac{1}{2})}{I(\theta_0 - \frac{1}{2} \leq x(1), x(n) \leq \theta_0 + \frac{1}{2})} \)

It is a f-n of \( T \)

\( T \) is minimal by Th. 6.12