Ch14 From Randomness to Probability

- Definition of probability
- Basic rules for probability
- Use Venn diagram to represent events and their probabilities
Motivation

• For a given coin, how can we know this coin is fair? (A coin is fair if it has equal chance to land on head or tail when it is flipped.)
  ❖ Flip the coin once to see the outcome?
  ❖ Flip the coin ten times to see the outcome?
    ❑ What if we get 9 heads and 1 tail?
    ❑ 9 tails and 1 head?
    ❑ 6 heads and 4 tails?
  ❖ “Chance/probability” of observing certain outcome of a random phenomena.
Motivation

- What is “chance/probability”?
  - What do we really mean by saying a coin is fair?
Probability

- Relative frequency (proportion of occurrences) of an outcome settles down to one value over the long run. That one value is then defined to be the probability of that outcome.

- This is guaranteed by Law of Large Numbers:
  For independent trials, the long-run relative frequency of repeated events gets closer and closer to a single value.
What is the probability of getting 9 heads and 1 tail after flipping a fair coin 10 times?

A simpler question:

- What is the probability of getting 2 heads and 1 tail after flipping a fair coin 3 times?

We can first list all possible outcomes ("H" denotes head and "T" denotes tail):

- HHH
- HHT
- HTH
- HTT
- THH
- THT
- TTH
- TTT

We can then list all outcomes which have 2 heads and 1 tail.
Probability: Terminology

- The **sample space** of a random phenomenon is the set of all possible outcomes. It is usually denoted by $S$.
- An **event** is an outcome or a set of outcomes (subset of the sample space).
- A **probability model** is a mathematical description of long-run regularity consisting of a sample space $S$ and a way of assigning probabilities to events.
Probability: Example

- Random phenomena: toss a fair coin 3 times
- Sample space:
  - HHH  HHT  HTH  HTT
  - THH  THT  TTH  TTT
- Events:
  - 2 heads and 1 tail; \{HHT, HTH, THH\}
  - 1 head and 2 tails; \{HTT, THT, TTH\}
  - At least 1 head; \{HTT, THT, TTH, HHT, HTH, THH, HHH\}
  - etc.
- Probabilities: each outcome has probability 0.125 of occurring.
Probability: Rules

- **Rule 1:**
  - A probability is a number between 0 and 1.
  - For any event A, \(0 \leq P(A) \leq 1\).
  - A probability can be interpreted as the *proportion* of times that a certain event can be expected to occur.
  - If the probability of an event is more than 1, then it will occur more than 100% of the time (Impossible!).
• **Rule 2:**
  
  - All possible outcomes together must have probability 1. \( P(S) = 1 \).
  
  - Because some outcome must occur on every trial, the sum of the probabilities for all possible outcomes must be exactly one.
  
  - If the sum of all of the probabilities is less than one or greater than one, then the resulting probability model will be incoherent.
Venn Diagram

• We can use graphs to represent probabilities and probability rules.
• Venn diagram:

area <-> probability

\[ P(S) = 1; \ 0 \leq P(A) \leq 1 \]
Probability: Rules

• Rule 3 (addition rule):

  ❖ If two events have no outcomes in common, they are said to be **disjoint** (or **mutually exclusive**).
  
  ❖ The probability that one or the other of two disjoint events occurs is the sum of their individual probabilities.

  \[ P(A \text{ or } B) = P(A) + P(B), \text{ provided that } A \text{ and } B \text{ are disjoint.} \]
• Addition rule: \( P(A \text{ or } B) = P(A) + P(B) \), A and B are disjoint.

• Mathematical notation: \( P(A \cup B) = P(A) + P(B) \), where \( \cup \) means union.
An application of Rule 3:

If the outcomes in a sample space are equally likely to occur, then the probability of an event is the number of outcomes in the event divided by the total number of possible outcomes.

\[
P(A) = \frac{\# \text{ outcomes in } A}{\# \text{ all possible outcomes}}
\]
Rule 4 (complement rule):

- The probability that an event does not occur is 1 minus the probability that the event does occur.
- The set of outcomes that are not in the event $A$ is called the complement of $A$, and is denoted by $A^C$.

$P(A^C) = 1 - P(A)$.

Example: what is the probability of getting at least 1 head in the experiment of tossing a fair coin 3 times?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>0.125</td>
</tr>
<tr>
<td>HHT</td>
<td>0.125</td>
</tr>
<tr>
<td>HTH</td>
<td>0.125</td>
</tr>
<tr>
<td>HTT</td>
<td>0.125</td>
</tr>
<tr>
<td>THH</td>
<td>0.125</td>
</tr>
<tr>
<td>THT</td>
<td>0.125</td>
</tr>
<tr>
<td>TTH</td>
<td>0.125</td>
</tr>
<tr>
<td>TTT</td>
<td>0.125</td>
</tr>
</tbody>
</table>
Venn Diagram

- Complement rule: $P(A^c) = 1 - P(A)$. 

![Venn Diagram](image)
Probability: Rules

- Rule 5 (multiplication rule):
  - Two events are said to be independent if the occurrence of one event does not influence the occurrence of the other.
  - For two independent events A and B, the probability that both A and B occur is the product of the probabilities of the two events.
  - \( P(A \text{ and } B) = P(A) \times P(B) \).
  - Mathematical notation: \( P(A \cap B) = P(A)P(B) \), where “\( \cap \)” means intersection.
• $P(\text{at least one}) = 1 - P(\text{none})$
Practice

• Consider tossing a fair die.
  ❖ What is the sample space?
  ❖ Let A be the event that the outcome is odd and let B be the event that the outcome is a multiple of 3.
    ❖ What is A or B?
    ❖ What is A and B?
    ❖ What is the complement of A?
    ❖ Are A and B disjoint?
Consider selecting a card from an ordinary deck of 52 cards. Assume the cards are well-shuffled. Let

- $A = \{\text{the selected is black}\}$
- $B = \{\text{the selected is a picture card}\}$
- $C = \{\text{the selected is an ace}\}$

- What is $A$ and $B$?
- What is $B$ or $C$?
- What is $A$ complement?
- Are $B$ and $C$ disjoint?
The American Red Cross says that about 45% of the U.S. population has Type O blood, 40% Type A, 11% Type B, and the rest Type AB. Someone volunteers to give blood. What is the probability that this donor

- has Type AB blood?
- has Type A or Type AB?
- is not Type O?
Practice

• The American Red Cross says that about 45% of the U.S. population has Type O blood, 40% Type A, 11% Type B, and the rest Type AB. Among four potential donors, what is the probability that
  ❖ all are Type O?
  ❖ no one is Type AB?
    \[ \Box = 1 - P(\text{all is Type AB}) \]
  ❖ they are not all Type A?
    \[ \Box = 1 - P(\text{all type A}) \]
  ❖ at least one person is Type B?
    \[ \Box = 1 - P(\text{no Type B}) \]
Suggested exercises from the textbook:

Ch14 1, 7, 9, 11, 13, 15, 19, 21, 23, 27, 29, 33, 35, 37, 41, 43