Solving Stochastic Partial Differential Equations as Stochastic Differential Equations in Infinite Dimensions - a Review

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Based on joint work with V. Mandrekar, B. Rajeev, P. Richard

Outline

Infinite Dimensional DE's

Two fundamental problems:

- Peano theorem is invalid in infinite dimansional Banach spaces

Theorem (Peano)

For each continuous function $f : \mathbb{R} \times B \to B$ *defined on some open set* $V \subset \mathbb{R} \times B$ *and for each point* $(t_0, x_0) \in V$ *the Cauchy problem*

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution which is defined on some neighborhood of t_0.

Theorem (Godunov, 1973)

Each Banach space in which Peano's theorem is true is finite dimensional.

- Appearence of unbounded operators in the equation $\Delta : W^{1,2} \to W^{1,2}$ (Sobolev space) is unbounded.

SPDE's to Infinite Dimensional SDE's

Most popular approaches:

- Semigroup solution, mild solution to a semiliner DE (Hille, DaPrato, Zabczyk).
- Solution in a multi-Hilbertian space, e.g. in a dual to a nuclear space (Itô, Kallianpur).
- Variational solution in Gelfand triplet (Agmon, Lions, Röckner).
- Solutions via Dirichlet forms (Albeverio, Osada (Itô Prize 2013))
- White noise apporach (Hida)
- Brownian sheet formulation (Walsh)
- Solutions in R^{∞} (Leha, Ritter)

(S)PDE's to Infinite Dimensional (S)DE's

Example (Abstract Cauchy Problem - Semilinear SDE)

- One–dimensional Heat Equation,

$$\begin{cases} u_t(t,x) &= u_{xx}(t,x), \ t>0 \\ u(0,x) &= \varphi(x) \end{cases} \Rightarrow \begin{cases} \frac{du(t)}{dt} &= \Delta u(t), \ t>0 \\ u(0) &= \varphi \in X \end{cases}$$

The Cauchy problem is equation is transformed to an abstract Cauchy problem in the Banach space X of bounded uniformly continuous functions. The differentiation is in the sense of the Banach space.
Solution:

$$u(t,x) = \left(G(t)\varphi\right)(x).$$

where $G(t)$ is the Gaussian semigroup on the Banach space X

$$\left(G(t)\varphi\right)(x) = \begin{cases} \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \exp\left\{-|x-y|^2/4t\right\} \varphi(y)\, dy, & t>0 \\ \varphi(x), & t=0. \end{cases}$$

SPDE's to Infinite Dimensional SDE's

Example (Fréchet nuclear space)

- Neuronal Models

$$dX_t = -A^* X_t + B(X_t) dW_t$$

If $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and A is an operator with a discrete spectrum, with eigenvalues and eigenvectors $\lambda_j > 0$ and $h_j \in H$, such that $\sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty$ for some $r_1 > 0$, then

$$\Phi = \left\{ \phi \in H : \quad \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, h_j \rangle_H^2 < \infty, \quad \forall r \geq 0 \right\}$$

is a Fréchet nuclear space, with $\{h_j\} \subset \Phi$ being a common orthogonal system.

SPDE's to Infinite Dimensional SDE's

- Let H_r, be the completion of Φ with respect to the Hilbertian norms $\|\cdot\|_r$ defined by the inner product

$$\langle f, g\rangle_r = \sum_{j=0}^{\infty} (1+\lambda_j)^{2r} \langle f, h_j\rangle_H \langle g, h_j\rangle_H, \quad f, g \in H.$$

$$\Phi' = \bigcup_{r>0} H_{-r}, \quad \text{so that} \quad \Phi \subset \ldots \subset H_r \subset \ldots \subset H \subset H_{-r} \subset \ldots \subset \Phi'$$

- For $\mathcal{S}'(\mathbb{R})$, take $A = t^2 - \frac{d^2}{dt^2} - I$, $H = L_2(\mathbb{R})$ and h_j, Hermite functions.
- For $p \in \mathbb{R}$ let S_p, be the completion of $\mathcal{S}$ with respect to the Hilbertian norms $\|\cdot\|_p$ defined by the inner product

$$\langle f, g\rangle_p = \sum_{k=0}^{\infty} (2k+1)^{2p} \langle f, h_k\rangle_{L_2} \langle g, h_k\rangle_{L_2}, \quad f, g \in \mathcal{S}.$$

Then $\mathcal{S} = \bigcap_{p>0} S_p$ and $\mathcal{S}' = \bigcup_{p>0} S_{-p}$. $A(\mathcal{S}) \subset \mathcal{S}$, $A^*(\mathcal{S}') \subset \mathcal{S}'$.

