1. [§8-13] In Example D of Section 8.4, the pdf of the population distribution is
\[ f(x|\alpha) = \begin{cases} 
\frac{1+\alpha x}{2} & -1 \leq x \leq 1, \\
0 & \text{otherwise}
\end{cases} \]
and the method of moments estimate was found to be \( \hat{\alpha} = 3\bar{X} \) (where \( \bar{X} \) is the sample mean of the random sample \( X_1, \ldots, X_n \)). In this problem, you will consider the sampling distribution of \( \hat{\alpha} \).

(a) Show that the estimate \( \hat{\alpha} \) is unbiased.
(b) Find \( \text{Var}[\hat{\alpha}] \). [Hint: What is \( \text{Var}\{\bar{X}\} \)？]
(b) Use the central limit theorem to deduce a normal approximation to the sampling distribution of \( \hat{\alpha} \). According to this approximation, if \( n = 25 \) and \( \alpha = 0 \), what is the \( P(|\hat{\alpha}| > .5) \)?

2. [§8-53] Let \( X_1, \ldots, X_n \) be i.i.d. uniform on \([0, \theta]\).

(a) Find the method of moments estimate of \( \theta \), and the mean, variance, bias, and MSE of the MME.
(b) The mle of \( \theta \) is \( \hat{\theta} = \max_{1 \leq i \leq n} X_i \). The pdf of \( \max_{1 \leq i \leq n} X_i \) (\( \text{How do we find this?} \)) is
\[ f(x|\theta) = \begin{cases} 
\frac{n x^{n-1}}{\theta^n} & 0 < x < \theta, \\
0 & \text{otherwise}
\end{cases} \]
Calculate the mean and variance of the mle. Compare the variance, the bias, and the mean squared error to those of the method of moments estimate.
(c) Find a modification of the mle that renders it unbiased.

3. [§8-7] Suppose that \( X \) follows a geometric distribution,
\[ P(X = k) = p(1 - p)^{k-1} \]
and assume \( X_1, \ldots, X_n \) is an i.i.d. sample of size \( n \). Find the asymptotic variance of the mle. (\( \text{The moments of geometric distribution can be found in P117.} \))
4. Consider an i.i.d. sample of random variables with density function

\[
f(x|\sigma) = \frac{1}{2\sigma} \exp \left( -\frac{|x|}{\sigma} \right), \quad -\infty < x < \infty, \quad \sigma > 0.
\]

Find the asymptotic variance of the mle.

5. The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

\[
f(x|x_0, \theta) = \theta x_0^\theta x^{-\theta-1}, \quad x \geq x_0, \theta > 1.
\]

Assume that \( x_0 > 0 \) is given and that \( X_1, X_2, \ldots, X_n \) is an i.i.d. sample.

(a) Find the method of moments estimate of \( \theta \).

(b) Find the mle of \( \theta \).

(c) Find the asymptotic variance of the mle.

6. Determine the values of the following quantities:

a. \( t_{0.15} \)  
b. \( t_{0.05.15} \)  
c. \( t_{0.1.25} \)  
d. \( t_{0.01.40} \)  
e. \( t_{0.001.40} \)  
f. \( \chi^2_{0.1.10} \)  
g. \( \chi^2_{0.1.20} \)  
h. \( \chi^2_{0.01.20} \)  
i. \( \chi^2_{0.005.20} \)  
j. \( \chi^2_{0.99.20} \)

7. Silicone implant augmentation rhinoplasty is used to correct congenital nose deformities. The success of the procedure depends on various biomechanical properties of the human nasal periosteum and fascia. The article “Biomechanics in Augmentation Rhinoplasty” (J. of Med. Engr. and Tech., 2005: 14-17) reported that for a sample of 15 (newly deceased) adults, the mean failure strain (%) was 25.0, and the standard deviation was 3.5. Assume that the distribution of failure strain is normal with mean \( \mu \).

(a) Find a 98% confidence interval for the parameter \( \mu \).

(b) If we know that the standard deviation \( \sigma \) of the distribution of failure strain is 3.5 (based on previous knowledge, for example), what is a 98% confidence interval for the parameter \( \mu \) now?
8. The following observations were made on fracture toughness of a base plate of 18% nickel maraging steel [“Fracture Testing of Weldments,” ASTM Special Publ. No. 381, 1965: 328-356 (in ksi \( \sqrt{\text{in}} \), given in increasing order)]:

\[
69.5 \quad 71.9 \quad 72.6 \quad 73.1 \quad 73.3 \quad 73.5 \quad 75.5 \quad 75.7 \\
75.8 \quad 76.1 \quad 76.2 \quad 76.2 \quad 77.0 \quad 77.9 \quad 78.1 \quad 79.6 \\
79.7 \quad 79.9 \quad 80.1 \quad 82.2 \quad 83.7 
\]

The sample mean and standard deviation are \( \bar{x} = 76.55 \) and \( s = 3.55 \), respectively. Assume the fracture toughness is normally distributed. Calculate a 99% CI for the standard deviation of the fracture toughness distribution.

9. [§8-7] Suppose that \( X \) follows a geometric distribution,

\[ P(X = k) = p(1 - p)^{k-1} \]

and assume \( X_1, \ldots, X_n \) is an i.i.d. sample of size \( n \). If \( \bar{X} = 2.5 \) and \( n = 60 \), find an approximate confidence interval for the parameter \( p \) with confidence level 98%.

(The mle for \( p \) and the asymptotic variance for the mle are found in previous homework.)

10. [§8-16] Consider an i.i.d. sample of random variables \( X_1, \ldots, X_n \) with density function

\[ f(x|\sigma) = \frac{1}{2\sigma} \exp \left( -\frac{|x|}{\sigma} \right), \quad -\infty < x < \infty, \quad \sigma > 0. \]

If \( \frac{1}{n} \sum_{i=1}^{n} |X_i| = 9 \) and \( n = 81 \), find a 90% approximate confidence interval for \( \sigma \).
11. The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

\[ f(x|x_0, \theta) = \theta x_0^\theta x^{-\theta-1}, \quad x \geq x_0, \theta > 1. \]

Assume that \( x_0 = 1 \) and that \( X_1, X_2, \ldots, X_n \) is an i.i.d. sample. If \( \frac{1}{n} \sum_{i=1}^{n} \log X_i = 10 \) and \( n = 100 \), find a 96% approximate confidence interval for \( \theta \).

12. Suppose that \( X_1, \ldots, X_n \) is a random sample from a Bernoulli distribution with parameter \( p \).

a. Find the mle of \( p \).

b. Show that mle of part (a) attains the Cramér-Rao lower bound.

13. Let \( X_1, \ldots, X_n \) be an i.i.d. sample from a Poisson distribution with mean \( \lambda \), and let \( T = \sum_{i=1}^{n} X_i \).

a. Show that the distribution of \( X_1, \ldots, X_n \) given \( T \) is independent of \( \lambda \), and conclude that \( T \) is sufficient for \( \lambda \).

b. Show that \( X_1 \) is not sufficient.