Plan for today: Quiz 2 will be held. The problem of the day is the Binomial problem. But first, a reminder of Class 6 material: random variables, distribution of a random variable, numerical characteristics: Expectation, Variance and Standard Deviation.

Any function $X$ defined on the sample space is called a random variable.

Let us introduce a random variable $X$ having the meaning of a win in a card game. $X$ is defined as follows:
If a card drawn is either Jack or Queen, then $X(s) = 5$; if $s$ is either King or Ace, then $X(s)=10$. Otherwise $X(s)=-4$.

Distribution of $X$:

<table>
<thead>
<tr>
<th>$X$</th>
<th>4</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>36/52</td>
<td>8/52</td>
<td>8/52</td>
</tr>
</tbody>
</table>

In the first row the possible values are given, in the second row the corresponding probabilities.
The Expectation (Expected value, Mean) is defined as follows:

\[ E(X) = \mu_X = \sum_{i=1}^{n} p_i x_i ; \text{ Expectation also is denoted by } E(X). \]

In our problem of drawing a card

\[ E(X) = \mu_X = \sum_{i=1}^{n} p_i x_i = \frac{8}{52} \cdot 5 + \frac{8}{52} \cdot 10 - 4 \cdot \frac{8}{52} = -0.4615. \]

The Variance is defined as

\[ V(X) = \sum_{i=1}^{n} p_i (x_i - \mu_X)^2 = \sum_{i=1}^{n} p_i x_i^2 - \mu_X^2. \]

The last expression (right-hand-side) is the so called a shortcut formula for variance.

Using it we find

\[ V(X) = \sum_{i=1}^{n} p_i x_i^2 - \mu_X^2 = 16 \cdot \frac{36}{52} + 25 \frac{8}{52} + 100 \cdot \frac{8}{52} - 0.4615^2 = 30.24. \]
\[ V(X) = 30.24, \sigma(X) = 5.5. \]
Exercise 4.12. (p.194) A random variable \( X \) has the following probability distribution:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>.1</td>
<td>.2</td>
<td>.4</td>
<td>.2</td>
<td>.1</td>
</tr>
</tbody>
</table>

a. List the possible values of \( X \).
b. What value of \( X \) is most probable?
c. Find \( P(X=7) \).
d. Find \( P(X >2) \).
e. Find \( E(X) \) and \( V(X) \). (Answer: \( E(X)=5 \), \( V(X)=4.8 \)).

In e.: The book says \( E(X) = 3.2 \), but it is 5. Please check!

The Binomial Problem

Students like the following simple formulation of the problem.

**A PROBLEM ON BINOMIAL.** A coin is tossed 5 times. TOSSES ARE INDEPENDENT. The probability of “H” is 0.3 (in each of the tosses), the probability of “T” is 0.7. What is the probability that “H” will occur 3 times?

The problem was posed 300 years ago by the Swiss mathematician Jacob Bernoulli.

The answer is: \( P(\text{Three H’s in 5 tosses}) = 10 \times 0.3^3 \times 0.7^2 \)

The coefficient 10 is the number of ways 3 positions (for H’s) can be chosen out of 5 positions.

The general problem solves out as follows:

\[
P(\text{“H” occurs k times in n tosses}) = \frac{n!}{k!(n-k)!} \times p^k(1-p)^{n-k}.
\]

The following problem mathematically is absolutely equivalent to the previous problem on the number of “H”-s.
Problem on Binomial. A student graduating from a university is applying for 5 jobs. And she believes that in each of the cases she has the probability 0.4 of getting an offer. Find the probability that she will get 2 offers.

A general table with answers to Binomial can be found on page 794 Appendix D. We give such a table right here.
<table>
<thead>
<tr>
<th>k</th>
<th>Individual probabilities</th>
<th>Cumulative probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.078</td>
<td>0.078</td>
</tr>
<tr>
<td>1</td>
<td>0.259</td>
<td>0.337</td>
</tr>
<tr>
<td>2</td>
<td>0.346</td>
<td>0.683</td>
</tr>
<tr>
<td>3</td>
<td>0.230</td>
<td>0.913</td>
</tr>
<tr>
<td>4</td>
<td>0.077</td>
<td>0.990</td>
</tr>
<tr>
<td>5</td>
<td>0.010</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Appendix, p.794 gives only the right-hand side column.

The first two columns actually represent the distribution of the random variable

\[ X = \{ \text{The number of “H”-s in 5 tosses, the probability of “H” in each toss is 0.4} \} \]
This is called a Binomial random variable with the probability of “success” 0.4 and the number of trials 5: \( X \sim \text{Bin}(5, 0.4) \). In general, we’ll consider

\( X \sim \text{Bin}(n,p) \), and

\[
P(X = k) = \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k}. \text{ The mean } E(X) = np,\]

\( \text{The variance equals } V(X) = np(1 - p). \)

Homework: Text to read pp.199-208, Exercises 4.40-4.46, 4.47.