Plan for Class 12:
1. Info on Quiz 3;
2. Normally distributed random variables,

Quiz 3 will be held on Wednesday, August 5. A tentative program is as follows (it will be refined on Monday).

5. The sum of Normally distributed random variables.
The most important class of distributions is that of Normal Distributions.

The density of a Normal distribution is given by the formula:

\[
f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.
\]

Since the anti-derivative of \(f\) is not an elementary function, for finding probabilities we will use either tables or software.

\(\mu\) plays the role of the mean, and \(\sigma^2\) of the variance.

Standard Normal distribution \(\text{N}(0,1)\) is that with zero mean and standard deviation 1.
Here are several exercises on using Table of the standard normal distribution. Z stands for a standard normal random variable.

\[ P(0 \leq Z \leq 1.33) = .4082 \]
\[ P(Z \leq 1) = .8413 \]
\[ P(-1 \leq Z \leq +1) = .6826 \]
\[ P(0 \leq Z \leq x) = .3023. What is x? (.85) \]
\[ P(Z \leq 1.33) = .9082 \]
\[ P(-2.4 \leq Z \leq -1.2) = .1069 \]
\[ P(Z \geq 1.42) = .0778 \]
Reduction of arbitrary Normal probability to that of Standard Normal.

Assume $X \sim N(\mu, \sigma^2)$. Then

$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right).$$

Exercise. Powerful business women in America.

Denote $X =$ The age of a randomly chosen powerful business woman

According to Forbes, $X \sim N(50, 6^2)$. Find $P(55 \leq X \leq 60)$.

According to the above reduction we get

$$P(55 \leq X \leq 60) = 0.1558$$

A problem on Normal.

The profit from an investment is normally distributed with a mean of $11,200 and standard deviation of $8,250.
a. What is the probability that there will be a loss rather than a profit? (0.9131)
b. What is the probability that the profit will be between $10,000 and 20,000? (0.4234)
Normal Approximation to Binomial with the continuity correction

Exercise. An advertising research study indicates that 40% of the viewers exposed to an advertisement try the product during the following four months. If 100 people are exposed to the ad, what is the probability that at most 35 of them will try the product in the following four months?

Denote $X = (The$ number of people out of 100 who will try the product in the following four months)

Obviously, $X \sim B(100, 0.4)$. We have to find $P(X \leq 35)$. 
The Normal approximation to Binomial says that if \( X \sim B(n, p) \), \( n \) is big, \( p \) is small, then

\[
P(X \leq k) \text{ appr.} = P\left(Z \leq \frac{k - np + 0.5}{\sqrt{np(1 - p)}}\right)
\]

In our case \( n = 100, \ p = 0.4 \). We have to calculate

\[
P(X \leq 35) = P\left(Z \leq \frac{35 - 40 + 0.5}{\sqrt{24}}\right) = P(Z \leq -0.918) = 0.1788.
\]

The straightforward Binomial table gives 0.179.
Problem. *On a sum of independent Normally distributed random variables.*

A restaurant has three sources of revenue: eat in orders, takeout orders, and the bar. The daily revenue from each source is normally distributed with mean and standard deviation shown in the table below.

<table>
<thead>
<tr>
<th>Source</th>
<th>Mean</th>
<th>Standard Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eat in</td>
<td>$5,780</td>
<td>$142</td>
</tr>
<tr>
<td>Takeout</td>
<td>641</td>
<td>78</td>
</tr>
<tr>
<td>Bar</td>
<td>712</td>
<td>72</td>
</tr>
</tbody>
</table>

a. Will the total revenue on a day be normally distributed?
b. What are the mean and standard deviation of the total revenue on a particular day?
c. What is the probability that the revenue will exceed $7,000 on a particular day?
Solution.  a. According to a math theorem, a sum of normally distributed independent random variables is a Normally distributed random variable. So, the answer is “Yes”.

b. \( E(X_1 + X_2 + X_3) = 5780 + 641 + 712 = 7133 \).

\[ V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3) = 31432. \]

\( \Sigma (X_1 + X_2 + X_3) = 177.3 \)

c. \( P(X_1 + X_2 + X_3 > 7000) = P(Z > 7000 - 7133/177.3) = P(Z > -0.75) = 0.7734. \)
Homework: Work out the warm-up exercises of Quiz 3 (see on the top of this lecture).
(The material of Class 11) We studied the simplest continuous distribution – uniform on \((c,d)\).

\[
f(x) = \frac{1}{d-c}, \text{ if } c \leq x \leq d.
\]

\[
\mu_x = \frac{c + d}{2}.
\]

\[
\sigma_x = \frac{d - c}{\sqrt{12}}.
\]

**Typical example is 4.27, pp.243-244.**

Suppose the research department of a steel manufacturer believes that one of the company’s rolling machines is producing sheets of steel of varying thickness. The thickness is a uniform random variable with values between 150 and 200 millimeters. Any sheets less than 160 millimeters must be scrapped because they are unacceptable to buyers. What is the probability that a randomly chosen sheet will be scrapped?
a. Calculate and interpret the mean and standard deviation of $X$, the thickness of the sheet produced by the machine.

b. Graph the probability distribution of $X$, and show the mean on the horizontal axis. Also show 1- and 2-standard deviation intervals around the mean.
The Exponential Distribution.

Example 4.28, p.245.

\( X = \) The length of time (in hours) between emergency arrivals at a certain hospital. The book says that \( X \) is subject to exponential with the mean \( \mathcal{G} = 2 \), i.e. 
\[
f(x) = \frac{1}{2} e^{-\frac{x}{2}}, \quad 0 \leq x < \infty .
\]

Find \( P(X \geq 5) \).

To work out this problem we give the necessary information about exponential:

\[
f(x) = \frac{1}{\mathcal{G}} e^{-\frac{x}{\mathcal{G}}}, \quad 0 \leq x < \infty ;
\]

\( \mu_x = \mathcal{G}, \quad \sigma_x = \mathcal{G} \).

\[
P(X \geq a) = e^{-\frac{a}{\mathcal{G}}} . \quad (1)
\]

The formula (1) will be used repeatedly.
Solution to problem 4.28. According to (1),

\[ P(X \geq 5) = \int_{5}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = e^{-\frac{5}{2}} = 0.08208. \]

We also worked out the following problems:

4.138. \( X \) is subject to exponential with mean 2. Find the probability that \( X \) is not within 3 standard deviation of the mean.

Solution.

\[ P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = P(X \leq 4\theta) = 1 - e^{-4} = 0.98. \]

We have found also the median of exponential \( X \) (in terms of the mean).

Some other questions on exponential. What is the median of \( X \)?

For the median we have the following equation:

\[ P(X > M) = 0.5 \text{ or } e^{-\frac{M}{2}} = 0.5; \quad M = -2 \ln 0.5 = 2 \cdot 0.69 = 1.38 \]
And the last problem: Find the probability
\[ P(1 \leq X \leq 3) \, . \]

Solution.

\[ P(1 \leq X \leq 3) = P(X \geq 1) - P(X \geq 3) = e^{-\frac{1}{2}} - e^{-\frac{3}{2}} = 0.61 - 0.14 = 0.47. \]