Analysis of Stochastic PDEs

CBMS-NSF Course at Michigan State University

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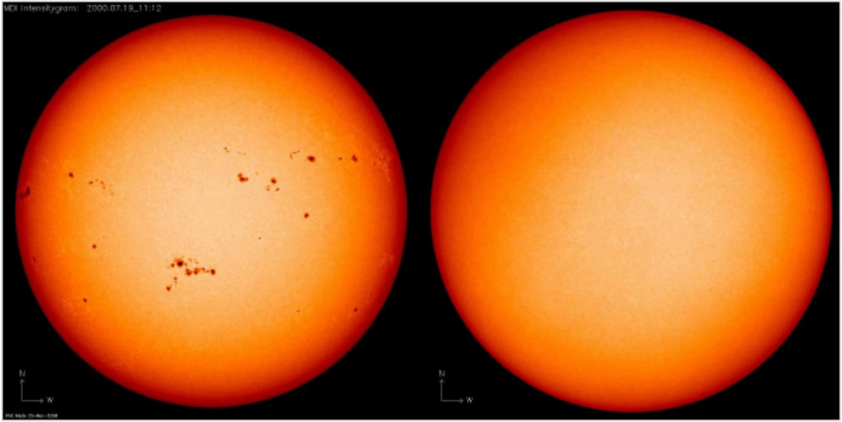
Nonlinear noise excitation (Lecture 1)

Is the Sun Missing Its Spots?



NASA

SUN GAZING These photos show sunspots near solar maximum on July 19, 2000, and near solar minimum on March 18, 2009. Some global warming skeptics speculate that the Sun may be on the verge of an extended slumber.

By KENNETH CHANG
Published: July 20, 2009

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$\partial_t u = \frac{1}{2}\partial_x^2 u + \lambda u \xi$ **on** $[0\,,1]$ **with Dirichlet BC**
$u_0(x) = \sin(\pi x)$ **[K-Kim, 2013]**

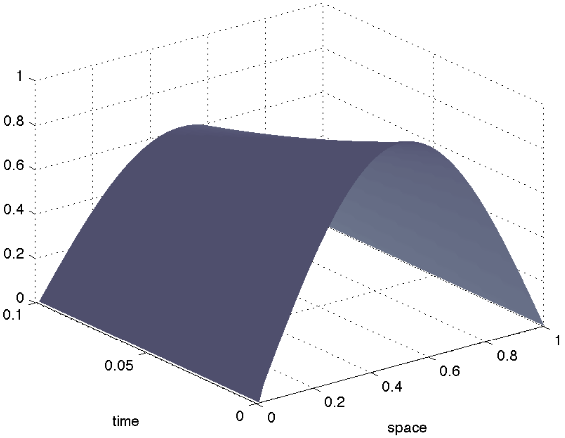

Figure: $\lambda = 0$; $u_t(x) = \sin(\pi x)\exp(-\pi^2 t/2)$

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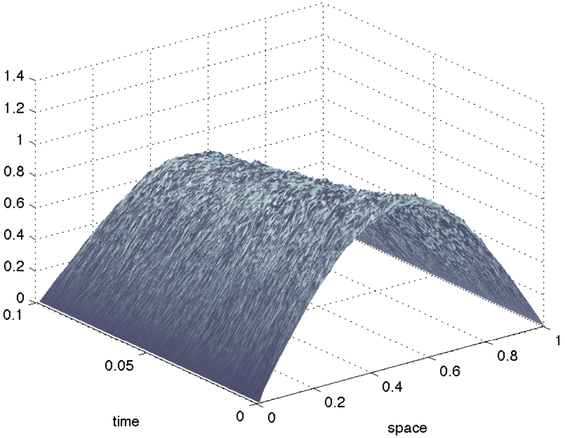

Figure: $\lambda = 0.1$; max. peak ≈ 1.4

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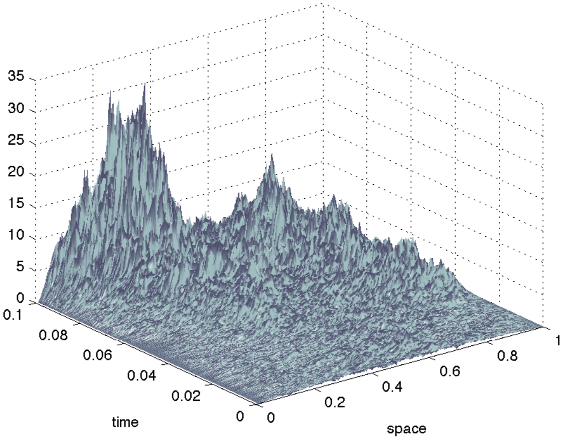

Figure: $\lambda = 2$; max. peak ≈ 35

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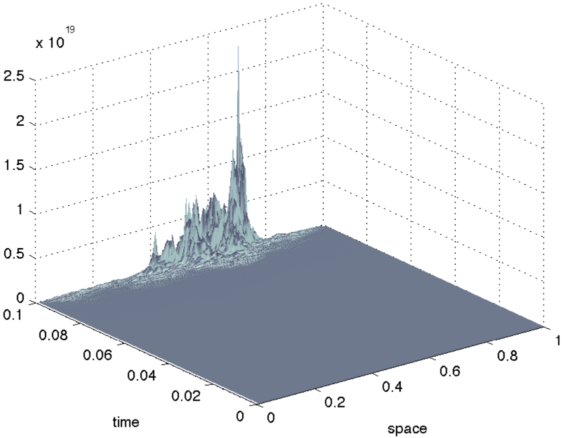

Figure: $\lambda = 5$; max. peak $\approx 2.5 \times 10^{19}$

Nonlinear noise excitation (Lecture 1)

Intermittency

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- Deep relations to fluid dynamics (Baxendale-Rozovskiĭ, 1993), turbulence (Mandelbrot, 1983; Majda, 1993; Gibbon and Titi, 2005), complex chemical reactions and the large-scale structure of galaxies (Molchanov, 1991; Shandarin-Zel′dovich, 1989; Zel′dovich et al, 1987, 1988, 1990) . . .

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- Complex problems in random media are associated to intermittency: As the systems feels more noise, it can begin to act erratically.
- Many field theories (SPDEs) yield intermittent solutions.

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- **Definition (Wiener, 1923)**
  White noise on $\mathbf{R}^m$ is a mean-zero set-indexed Gaussian random field [GRF] $\{\xi(A)\}_{A\in\mathscr{L}(\mathbf{R}^m)}$ with

  $$\mathrm{Cov}(\xi(A_1)\,,\xi(A_2)) = |A_1 \cap A_2| \qquad (A_i \in \mathscr{L}(\mathbf{R}^m)),$$

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  - Easy fact: White noise exists and is an $L^2(\Omega)$-valued countably-additive measure on $\mathscr{L}(\mathbf{R}^m)$.

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  - Suppose $h : \mathbf{R}^m \to \mathbf{R}$ is elementary; i.e., $h(x) := \sum_{i=1}^n c_i \mathbf{1}_{A_i}(x)$ where $A_i \in \mathscr{L}(\mathbf{R}^n)$ are disjoint and $c_i \in \mathbf{R}$. Then,

$$\xi(h) := \int h \, \mathrm{d}\xi := \int h(x) \, \xi(\mathrm{d}x) := \sum_{i=1}^n c_i \xi(A_i)$$

is defined unambiguously; the preceding is: (a) Linear in h [a.s.]; (ii) A GRF indexed by all elementary functions h; and (iii)

$$\mathrm{E}\left[\xi(h)\right] = 0 \quad \text{and} \quad \mathrm{E}\left(|\xi(h)|^2\right) = \|h\|^2_{L^2(\mathbf{R}^m)}.$$

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Gaussian random fields [GRFs] (Lecture 2)

Wiener integrals

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  - $\mathrm{Cov}(\int h \, \mathrm{d}\xi \, , \int g \, \mathrm{d}\xi) = \int_{\mathbf{R}^m} h(x)g(x) \, \mathrm{d}x \qquad \forall h, g \in L^2(\mathbf{R}^m)$;

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  - If $h, g \in L^2(\mathbf{R}^m)$ and $h \perp g$ then $\int h \, \mathrm{d}\xi$ is independent from $\int g \, d\xi$.