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Example (Gelfand Triplet - Variational Solutions)

- Diffusion Models

$$dX_t = AX(t) + B(X_t)dW_t$$

with $A : V = W_0^{1,2} \to V^* = W_0^{-1,2}$ and $B : V \to \mathcal{L}(\mathbb{R}, H) = H = L^2$ by

$$\begin{aligned} Av &= \alpha^2 \frac{d^2 v}{dx^2} + \beta \frac{dv}{dx} + \gamma v + g, \quad v \in V, \\ Bv &= \sigma_1 \frac{dv}{dx} + \sigma_2 v, \quad v \in V. \end{aligned}$$

where $H = L^2((-\infty, \infty))$, $V = W_0^{1,2}((-\infty, \infty))$, with the usual norms

$$\begin{aligned} \|v\|_H &= \left(\int_{-\infty}^{+\infty} v^2\, dx\right)^{1/2}, \quad v \in H, \\ \|v\|_V &= \left(\int_{-\infty}^{+\infty} \left(v^2 + \left(\frac{dv}{dx}\right)^2\right) dx\right)^{1/2}, \quad v \in V. \end{aligned}$$

SPDE's to Infinite Dimensional SDE's

- Variational Method - Gelfand triplet,

$$V \hookrightarrow H \hookrightarrow V^*,$$

V, H, V^* are real separable Hilbert spaces, H is identified with its dual H^*. Embeddings are continuous, dense and compact (or Hilbert-Schmidt)

Semilinear SDE/SDE if $A = 0$

$$
\begin{aligned}
dX(t) &= (A(X(t)) + F(t, X))\,dt + B(t, X)\,dW_t \quad \text{Wiener} \\
dX(t) &= (A(X(s)) + F(s, X))\,ds + \int_U B(s, X, u)\,q(ds, du) \text{Poisson} \\
X(0) &= \xi_0 - \mathcal{F}_0 \text{ meas.}
\end{aligned}
$$

$$
\begin{aligned}
A : \mathcal{D}(A) \subset H &\to H \quad \text{generator of a } C_0\text{–semigroup} \\
F : [0, T] \times C([0, T], H) &\to H \\
B : [0, T] \times C([0, T], H) &\to \mathcal{L}_2(K_Q, H)(\text{ Wiener}) \\
B : [0, T] \times \mathcal{D}([0, T], H) &\to L_2(U, H)(\text{ Poisson})
\end{aligned}
$$

W_t is a K–valued Q–Wiener process, $q(ds\,du) = N(ds\,du) - ds\,\mu(du)$ is compensated Poisson random measure (cPrm).

Semilinear SDE

Mild solution (if $A = 0$, strong or weak solution) in $C([0,T],H)$ or $\mathcal{D}([0,T],H)$

$$
\begin{aligned}
X(t) &= S(t)\xi_0 + \int_0^t S(t-s)F(s,X)\,dt + \int_0^t S(t-s)B(s,X)\,dW_s \\
X(t) &= S(t)\xi_0 + \int_0^t S(t-s)F(s,X)\,dt \\
&\quad + \int_0^t \int_U S(t-s)B(s,X,u)\,q(ds\,du)
\end{aligned}
$$

Motivation for the Mild Solution

- Inhomogeneous Initial Value Problem

$$\frac{du(t)}{dt} = Au(t) + f(t),\ t > 0\ u(0) = x \in \mathcal{D}(A)$$

If u is a solution, then

$$\frac{dT(t-s)u(s)}{ds} = T(t-s)f(s)$$

and by integrating

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds$$

Motivation for the Mild Solution

- Stochastic Convolution

$$\int_0^t S(t-s)f(s)\,ds \text{ is replaced by } S\star\Phi(t)=\int_0^t S(t-s)\Phi(s)\,dW_s.$$

(see §5.3 D. Khoshnevisan Course notes and Theorem 3.1 in

Theorem

Assume that A is an infinitesimal generator of a C_0–semigroup of operators $S(t)$ on H, and W_t is a K–valued Q–Wiener process. (a) For $h\in\mathcal{D}(A^*)$, then

$$\langle X(t),h\rangle_H=\int_0^t\langle X(s),A^*h\rangle_H\,ds+\left\langle\int_0^t\Phi(s)\,dW_s,h\right\rangle_H,\quad P\text{–a.s.},\qquad(2.1)$$

iff $X(t)=S\star\Phi(t)$. (b) If $\Phi\in\Lambda_2(K_Q,H)$, $\Phi(K_Q)\subset\mathcal{D}(A)$, and $A\Phi\in\Lambda_2(K_Q,H)$, then $S\star\Phi(t)$ is a strong solution.