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- **Proposition (A stochastic Young inequality)**
  If $f \in L^2(\mathbf{R}^m)$, then there is a modification of $f * \xi$ that is measurable [jointly in $(x\,,\omega)$]. Moreover, for all Borel measures μ on $\mathbf{R}^m$,

$$\mathrm{E}\left(\left|\int_{\mathbf{R}^m}(f*\xi)(x)\,\mu(\mathrm{d}x)\right|^2\right) \leqslant [\mu(\mathbf{R}^m)]^2 \cdot \|f\|^2_{L^2(\mathbf{R}^m)}.$$

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- In that case, the Wiener isometry shows that

$$\begin{aligned}\mathrm{E}\left(|(f*\xi)(x)-(f*\xi)(y)|^2\right) &= \int_{\mathbf{R}^m} |f(x-z)-f(y-z)|^2\,\mathrm{d}x\\ &\leqslant \text{const}\cdot\|x-y\|^2.\end{aligned}$$

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- The rest follows from the Cauchy–Schwarz inequality. $\square$

Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

- **Definition (Čencov, 1956)**

  A *Brownian sheet* over $\mathbf{R}^m$ is a mean-zero GRF $\{B(x)\}_{x\in\mathbf{R}^m}$ with

$$\mathrm{Cov}(B(x)\,, B(y)) = \prod_{j=1}^m \min(|x_j|\,, |y_j|)\mathbf{1}_{[0,\infty)}(x_jy_j) \qquad (x, y \in \mathbf{R}^m).$$

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  - When $m=1$, B is a [two-sided] Brownian motion.
  - B has a continuous modification [Kolmogorov continuity thm].

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  - In particular, if $\xi = \xi(t\,,x)$ is space-time white noise and W is Br. sheet on $\mathbf{R}^2$ then

$$\frac{\partial^2}{\partial t \partial x} W(t\,,x) = \xi(t\,,x).$$

Proof in the case that $m = 1$

- Goal. $\forall \phi \in C^\infty(\mathbf{R})$, supported in $(0\,,1)$: $\int \phi \, \mathrm{d}\xi = -\int_0^1 \phi' B$ [a.s.]

Example: Brownian sheet (Lecture 2)

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$$\int \phi_n \, \mathrm{d}\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi\left(\left[\frac{j}{n}\,, \frac{j+1}{n}\right]\right)$$

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&= \sum_{j=0}^{n-1} \phi(j/n)\left\{B((j+1)/n) - B(j/n)\right\} \\
&= -\sum_{j=1}^{n} B(j/n)\left\{\phi\left(\frac{j}{n}\right) - \phi\left(\frac{j-1}{n}\right)\right\} \\
&\to -\int_0^1 B(t)\phi'(t)\,\mathrm{d}t \qquad (n \to \infty).
\end{aligned}$$

- LHS $\to \int \phi\,\mathrm{d}\xi$ by definition. $\square$

Example: Brownian sheet

- **Corollary (Stochastic Fubini)**
  If $f \in L^2(\mathbf{R}^m)$ and μ is a finite Borel measure on $\mathbf{R}^m$, then

$$\int_{\mathbf{R}^m} (f * \xi)\, \mathrm{d}\mu = \int \left(\breve{f} * \mu\right) \mathrm{d}\xi \qquad\qquad (\breve{f}(x) := f(-x)).$$

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    $$\text{LHS}=(-1)^m \int_{\mathbf{R}^m} \mu(\mathrm{d}x) \int_{\mathbf{R}^m} \mathrm{d}y\ B(y) \frac{\partial^m f(x-y)}{\partial y_1 \cdots \partial y_m}$$

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  - By the Young inequality, $\breve{f}_n * \mu \to \breve{f} * \mu$ in $L^2(\mathbf{R}^m)$; therefore,

$$\int \left(\breve{f}_n * \mu\right) \mathrm{d}\xi \to \int \left(\breve{f} * \mu\right) \mathrm{d}\xi \text{ in } L^2(\Omega),$$

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  - By Step 1, LHS $= \int (f_n * \xi)\, \mathrm{d}\mu$; owing to the stochastic Young inequality, it suffices to prove that $f_n * \xi \to f * \xi$ in $L^2(\Omega)$, but this holds also by the Wiener isometry. $\square$

Gaussian random fields [GRFs] (Lecture 2)

Example: fractional Brownian motion [fBm]

- **Definition (Mandelbrot–Van Ness, 1968)**

  An fBm *with index H* is a centered Gaussian process $\{X_t\}_{t \geqslant 0}$ with $X_0 = 0$ and $\mathrm{E}\left(|X_t - X_s|^2\right) = |t - s|^{2H}$ $(s, t \geqslant 0)$.

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  - fBm$(1/2)$ = BM.
  - fBm(H) has a Hölder-continuous modification [Kolmogorov continuity thm] of index $< H$.

- Define for all $t \geqslant 0$ and $H, s \in \mathbf{R}$,

$$f_H(t\,,s) := (t-s)_+^{H-(1/2)} - (-s)_+^{H-(1/2)}$$

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- Then,

$$\|f_H(t\,,\bullet)\|^2_{L^2(\mathbf{R})} = \int_0^t (t-s)^{2H-1}\,\mathrm{d}s + \int_0^\infty \left[(t+s)^{H-(1/2)} - s^{H-(1/2)}\right]^2 \mathrm{d}s$$

is finite iff $H \in (0\,,1)$. And $\forall H \in (0\,,1)$:
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  is finite iff $H \in (0\,,1)$. And $\forall H \in (0\,,1)$:
  $\|f_H(t\,,\bullet)\|^2_{L^2(\mathbf{R})} = C_H t^{2H}$.

- Then we can construct fBm(H) as

$$X_t := \frac{1}{\sqrt{C_H}} \int f_H(t\,,s)\,\xi(\mathrm{d}s) \qquad (t>0).$$

- Let $\{X_t\}_{t \geqslant 0}$ be a fBm(H).

Gaussian random fields [GRFs] (Lecture 2)

Example: fractional Brownian motion [fBm] (some background facts)

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- **Theorem (Marcus, 1968; Shao, 1996; ...)**

  With probability one:

$$\limsup_{\varepsilon\downarrow 0} \frac{X_{t+\varepsilon}-X_t}{\varepsilon^H\sqrt{2\ln\ln(1/\varepsilon)}} = 1 \quad \forall t \geqslant 0; \quad \textit{and}$$

$$\lim_{n\to\infty} \sum_{a2^n\leqslant j\leqslant b2^n} \left|X_{(j+1)/2^n}-X_{j/2^n}\right|^{1/H} = \mathrm{E}\left(|\mathcal{N}|^{1/H}\right)\cdot(b-a);$$

  $\forall 0 \leqslant a < b < \infty$, *where* $\mathcal{N}$ *is a standard normal r.v.*

- Let μ be a finite signed Borel measure on $\mathbf{R}$. Want to solve the initial-value problem

$$\frac{\partial}{\partial t}u = \frac{\nu}{2}\frac{\partial^2}{\partial x^2}u + \mu, \tag{HE}$$

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- **Definition**

  We say that $u = u_t(x)$ is a *weak solution* to (HE) if $u \in L^1_{loc}(\mathbf{R}_+ \times \mathbf{R})$ and

$$-\int_{\mathbf{R}_+\times\mathbf{R}} u\frac{\partial}{\partial t}\varphi \, \mathrm{d}t\, \mathrm{d}x = \frac{\nu}{2}\int_{\mathbf{R}_+\times\mathbf{R}} u\frac{\partial^2}{\partial x^2}\varphi\, \mathrm{d}t\, \mathrm{d}x + \int \varphi\, \mathrm{d}\mu,$$

  for all $\varphi \in C_c^\infty((0\,,\infty)\times\mathbf{R})$.

- Define the heat kernel

$$p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left(-\frac{x^2}{2\nu t}\right) \qquad (t > 0,\ x \in \mathbf{R}).$$

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- **Theorem (Duhamel's principle)**

  The following is the unique weak solution to (HE)*:*

$$u_t(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y - x)\, \mu(\mathrm{d}s\, \mathrm{d}y).$$

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- "Mild ⇒ weak," even when μ is signed and σ-finite, as long as μ has "bounded thermal energy" (Watson, 1974).

A Linear Heat Equation (Lecture 3)

A non-random heat equation ($\partial_t u = (\nu/2)\partial_x^2 u + \mu$)

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- $\mu \equiv 0 \Rightarrow\ u_t(x) = (p_t * u_0)(x)$ solves (HE).

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A Linear Heat Equation (Lecture 3)

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- Now we study the "linear stochastic heat equation,"

$$\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi, \qquad \text{(SHE)}$$

  subject to $u_0 :=$ nice and non random; $\xi :=$ space-time white noise. $[\xi = \partial_t \partial_x W]$

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A random heat equation $(\partial_t u = (\nu/2)\partial_x^2 u + \xi)$

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  - The solution is a GRF!

- Immediate goal: Describe the local behavior of the solution to (SHE). Let

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- This solves $\partial_t Z = (\nu/2)\partial_x^2 Z + \xi$ subject to $Z_0(x) \equiv 0$. The general case is $u_t(x) = (p_t * u_0)(x) + Z_t(x)$, so the real issue is to understand the structure of Z.

A Linear Heat Equation (Lecture 3)

Structure theory

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- This solves $\partial_t Z = (\nu/2)\partial_x^2 Z + \xi$ subject to $Z_0(x) \equiv 0$. The general case is $u_t(x) = (p_t * u_0)(x) + Z_t(x)$, so the real issue is to understand the structure of Z.
- We are about to see that $t \mapsto Z_t(x)$ is a smooth perturbation of fBm($1/4$) and $x \mapsto Z_t(x)$ is a smooth perturbation of a two-sided Br. motion. In particular, the local behavior of Z looks like fBm($1/4$) in time and Br. motion in space, viz.

A Linear Heat Equation (Lecture 3)

Structure theory

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- 2-sided BM is any GRF $\{B(x)\}_{x\in\mathbf{R}}$ with $\mathrm{E}(|B(x)-B(y)|^2) \propto |x-y|$.

Corollary (Swanson, 2007; Pospìšil–Tribe, 2007)

With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< 1/4$ and $x \mapsto Z_t(x)$ is Hölder continuous of index $< 1/2$. Moreover:

$$\limsup_{\varepsilon\downarrow 0} \frac{Z_t(x+\varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln\ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \qquad (\forall t > 0,\, x \in \mathbf{R});$$

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A Linear Heat Equation (Lecture 3)

Structure theory (some corollaries; $Z_t(x) := \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,\xi(\mathrm{d}s\,\mathrm{d}y)$.)

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A Linear Heat Equation (Lecture 3)

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$$\lim_{m\to\infty} \sum_{c2^m \leqslant i \leqslant d2^m} \left[Z_{(i+1)/2^m}(x) - Z_{i/2^n}(x)\right]^4 = \frac{6(d-c)}{\nu\pi} \qquad (0 < c < d,\ x \in \mathbf{R}).$$

A Linear Heat Equation (Lecture 3)

Structure theory ($Z_t(x) := \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,\xi(\mathrm{d}s\,\mathrm{d}y)$)

- **Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)**

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  (i) $\forall$ *fixed* $x \in \mathbf{R}$, $\exists\,\mathrm{fBm}(1/4)$ $\{X_t\}_{t\geqslant 0}$ *such that*

  $$Z_t(x) - \left(\frac{2}{\nu\pi}\right)^{1/4} X_t \qquad (t \geqslant 0)$$

  is a mean-zero continuous Gaussian process that is C^∞ *on* $(0\,,\infty)$.

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A Linear Heat Equation (Lecture 3)

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- A version of (ii) was first found in Walsh (1986).

A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $Z_t(x) := \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,\xi(\mathrm{d}s\,\mathrm{d}y)$]

- Expand
$$Z_{t+\varepsilon}(x) - Z_t(x) = J_1 + J_2,$$
where
$$J_1 := \int_{(0,t)\times\mathbf{R}} \left[p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)\right] \xi(\mathrm{d}s\,\mathrm{d}y);$$
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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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- We compute these terms separately, using Wiener's isometry.

A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $\mathrm{E}(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = \mathrm{E}(J_1^2) + \mathrm{E}(J_2^2)$]

- Since $J_2 = \int_{(t,t+\varepsilon)\times\mathbf{R}} p_{t+\varepsilon-s}(y-x)\,\xi(\mathrm{d}s\,\mathrm{d}y)$,

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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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Chapman–Kolm

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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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- By Plancherel's theorem,

$$\int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, [p_{\varepsilon+s}(y) - p_s(y)]^2 = \sqrt{\frac{\varepsilon}{\nu\pi}}\left(\sqrt{2} - 1\right) = \sqrt{\frac{2\varepsilon}{\nu\pi}} - \mathrm{E}(J_2^2).$$

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- Let η be an independent white noise on $\mathbf{R}$, and define $T_t := (2\nu\pi)^{-1/2} \int_{-\infty}^{\infty} z^{-1}(1 - \exp\{-\nu t z^2/2\}) \, \eta(\mathrm{d}z)$. Then T is smooth, and similar computations show that

$$\mathrm{E}\left(|T_{t+\varepsilon} - T_t|^2\right) = \int_t^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; [p_{s+\varepsilon}(y) - p_s(y)]^2 \, .$$

- $\mathrm{E}(J_1^2) = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{\varepsilon+s}(y) - p_s(y)]^2$.
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- Therefore,
  $$\mathrm{E}\left(|Z_{t+\varepsilon}(x) + T_{t+\varepsilon} - Z_t(x) + T_t|^2\right) = \sqrt{\frac{2\varepsilon}{\nu\pi}} \qquad [\text{fBm}(1/4)].$$

A Linear Heat Equation (Lecture 3)

Ideas of proof [Foondun–K–Mahboubi]

- $Z_t(x+\varepsilon)-Z_t(x) = \int_{(0,t)\times\mathbf{R}} [p_{t-s}(y-x-\varepsilon)-p_{t-s}(y-x)]\,\xi(\mathrm{d}s\,\mathrm{d}y) \Rightarrow$

A Linear Heat Equation (Lecture 3)

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- By Wiener+Plancherel,

$$\mathrm{E}\left(|Z_t(x+\varepsilon)-Z_t(x)|^2\right)=\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \ [p_s(y-\varepsilon)-p_s(y)]^2$$

A Linear Heat Equation (Lecture 3)

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A Linear Heat Equation (Lecture 3)

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$$
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&= \frac{\varepsilon}{\nu\pi}\int_{-\infty}^{\infty}\left(1-\mathrm{e}^{-t\nu(w/\varepsilon)^2}\right)\left(\frac{1-\cos w}{w^2}\right)\mathrm{d}w \\
&= \frac{\varepsilon}{\nu} - \frac{\varepsilon}{\nu\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-t\nu(w/\varepsilon)^2}\left(\frac{1-\cos w}{w^2}\right)\mathrm{d}w
\end{aligned}
$$

A Linear Heat Equation (Lecture 3)

Ideas of proof [Foondun–K–Mahboubi]

- $Z_t(x+\varepsilon)-Z_t(x) = \int_{(0,t)\times\mathbf{R}} [p_{t-s}(y-x-\varepsilon)-p_{t-s}(y-x)]\,\xi(\mathrm{d}s\,\mathrm{d}y) \;\Rightarrow$
- By Wiener+Plancherel,

$$
\begin{aligned}
\mathrm{E}\left(|Z_t(x+\varepsilon)-Z_t(x)|^2\right) &= \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \;\; [p_s(y-\varepsilon)-p_s(y)]^2 \\
&= \frac{\varepsilon}{\nu\pi} \int_{-\infty}^{\infty} \left(1-\mathrm{e}^{-t\nu(w/\varepsilon)^2}\right)\left(\frac{1-\cos w}{w^2}\right)\mathrm{d}w \\
&= \frac{\varepsilon}{\nu} - \frac{\varepsilon}{\nu\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-t\nu(w/\varepsilon)^2}\left(\frac{1-\cos w}{w^2}\right)\mathrm{d}w \\
&= \frac{\varepsilon}{\nu} - \mathrm{E}\left(|S(x+\varepsilon)-S(x)|^2\right),
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where $S(x) := \displaystyle\int_{(t,\infty)\times\mathbf{R}} [p_s(w-x)-p_s(w)]\,\zeta(\mathrm{d}s\,\mathrm{d}w)$, for an independent space-time white noise ζ.