Multi–Hilbertian (Fréchet nuclear) space SDE

$$
\begin{aligned}
dX(t) &= F(t, X(t)))\, dt + B(t, X(t))\, dW_t \quad \text{Wiener} \\
X(t) &= \int_0^t F(s, X(s)))\, ds + \int_0^t \int_U B(s, X(s-), u)\, q(ds, du) \text{Poisson} \\
X(0) &= \xi_0 - -\mathcal{F}_0 \text{ meas.}
\end{aligned}
$$

$$
\begin{aligned}
F : [0, T] \times \Phi' &\to \Phi' \\
B : [0, T] \times \Phi' &\to L(\Phi', \Phi')(\text{ Wiener}) \\
B : [0, T] \times \Phi' \times U &\to \Phi'(\text{ Poisson})
\end{aligned}
$$

Solution, continuous or cadlag, is Φ'–valued, but found in H_{-p}, some $p > 0$.

$$
\begin{aligned}
\langle \phi, X(t) \rangle &= \langle \phi, \xi_0 \rangle + \int_0^t \langle \phi, F(s, X(s))) \rangle\, dt + \langle \phi, \int_0^t B(s, X(s))\, dW_s \rangle \\
\langle \phi, X(t) \rangle &= \langle \phi, \xi_0 \rangle + \int_0^t \langle \phi, F(s, X) \rangle\, dt + \int_0^t B(s, X(s-), u)[\phi]\, q(du\, ds)
\end{aligned}
$$

Variational Method

Variational SDE

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

with the coefficients

$$A : [0, T] \times V \to V^* \qquad \text{and} \qquad B : [0, T] \times V \to \mathcal{L}_2(K_Q, H)$$

and an H–valued $\mathcal{F}_0$–measurable initial condition $\xi_0 \in L^2(\Omega, H)$.

$$\text{(W)} \quad X(t) = \xi_0 + \int_0^t A(s, X(s))\, ds + \int_0^t B(s, X(s))\, dW_s, \; P\text{–a.s.}$$

$$\text{(P)} \quad X(t) = \xi_0 + \int_0^t A(s, X(s))\, ds + \int_0^t \int_U B(s, X(s-), u)\, q(ds\, du),$$

P–a.s

The integrants A and B are evaluated at a V–valued $\mathcal{F}_t$–measurable version of $X(t)$ in $L^2([0, T] \times \Omega, V)$.

Solving the Equation

A, B	Martingale Rep. Thm. Characterize the Lim. $V \hookrightarrow H \hookrightarrow V^*$	X weak soln. in C or $\mathcal{D}([0,T],V^*)$
$\uparrow$		$\Uparrow$
a^n, b^n $x \in \mathbb{R}^n$, $a^n(t,x)_j = \langle \phi_i, A(t, \sum_{k=1}^{n} x_k \phi_k \rangle_j$		X^n weak soln. in C or $\mathcal{D}([0,T],\mathbb{R}^n)$
$\uparrow$	$\mathbb{R}^n \hookrightarrow \mathbb{R}^n \hookrightarrow \mathbb{R}^n$	$\Uparrow$
$a^{n,l}, b^{n,l},\ l \to \infty$ Lip. Approx.		$X^{n,l}$ strong soln.
		$\uparrow$ in L^2
$E \sup_t \left\lVert X^{n,l,k}(t) - X^{n,l}(t) \right\rVert^2_{\mathbb{R}^n}$	Pickard	$X^{n,l,k}$

Semilinear SDE/SDE when $A = 0$ - Coefficients

(M) F and B are jointly measurable, and for every $0 \leq t \leq T$, they are measurable with respect to the product σ–field $\mathcal{F}_t \otimes \mathcal{C}_t$ on $\Omega \times C([0, T], H)$, where $\mathcal{C}_t$ is a σ–field generated by cylinders with bases over $[0, t]$.

(JC) F and B are jointly continuous.

(G-F-B) There exists a constant ℓ, such that $\forall x \in C([0, T], H)$

$$\|F(\omega, t, x)\|_H + \|B(\omega, t, x)\|_{\mathcal{L}_2(K_Q, H)} \leq \ell \left(1 + \sup_{0 \leq s \leq T} \|x(s)\|_H\right),$$

for $\omega \in \Omega$, $0 \leq t \leq T$.