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  - There are, more interesting, nonlinear versions as well.
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  - The preceding says that we can think of the linear stochastic heat equation as the infinite-density limit of a system of interacting BMs with nearest-neighbor gravitational attraction.

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- The same sort of remark applies to space-time white noise, as long as we "regularize the Laplacian" [e.g., replace it with $-(-\Delta)^{1+\delta}$ for a suitable $\delta > 0$].

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- Similar issues arise in the Itô theory of SDEs.

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- $\mathscr{F}_t(h) :=$ the sigma-algebra generated by the completions of all $\mathscr{F}_t^{0,x}(h)$, over all $x \in \mathbf{R}$.
- $\mathscr{F}_t :=$ the sigma-algebra generated by $\mathscr{F}_t(h)$. Intuitively, $\mathscr{F}_t$ contains all the white noise information by time t.

Walsh–Dalang integrals (Lecture 4)

Stochastic integration

- Wish to construct an Itô-type integral $\int \Phi_t(x)\,\xi(\mathrm{d}t\,\mathrm{d}x)$ when Φ is a "predictable" random field. More convenient form: $\int h_t(x)\Phi_t(x)\,\xi(\mathrm{d}t\,\mathrm{d}x)$ for meas. non-random h.
- Need a filtration first … Answer: Brownian filtration.
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- $\{\mathscr{F}_t\}_{t\geqslant 0}$ the "Brownian filtration."

Walsh–Dalang integrals (Lecture 4)

The stochastic integral

- $(t\,,x) \mapsto \Phi_t(x)$ is an *elementary* random field when $\exists 0 \leqslant a < b$ and an $\mathscr{F}_a$-meas. $X \in L^2(\Omega)$ and $\phi \in L^2(\mathbf{R})$ such that

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- The stochastic integral is Wiener's; well-defined iff $h_t(x)\phi(x) \in L^2([a\,,b] \times \mathbf{R})$.

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- $\int h\Phi \, d\xi$ is defined unambiguously, as a result.

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- Note that if Φ is any simple random field, then

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- **Definition**
  Let $\mathcal{L}^{\beta,2} :=$ the completion of all simple random fields in norm $\mathcal{N}_{\beta,2}$.

- If $\Phi \in \mathcal{L}^{\beta,2}$, then $I := \int h\Phi \, \mathrm{d}\xi$ well-defined, and $\mathrm{E}(I^2) \leqslant [\mathcal{N}_{\beta,2}(\Phi)]^2 \int_0^\infty \mathrm{e}^{\beta s} \, \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \; [h_s(y)]^2$, provided that h is meas. and the preceding integral converges.

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- We have the Walsh isometry for all $\Phi \in \mathcal{L}^{\beta,2}$ and h as above: $\mathrm{E}(I^2) = \int_0^\infty \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}x\ [h_t(x)]^2 \mathrm{E}(|\Phi_t(x)|^2)$.

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- **Remark**
  $M_t := \int_{(0,t)\times\mathbf{R}} h\Phi \, \mathrm{d}\xi$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t = \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \; [h_s(y)]^2 |\Phi_s(y)|^2$.

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- **Remark**
  $M_t := \int_{(0,t)\times\mathbf{R}} h\Phi\, d\xi$ *is a continuous* $L^2(\Omega)$*-martingale with quadratic variation* $\langle M\rangle_t = \int_0^t ds \int_{-\infty}^\infty dy\ [h_s(y)]^2 |\Phi_s(y)|^2$.
  - Proof: Check when Φ is simple; appeal to Doob's maximal inequality when $\Phi \in \mathcal{L}^{\beta,2}$. □

- **Theorem (Burkholder–Davis–Gundy, 1966, 1970, 1972; Carlen–Kree, 1991; Davis, 1970, 1976)**

  If M_t is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M\rangle_t$, then for all real numbers $k \in [2\,,\infty)$,

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  - See the lecture notes [Appendix B] for proofs etc.

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  - See the lecture notes [Appendix B] for proofs etc.
  - Equivalently, $\mathrm{E}(|M_t|^k) \leqslant (4k)^{1/2}\mathrm{E}(\langle M\rangle_t^{k/2})$.

- **Proposition**

  If Φ and h are as before, then $\forall t \geqslant 0$,

$$\left\| \int_{(0,t)\times\mathbf{R}} h\Phi \, \mathrm{d}\xi \right\|_k^2 \leqslant 4k \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [h_s(y)]^2 \|\Phi_s(y)\|_k^2.$$

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  - Proof. The quadratic variation of $M_t := \int_{(0,t)\times\mathbf{R}} h\Phi \, \mathrm{d}\xi$ is $\langle M\rangle_t = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} [h_s(y)]^2 [\Phi_s(y)]^2$, whence by BDG,

$$\|M_t\|_k^2 \leqslant 2k \left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} [h_s(y)]^2 [\Phi_s(y)]^2 \right\|_{k/2}.$$

  Apply Minkowski's inequality. $\square$

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- **Proposition**

  Suppose Φ is a space-time random field that is continuous in $L^2(\Omega)$ and $\mathcal{N}_{\beta,2}(\Phi) < \infty$, then $\Phi \in \cap_{\alpha>\beta}\mathcal{L}^{\alpha,2}$.

- Approximate Φ with

$$S_t^{n,N}(x) := \Phi_{\lfloor nt \rfloor / n}(\lfloor nx \rfloor / n) \cdot \mathbf{1}_{[0,N]\times\mathbf{R}}(t\,, x),$$

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- Prove that $\mathcal{N}_{\alpha,2}\left(S^{n,N} - \Phi\right) \to 0$ as $n, N \to \infty$, for all $\alpha > \beta$. $\square$

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  - $\therefore \quad p \circledast \bullet : \cup_{\beta>0}\mathcal{L}^{\beta,2} \to \cup_{\beta>0}\mathcal{L}^{\beta,2}$.

A Nonlinear Heat Equation (Lecture 5)

The key step of the proof [see the lecture notes for the rest]

- If $\beta > 0$ and $k \in [2\,,\infty)$, then

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A Nonlinear Heat Equation (Lecture 5)

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- **Proposition (Foondun–K, 2009; Conus–K, 2010)**
  For all $\beta > 0$, $k \in [2\,,\infty)$, and $\Phi \in \mathcal{L}^{\beta,2}$,

$$\mathcal{N}_{\beta,k}(p \circledast \Phi) \leqslant \frac{k^{1/2}}{(\nu\beta/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi).$$

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$$\|p \circledast \Phi\|_k^2 \leqslant 4k \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [p_{t-s}(y-x)]^2 \|\Phi_s(y)\|_k^2$$

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A Nonlinear Heat Equation (Lecture 5)

Proof of the stochastic Young inequality $[\mathcal{N}_{\beta,k}(p \circledast \Phi) \leqslant \frac{k^{1/2}}{(\nu\beta/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi)]$

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- Do the remaining arithmetic. $\square$

A Nonlinear Heat Equation (Lecture 5)

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$$|\sigma(x)| \vee |b(x)| \leqslant \mathrm{Lip}(1 + |x|) \qquad \forall x \in \mathbf{R}.$$

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- **Theorem**
  There exists a random field $u \in \cup_{\beta>0}\mathcal{L}^{\beta,2}$ that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite L such that

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  - *We will see soon that the exponent bound of k^3 is not artificial.* □

- We consider the more important contribution of diffusion; thus, $b \equiv 0$ from now on; it is easy to adapt the proof to work for general drift function b.

A Nonlinear Heat Equation (Lecture 5)

Existence; sketch of proof

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- We will have to prove *a priori* that $u^{(n)}$'s are all well defined etc.

- *A priori bound #1*:

$$\mathcal{N}_{\beta,k}\left(u^{(n+1)}\right) \leqslant \mathcal{N}_{\beta,k}\left(p_\bullet * u_0\right) + \mathcal{N}_{\beta,k}\left(p \circledast \sigma(u^{(n)})\right)$$

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- $\mathrm{E}\left(\left|u_t^{(n)}(x)\right|^k\right) \leqslant C^k \exp\left\{16k^3 t\right\}.$

Existence; sketch of proof

$[u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s^{(n)}(y))\,\xi(\mathrm{d}s\,\mathrm{d}y).]$

- *A priori bound #2* (similar): $\exists D < \infty$—independently of k, n—such that $\mathcal{N}_{16k^2,k}\left(u^{(n+1)} - u^{(n)}\right) \leqslant D\mathrm{e}^{-n/D}$.

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  uniformly for $x, x' \in \mathbf{R}$ and $t, t' \in [0\,,T]$.
- $\mathcal{N}_{16k^2,k}(u^{(n)}) < \infty$ and continuity in $L^2(\Omega) \Rightarrow u^{(n)} \in \mathcal{L}^{64,2}$ $\forall n$, and existence works, as in PDEs, by taking Cauchy limits. $\square$

- Suppose $\exists\alpha > 0$ and $u, v \in \mathcal{L}^{\alpha,2}$, so that $\sup_{x\in\mathbf{R}}\{\mathrm{E}(|u_t(x)|^2) \vee \mathrm{E}(|v_t(x)|^2)\} \leqslant L\exp\{\alpha t\}\ \forall t > 0$ and

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- $\forall\beta > \alpha$,

$$\mathcal{N}_{\beta,2}\,(u - v) \leqslant \mathcal{N}_{\beta,2}\,(p \circledast [\sigma(u) - \sigma(v)])$$

A Nonlinear Heat Equation (Lecture 5)

Uniqueness; sketch of proof

- Suppose $\exists\alpha > 0$ and $u, v \in \mathcal{L}^{\alpha,2}$, so that $\sup_{x\in\mathbf{R}}\{\mathrm{E}(|u_t(x)|^2) \vee \mathrm{E}(|v_t(x)|^2)\} \leqslant L\exp\{\alpha t\}$ $\forall t > 0$ and

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- Choose β large to see that $\mathcal{N}_{\beta,2}(u-v) = 0$. $\square$

A Nonlinear Heat Equation (Lecture 5)

Addenda

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A Nonlinear Heat Equation (Lecture 5)
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- Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.
- "Stopping-time arguments" can be used to replace the Lipschitz-continuity hypotheses on σ and b to the following:

A Nonlinear Heat Equation (Lecture 5)

Addenda

- Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.
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Suppose $b \geqslant 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geqslant C|y|^{1+\delta}$ for all y. Then there is no solution that satisfies $\sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^2) \leqslant L\exp\{Lt\}$ for all $t > 0$.

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- Similarly for b, as we will see next (easier).

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- $\therefore J(t) := \inf_y \mathrm{E}(|u_t(y)|)$ is a supersolution to the ODE $f'(t) = q[f(t)]^{1+\delta}$ s.t. $f(0) = \ell$. Easy: $f(t) = \infty$ for all $t \geqslant (\delta q \ell^\delta)^{-1} \Rightarrow J(t) = \infty$ for all $t \geqslant (\delta q \ell^\delta)^{-1}$. □

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- Apply Itô's formula—à la Tanaka (1962/1963)—to the function $f(z) = |z - x|$ to see that

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- $L_t^x(X) := \int_0^t \delta_x(X_s)\,\mathrm{d}s$ is a continuous random field. local times

Intermezzo: PAM (Lecture 6)

- Consider the following *parabolic Anderson model*:

$$\frac{\partial}{\partial t}u = \frac{\nu}{2}\frac{\partial^2}{\partial x^2}u + u\xi, \tag{PAM}$$

subject to $u_0(x) := 1$.

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- **Theorem (Bertini–Cancrini, 1995)**
  For all integers $k\geqslant 1$, and all reals $t\geqslant 0$ and $x\in\mathbf{R}$,

$$\mathrm{E}\left(|u_t(x)|^k\right) \geqslant \exp\left(\frac{k(k^2-1)t}{24\nu}\right).$$

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- Because $(p_t * 1)(x) = 1$ and u is mild, $u = 1 + (p \circledast u)$, whence Feynman $u = 1 + (p \circledast 1) + (p \circledast 1 \circledast 1) + \cdots$.

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- By the Feynman–Kac formula,
$$U_t(x) = \mathrm{E}\left[\exp\left(\int_0^t G_{t-s}(B_s + x)\,\mathrm{d}s\right)\right],$$
where $\{G_t\}_{t\geqslant 0}$ is BM with speed ν.

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- Intuitively, ξ is a centered GRF with

$$\mathrm{Cov}(\xi_t(x)\,, \xi_s(y)) = \delta_0(t-s)\delta_0(x-y).$$

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- Now consider the case where G is a smooth approximation to ξ: A GRF with $\mathrm{E}G_t(x) = 0$ and $\mathrm{Cov}(G_t(x)\,, G_s(y)) = p_\varepsilon(s-t)p_\eta(x-y)$, where $\varepsilon, \eta \approx 0$ are positive.

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Intermezzo: PAM (Lecture 6)

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Theorem (Bertini–Cancrini, 1995; Hu–Nualart, 2009; Conus, 2011)

For all integers $k \geqslant 2$ and all reals $t \geqslant 0$ and $x \in \mathbf{R}$,

$$\mathrm{E}\left(|u_t(x)|^k\right) = \mathrm{E}\exp\left(\frac{1}{\nu}\sum_{1\leqslant i<j\leqslant k} L_{\nu t}^0\left(b^{(j)} - b^{(i)}\right)\right),$$

where the $b^{(i)}$'s are i.i.d. BMs with speed one.

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- Thus, $\langle M\rangle_t = \sum_{j=1}^k \int_0^t \left[\sum_{i=1}^k \operatorname{sgn}\left(b_s^{(j)} - b_s^{(i)}\right)\right]^2 \mathrm{d}s$

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  - $\mathrm{E}(|u_t(x)|^k) \geqslant \mathrm{E}\exp\{(2\nu)^{-1} M_{\nu t}\} = \exp\{\sigma^2 t/(8\nu)\}$. $\square$

Intermittency (Lecture 7)

Some motivation

- Let $\boldsymbol{\psi} := \{\psi_\ell(z)\}_{\ell \geqslant 0, z \in \mathbf{Z}}$ be a [discrete] space- [discrete] time *non negative* random field and $z \mapsto \psi_\ell(z)$ is i.i.d. <u>mean one</u> $\forall \ell \geqslant 0$.

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  - If $2 \leqslant k \leqslant K$, then $\frac{\gamma(k)}{k} \leqslant \frac{\gamma(K)}{K}$, by Jensen's inequality. The issue is with strict inequalities.

    $k^{-1} \log \mathrm{E}([\psi_\ell(z)]^k) \leqslant K^{-1} \log \mathrm{E}([\psi_\ell(z)]^K)$ when $k < K$

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  The function $k \mapsto k^{-1}\gamma(k)$ is well-defined and convex on $(0\,,\infty)$. Moreover, if $\gamma(k_0) > 0$ for some $k_0 > 1$, then $k \mapsto k^{-1}\gamma(k)$ is strictly increasing on $[k_0\,,\infty)$.

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- Because $\gamma(1) = 0$, convexity yields

$$\gamma(k) \leqslant \alpha\gamma(K) + (1-\alpha)\gamma(1) = \frac{k-1}{K-1}\gamma(K).$$

Rearrange, using the facts that: (i) $\gamma(k) > 0$ for all $k \geqslant k_0$; and (ii) $(k-1)/(K-1) < k/K$. □

Intermittency (Lecture 7)

Separation of scales

- **Lemma (Paley–Zygmund, 1932)**
  Fix reals $n > m \geqslant 2$*, and let* $X \in L^n(\Omega)$ *be non negative with* $\mathrm{P}\{X > 0\} > 0$*. Then* $\forall \delta \in (0\,,1)$*,*
  $$\mathrm{P}\left\{X \geqslant \delta \|X\|_m\right\} \geqslant (1 - \delta^m)^{n/(n-m)} \cdot \frac{[\mathrm{E}(X^m)]^{n/(n-m)}}{[\mathrm{E}(X^n)]^{m/(n-m)}}.$$

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  Solve to finish. $\square$

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Take logs etc. □

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Intermittency (Lecture 7)

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\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant z\leqslant \exp(\theta N)} \psi_N(z) \geqslant \|\psi_N(0)\|_m\right\} \\
&\leqslant \sum_{1\leqslant z\leqslant \exp(\theta N)} \mathrm{P}\left\{\psi_N(0) \geqslant \|\psi_N(0)\|_m\right\} \leqslant \left(1 + \left\lfloor \mathrm{e}^{\theta N}\right\rfloor\right) \frac{\mathrm{E}(\psi_N(0))}{\|\psi_N(0)\|_m} \\
&= \exp\left(-N\left[\frac{\gamma(m)}{m} - \theta\right] + o(N)\right).
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Intermittency (Lecture 7)

Separation of scales

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- Borel–Cantelli: $\exists 0<\theta_1<\theta_2<\cdots$ such that a.s. $\forall i$,

$$\begin{aligned}
0 &< \limsup_{N\to\infty} \frac{1}{N} \max_{1\leqslant z\leqslant \exp(\theta_i N)} \log \psi_N(z) \\
&< \liminf_{N\to\infty} \frac{1}{N} \max_{1\leqslant z\leqslant \exp(\theta_{i+1} N)} \log \psi_N(z) < \infty.
\end{aligned}$$

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Back to SPDEs

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- Fact. If $\gamma_2(x) > 0$ $\forall x$ then $k \mapsto k^{-1}\gamma_k(x)$ is strictly increasing; same for upper L. exponents.

Intermittency (Lecture 7)

Back to SPDEs

- **Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)**

  If $\inf|u_0|>0$ *then* $\inf_{x\in\mathbf{R}}\gamma_2(x)\geqslant(4\nu)^{-1}\inf_{z\in\mathbf{R}\setminus\{0\}}\left|\frac{\sigma(z)}{z}\right|^4$.

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Back to SPDEs

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Intermittency (Lecture 7)
Back to SPDEs

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- Apply the "key renewal theorem" to see that $I(t) \geqslant c_3 \exp\{c_4 t\}$. □

Intermittency Fronts (Lecture 8)

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- $\sigma : \mathbf{R} \to \mathbf{R}$ Lipschitz and $\sigma(0) = 0$.
- Synopsis of behavior. A kind of weak intermittency occurs. Roughly, tall peaks arise as $t \to \infty$, but the farthest peaks move roughly linearly with time away from the origin.

interm. fronts

Intermittency Fronts (Lecture 8)

- Define, for all $\alpha > 0$,

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- If there exists α_* that is both a lower front and an upper front then α_* is the intermittency front. phase transition

Intermittency Fronts (Lecture 8)

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  Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, $\mathscr{S}(\alpha) < 0 \quad \textit{if} \quad \alpha > \frac{1}{2}\mathrm{Lip}_\sigma^2$.
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  - The existence of an intermittency front has been proved recently by Le Chen–Dalang; in fact, they proved that the intermittency front is at $C^2/2$.
  - A closely-related result: Because $\sigma(0) = 0$ and $u_0 \in L^2(\mathbf{R})$, $u_t \in L^2(\mathbf{R})$ a.s. for all $t > 0$ [Dalang–Mueller, 2003].

Sketch of proof

- $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geqslant 0} \sup_{x \in \mathbf{R}} \left[\mathrm{e}^{-\beta t + cx} \mathrm{E}\left(|\Phi_t(x)|^2 \right) \right]^{1/2}.$

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- **Proposition**

  For all $c \in \mathbf{R}$, $\beta > c^2\nu/4$, and $\Phi \in \mathcal{L}^{\beta,2}$,

$$\mathcal{N}_{\beta,c}(p \circledast \Phi) \leqslant \frac{\mathcal{N}_{\beta,c}(\Phi)}{(\nu(4\beta - c^2\nu))^{1/4}}.$$

Intermittency Fronts (Lecture 8)

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- Because $\sigma(0) = 0$, $|\sigma(u_t^{(n)}(x))| \leqslant \mathrm{Lip}_\sigma |u_t^{(n)}(x)| \Rightarrow$

$$\mathcal{N}_{\beta,c}\left(u^{(n+1)}\right) \leqslant \mathcal{N}_{\beta,c}\left(p_\bullet * u_0\right) + \frac{\mathrm{Lip}_\sigma}{(\nu(4\beta - c^2\nu))^{1/4}}\mathcal{N}_{\beta,c}\left(u^{(n)}\right).$$

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- Also,

$$e^{-\beta t+cx}(p_t*|u_0|)(x)=e^{-\beta t}\int_{-\infty}^{\infty}p_t(y-x)e^{-c(y-x)}e^{cy}|u_0(y)|\,dy$$

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- Because $\sigma(0) = 0$, $|\sigma(u_t^{(n)}(x))| \leqslant \mathrm{Lip}_\sigma |u_t^{(n)}(x)| \Rightarrow$

$$\mathcal{N}_{\beta,c}\left(u^{(n+1)}\right) \leqslant \mathcal{N}_{\beta,c}(p_\bullet * u_0) + \frac{\mathrm{Lip}_\sigma}{(\nu(4\beta - c^2\nu))^{1/4}}\mathcal{N}_{\beta,c}\left(u^{(n)}\right).$$

- Also,

$$e^{-\beta t+cx}(p_t * |u_0|)(x) = e^{-\beta t}\int_{-\infty}^{\infty} p_t(y-x)e^{-c(y-x)}e^{cy}|u_0(y)|\,\mathrm{d}y$$

Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t\geqslant 0}\sup_{x\in\mathbf{R}}\left[\mathrm{e}^{-\beta t+cx}\mathrm{E}\left(|\Phi_t(x)|^2\right)\right]^{1/2}$

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- Also,

$$\begin{aligned}\mathrm{e}^{-\beta t+cx}(p_t * |u_0|)(x) &= \mathrm{e}^{-\beta t}\int_{-\infty}^{\infty} p_t(y-x)\mathrm{e}^{-c(y-x)}\mathrm{e}^{cy}|u_0(y)|\,\mathrm{d}y \\ &\leqslant \mathcal{N}_{0,c}(u_0)\exp\left(-t\left[\beta - \frac{c^2\nu}{2}\right]\right).\end{aligned}$$

- Apply this with $\beta := c^2\nu/2$ to see that

$$\mathcal{N}_{c^2\nu/2,c}\left(u^{(n+1)}\right) \leqslant \mathcal{N}_{0,c}(u_0) + \frac{\mathrm{Lip}_\sigma}{\sqrt{|c|\nu}}\mathcal{N}_{c^2\nu/2,c}\left(u^{(n)}\right).$$

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- The exponent is <0 iff $\alpha > c\nu/2$ for any $c>\mathrm{Lip}_\sigma^2/\nu$.
- $\therefore \mathscr{S}(\alpha)<0$ if $\alpha>\mathrm{Lip}_\sigma^2/2$. 1/2 of the thm $\square$

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&\geqslant |(p_t * u_0)(x)|^2 + \mathrm{L}_\sigma^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|u_s(y)|^2\right).
\end{aligned}$$

- $\mathbf{1}_{[\alpha t,\infty)}(x) \geqslant \mathbf{1}_{[\alpha(t-s),\infty)}(y-x) \cdot \mathbf{1}_{[\alpha s,\infty)}(y)$.

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- $\mathbf{1}_{[\alpha t,\infty)}(x) \geqslant \mathbf{1}_{[\alpha(t-s),\infty)}(y-x) \cdot \mathbf{1}_{[\alpha s,\infty)}(y)$.
- $\therefore$

$$\begin{aligned}&\int_{\alpha t}^{\infty} \mathrm{d}x \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|u_s(y)|^2\right) \\ &\geqslant \int_0^t \mathrm{d}s \left(\int_{\alpha(t-s)}^{\infty} [p_{t-s}(z)]^2 \mathrm{d}z\right)\left(\int_{\alpha s}^{\infty} \mathrm{E}\left(|u_s(y)|^2\right) \mathrm{d}y\right)\end{aligned}$$

Intermittency Fronts (Lecture 8)

Sketch of proof

- $\therefore M_+(t) := \int_{\alpha t}^{\infty} \mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}x$ solves

$$M_+(t) \geqslant \int_{\alpha t}^{\infty} |(p_t * u_0)(x)|^2 \, \mathrm{d}x + \mathrm{L}_\sigma^2 (T * M_+)(t),$$

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- Similarly, $M_-(t) := \int_{-\infty}^{-\alpha t} \mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}x$ solves

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- Laplace transform: $(\mathscr{L}\phi)(\beta) := \int_0^\infty \mathrm{e}^{-\beta t} \phi(t) \, \mathrm{d}t.$

Intermittency Fronts (Lecture 8)

Sketch of proof

- $M(t) := \int_{|x|>\alpha t} \mathrm{E}(|u_t(x)|^2)|\,\mathrm{d}x$, $T(t) = \int_{\alpha t}^{\infty} [p_t(z)]^2\,\mathrm{d}z$.

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-

$$(\mathscr{L}M)(\beta) \geqslant \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{d}t \int_{|x|>\alpha t} \mathrm{d}x\ |(p_t * u_0)(x)|^2 + \mathrm{L}_\sigma^2 (\mathscr{L}T)(\beta)(\mathscr{L}M)(\beta).$$

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- Direct computation:
$$(\mathscr{L}T)(0) = \frac{1}{2\nu\pi} \int_0^\infty \frac{\mathrm{d}t}{t} \int_{\alpha t}^\infty \mathrm{d}z\ \mathrm{e}^{-z^2/(\nu t)} \to \infty \text{ as } \alpha \downarrow 0.$$

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Intermittency Fronts (Lecture 8)

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- $$(\mathscr{L}M)(\beta) \geqslant \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{d}t \int_{|x|>\alpha t} \mathrm{d}x\ |(p_t * u_0)(x)|^2 + \mathrm{L}_\sigma^2(\mathscr{L}T)(\beta)(\mathscr{L}M)(\beta).$$
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- Therefore, $\exists \alpha, \beta > 0$ such that $(\mathscr{L}M)(\beta) = \infty$.
- Argue by contradiction to see that for this choice of α, β, $\mathscr{S}(\alpha) = \limsup_{t\to\infty} t^{-1} \sup_{|x|>\alpha t} \log \mathrm{E}(|u_t(x)|^2) \geqslant \beta > 0$. □

Aside: Regularity Theory (Appendix C)

Kolmogorov's Continuity Theorem

- $\{X_t\}_{t \in T}$ a stochastic process, where $T \subset \mathbf{R}^m$ is meas. and bdd.

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Theorem (KCT)

Suppose $\exists$ finite $C > 0$ and $k > H := \sum_{j=1}^m \alpha_j^{-1}$ so that

$$\|X_t - X_s\|_k \leqslant C\,\varrho(t-s) \qquad \forall s,t \in T.$$

Then X has a continuous modification $\bar{X}$ that is Hölder continuous. In fact, $\forall q \in (0\,,1-(H/k))$,

$$\mathrm{E}\left(\sup_{\substack{s,t\in T:\\ s\neq t}} \left|\frac{\bar{X}_t - \bar{X}_s}{[\varrho(t-s)]^q}\right|^k\right) < \infty.$$

Aside: Regularity Theory (Appendix C)

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- $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous, strictly increasing function with $\mu(0) = 0$.
- Garsia's integral[s]:

$$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \qquad \forall k \geqslant 1.$$

Aside: Regularity Theory (Appendix C)

$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

Theorem (Garsia's theorem)

If $\mathcal{I}_k < \infty$ for some $k \in [1\,, \infty)$ and $\int_0^{r_0} |\mathrm{B}_\varrho(r)|^{-2/k}\,\mathrm{d}\mu(r) < \infty$, then $f = \bar{f}$ a.e., where $\bar{f} : \mathbf{R}^m \to \mathbf{R}$ satisfies

$$\left|\bar{f}(s) - \bar{f}(t)\right| \leqslant 12\mathcal{I}_k^{1/k} \cdot \int_0^{\varrho(s-t)} |\mathrm{B}_\varrho(r)|^{-2/k}\,\mathrm{d}\mu(r),$$

for all $s, t \in \mathbf{R}^m$ that satisfy $\varrho(s-t) \leqslant r_0$.

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

- $\forall$ meas. $Q \subset \mathbf{R}^m$ with $|Q| > 0$, define

$$\bar{f}_Q(x) := \frac{1}{|Q|} \int_Q f(x+z)\,\mathrm{d}z \qquad (x \in \mathbf{R}^m).$$

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$$\bar{f}_Q(x) := \frac{1}{|Q|} \int_Q f(x+z)\,\mathrm{d}z \qquad (x \in \mathbf{R}^m).$$

- **Lemma (Garsia's lemma)**
  $\forall$ $k \geqslant 1$ and bounded and meas. $Q \subset Q' \subset \mathbf{R}^m$ with $|Q| > 0$,

$$\sup_{x \in \mathbf{R}^m} \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right| \leqslant \sup_{a \in Q, b \in Q'} \mu(\varrho(a-b)) \cdot \left(\frac{\mathcal{I}_k}{|Q|^2} \right)^{1/k}.$$

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

- Jensen's inequality:

$$\left|\bar{f}_Q(z) - \bar{f}_{Q'}(z)\right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_Q \mathrm{d}x \int_{Q'} \mathrm{d}y \ \ (f(x+z) - f(y+z)) \right|^k$$

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

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$$\leqslant \frac{1}{|Q|^2} \int_Q \mathrm{d}x \int_{Q'} \mathrm{d}y \, |f(x+z) - f(y+z)|^k \, .$$

- Choose and fix $\alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a-b)) \Rightarrow$

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

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- Choose and fix $\alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a-b)) \Rightarrow$

$$\left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right| \leqslant \alpha \left(\frac{\mathcal{I}_k}{|Q|^2} \right)^{1/k} .$$

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- Choose and fix $\alpha > \sup_{a\in Q} \sup_{b \in Q'} \mu(\varrho(a-b)) \Rightarrow$

$$\left|\bar f_Q(z) - \bar f_{Q'}(z)\right| \leqslant \alpha \left(\frac{\mathcal{I}_k}{|Q|^2}\right)^{1/k} .$$

- Let $\alpha \downarrow \sup_{a\in Q} \sup_{b \in Q'} \mu(\varrho(a-b))$ to finish. $\square$

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

- Define $r_n \downarrow 0$ via: $r_0 > 0$ fixed; $\mu(2r_n) = 2^{-n}\mu(2r_0)$; equivalently,

$$r_{n+1} := \sup\left\{ r > 0 : \ \mu(2r) = \frac{1}{2}\mu(2r_n) \right\}.$$

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- Define

$$\bar{f}_n(z) := \bar{f}_{\mathrm{B}_\varrho(r_n)}(z)$$

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- Define

$$\bar{f}_n(z) := \bar{f}_{\mathrm{B}_\varrho(r_n)}(z) = \frac{1}{|\mathrm{B}_\varrho(r_n)|} \int_{\mathrm{B}_\varrho(r_n)} f(x+z)\,\mathrm{d}x.$$

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

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- **Lemma**
  Suppose $\exists k \in [1\,,\infty)$ *so that:*

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- **Lemma**
  Suppose $\exists k \in [1\,,\infty)$ *so that:*
    - $\mathcal{I}_k < \infty$*: and*

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  Suppose $\exists k \in [1\,,\infty)$ *so that:*
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  Then, $\bar{f} := \lim_{n\to\infty} \bar{f}_n$ *exists, and*

$$\sup_{z\in\mathbf{R}^m} \left|\bar{f}(z) - \bar{f}_\ell(z)\right| \leqslant 4\mathcal{I}_k^{1/k} \cdot \int_0^{r_{\ell+1}} |\mathrm{B}_\varrho(r)|^{-2/k}\,\mathrm{d}\mu(r) \quad \forall \ell \in \{1\,,2\,,\ldots\}.$$

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  Consequently, $f = \bar{f}$ *a.e.*

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- Proof. If $a \in \mathrm{B}_\varrho(r_n)$ and $b \in \mathrm{B}_\varrho(r_{n+1})$, then

$$\varrho(a-b) \leqslant \varrho(a) + \varrho(b)$$

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$$\varrho(a-b) \leqslant \varrho(a) + \varrho(b) \leqslant r_n + r_{n+1} \leqslant 2r_n.$$

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- Garsia's lemma $\Rightarrow$

$$\left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| \leqslant \sum_{n=\ell}^{\ell+L-1} \left|\bar{f}_{n+1}(z) - \bar{f}_n(z)\right|$$

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$$\begin{aligned}\left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| &\leqslant \sum_{n=\ell}^{\ell+L-1} \left|\bar{f}_{n+1}(z) - \bar{f}_n(z)\right| \\ &\leqslant \mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_n)}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \quad \forall \ell, L \geqslant 0.\end{aligned}$$

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- Since $\mu(2r_n) = 2\mu(2r_{n+1})$, we can write $\mu(2r_n) = 4\{\mu(2r_{n+1}) - \mu(2r_{n+2})\}$.

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$\mathcal{I}_k := \int_{\mathbf{R}^m} \mathrm{d}x \int_{\mathbf{R}^m} \mathrm{d}y \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k$

$$\therefore \; \left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| \leqslant 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}}$$

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$$= 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{\mathrm{d}\mu(r)}{\left|\mathrm{B}_\varrho(r_{n+1})\right|^{2/k}}$$

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$$\begin{aligned}
\therefore\ \left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| &\leqslant 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\
&= 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{\mathrm{d}\mu(r)}{\left|\mathrm{B}_\varrho(r_{n+1})\right|^{2/k}} \\
&\leqslant 4\mathcal{I}_k^{1/k} \cdot \int_0^{r_{\ell+1}} \frac{\mathrm{d}\mu(r)}{\left|\mathrm{B}_\varrho(r)\right|^{2/k}}.
\end{aligned}$$

Aside: Regularity Theory (Appendix C)

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$$\begin{aligned}
\therefore \left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| &\leqslant 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\
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\end{aligned}$$

Therefore, $\ell \mapsto \bar{f}_\ell$ is uniformly Cauchy and $\bar{f} := \lim_{\ell\to\infty} \bar{f}_\ell \exists$

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\therefore\ \left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| &\leqslant 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\
&= 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{\mathrm{d}\mu(r)}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\
&\leqslant 4\mathcal{I}_k^{1/k} \cdot \int_0^{r_{\ell+1}} \frac{\mathrm{d}\mu(r)}{|\mathrm{B}_\varrho(r)|^{2/k}}.
\end{aligned}$$

Therefore, $\ell \mapsto \bar{f}_\ell$ is uniformly Cauchy and $\bar{f} := \lim_{\ell\to\infty} \bar{f}_\ell \exists$

- Remains: $f = \bar{f}$ a.e.: $\forall$ nice ϕ,

$$\int \phi(x)\bar{f}(x)\,\mathrm{d}x = \lim_{\ell\to\infty} \int \phi(x)\bar{f}_\ell(x)\,\mathrm{d}x$$

Aside: Regularity Theory (Appendix C)

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$$\begin{aligned} \therefore\ \left|\bar{f}_{\ell+L}(z) - \bar{f}_\ell(z)\right| &\leqslant 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\ &= 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{\mathrm{d}\mu(r)}{|\mathrm{B}_\varrho(r_{n+1})|^{2/k}} \\ &\leqslant 4\mathcal{I}_k^{1/k} \cdot \int_0^{r_{\ell+1}} \frac{\mathrm{d}\mu(r)}{|\mathrm{B}_\varrho(r)|^{2/k}}. \end{aligned}$$

Therefore, $\ell \mapsto \bar{f}_\ell$ is uniformly Cauchy and $\bar{f} := \lim_{\ell\to\infty} \bar{f}_\ell \exists$

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Under the preceding integrability conditions, $\bar{f}$ is continuous. In fact, if $s, t \in \mathbf{R}^m$ satisfy $\varrho(s-t) \leqslant r_0$, then

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- This concludes the proof of Garsia's theorem. □

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Intermittency Islands (Lecture 9)

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- $\mathrm{E}u_t(x) = 1 = \|u_t(x)\|_1$. Therefore, by Fatou's lemma, $\liminf_{|x|\to\infty} u_t(x) < \infty$. So the lim sup is not a lim.

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- So far (Conus–Joseph–K, 2012; Mueller–Nualart, 2008):

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  - The probability bound follows from the expectation bound and the Chebyshev inequality.

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Proof of the expectation bound

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- Choose α small so that $c > 1$. $\square$

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\mathrm{P}\left\{X \geqslant \tfrac{1}{2}\|X\|_m\right\} &\geqslant \left(1 - 2^{-m}\right)^2 \frac{[\mathrm{E}(X^m)]^2}{\mathrm{E}(X^{2m})} \\
&\geqslant (1 - 2^{-2})^2 \frac{L^{2m}\exp(2Lm^3 t)}{\exp(8\ell m^3 t)} \\
&\geqslant \frac{9}{16} L^{2m} \exp\left(-Qm^3 t\right) \qquad Q := 8\ell - 2L.
\end{aligned}$$

- $\Rightarrow Q > 0 \Rightarrow$ solve.

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  - If $u_t(x) \gg 1$ then $|x| \gg 1$; therefore, the collection of y "near" x with $u_t(y) \gg 1$ should be "relatively small." Anderson localization

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  Fix $t > 0$ and $N \in \{0\,,1\,,\cdots\}$. Then $\exists\ C = C(t)$ and $\{Y(x)\}_{x\in\mathbf{R}}$ such that:
  - $\mathrm{E}\left(|u_t(x) - Y(x)|^N\right) \leqslant C^N \mathrm{e}^{-CN^3}$; *and*
  - *If $x_1, x_2, \ldots \in \mathbf{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.*

- Step 1. Solve for all $\beta > 0$,

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t)\times[x\pm\sqrt{\beta t}]} p_{t-s}(y-x) U_s^{(\beta)}(y)\, \xi(\mathrm{d}s\,\mathrm{d}y).$$

  - Existence, uniqueness, …, all ok.
  - $U_t^{(\beta,n+1)}(x) = 1 + \int_{(0,t)\times[x\pm\sqrt{\beta t}]} p_{t-s}(y-x) U_s^{(\beta,n)}(y)\, \xi(\mathrm{d}s\,\mathrm{d}y).$
  - If $\beta \gg 1$ then $U^{(\beta)} \approx u$. $\beta \propto k^2$

Correlation length (Lecture 10)

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