(A4) For all $x, y \in C([0, T], H)$, $\omega \in \Omega$, $0 \leq t \leq T$, there exists $\mathcal{K} > 0$, such that

$$\|F(\omega, t, x) - F(\omega, t, y)\|_H + \|B(\omega, t, x) - B(\omega, t, y)\|_{\mathcal{L}_2(K_Q, H)} \leq \mathcal{K} \sup_{0 \leq s \leq T} \|x(s) - y(s)\|_H.$$

Infinite Dimensional SDE ($A = 0$) - Solutions

- Lipschitz case - strong solutions exist and are unique (Pickard)
- Continuous case - Lipschitz Approximation

$$F_n(t,x) = \int \cdots \int F(t, (\gamma_n(\cdot, x_0, ..., x_n), \underline{e})) \times \text{smoothing kernel}$$

Existence result for $dX(t) = F(t, X))\, dt + B(t, X)\, dW_t$

Theorem

Let H_{-1} be a real separable Hilbert space. Let the coefficients F, B of the SDE satisfy conditions (M), (JC), (G-F-B) on H_{-1}. Assume that there exists a Hilbert space H such that the embedding $J : H \hookrightarrow H_{-1}$ is a compact operator (failure of the Peano theorem) and that F, B restricted to H satisfy

$$\begin{aligned} F &: [0,T] \times C([0,T], H) \to H, \\ B &: [0,T] \times C([0,T], H) \to \mathcal{L}(K, H), \end{aligned}$$

and the linear growth condition (G-F-B). Then the SDE has a weak solution $X(\cdot) \in C([0,T], H_{-1})$.

Smoothing Kernel

Let $\{e_n\}_{n=1}^{\infty}$ be an ONB in H. Denote

$$
\begin{aligned}
f_n(t) &= (\langle x(t), e_1\rangle_H, \langle x(t), e_2\rangle_H, \dots, \langle x(t), e_n\rangle_H) \in R^n, \\
\Gamma_n(t) &= f_n(kT/n) \quad \text{at } t = kT/n \text{ and linear otherwise,} \\
\gamma_n(t, x_0, \dots, x_n) &= x_k \quad \text{at } t = \frac{kT}{n} \text{ and linear otherwise, with } x_k \in R^n, \\
& \qquad k = 0, 1, ..., n.
\end{aligned}
$$

Let $g : R^n \to R$ be non-negative, vanishing for $|x| > 1$, possessing bounded derivative, and such that $\int_{R^n} g(x)dx = 1$. Let $\varepsilon_n \to 0$. We define

$$
\begin{aligned}
F_n(t, x) &= \int \cdots \int F(t, (\gamma_n(\cdot, x_0, ..., x_n), \underline{e})) \\
&\quad \times \exp\left\{-\frac{\varepsilon_n}{n}\sum_{k=0}^{n} x_k^2\right\} \prod_{k=0}^{n}\left(g\left(\frac{f_n(\frac{kT}{n} \wedge t) - x_k}{\varepsilon_n}\right)\frac{dx_k}{\varepsilon_n}\right)
\end{aligned}
\tag{3.1}
$$

Above, $(\gamma_n(\cdot, x_0, \dots, x_n), \underline{e}) = \gamma_n^1 e_1 + \dots + \gamma_n^n e_n$, where $\gamma_n^1, \dots, \gamma_n^n$ are the coordinates of the vector γ_n in R^n, and $x_k^2 = \sum_{i=1}^{n}(x_k^i)^2$, $dx_k = dx_k^1 \dots dx_k^n$.

Semilinear SDE ($A \neq 0$) - Solutions

- Lipschitz case - strong solutions exist and are unique
- Continuous case - Lipschitz Approximation

Theorem

Assume that A is an infinitesimal generator of a compact C_0–semigroup $S(t)$ (Peano) on a real separable Hilbert space H. Let the coefficients of the Semilinear SDE satisfy conditions (M), (JC), (G-F-B). Then the Semilinear SDE

$$dX(t) = (AX(t) + F(t, X))\, dt + B(t, X)\, dW_t$$

has a martingale solution (i.e. weak mild solution).

Multi-Hilbertian Space; Coefficients are Differential Operators.

- Consider an SDE

$$dX_t = L(X_t)dt + A(X_t)dB_t \tag{3.2}$$

- For $1 \le i \le d$, let $\partial_i : \mathcal{S} \to \mathcal{S}$ be the differentiation operators. Then ∂_i extends in the usual manner as an operator $\partial_i : \mathcal{S}' \to \mathcal{S}'$. Let ∂_i^* denote the transpose of ∂_i. Then $\partial_i^* : \mathcal{S}' \to \mathcal{S}'$ is given by $\partial_i^* u = -\partial_i u$, $u \in \mathcal{S}'$.
- Define $A : \mathcal{S}' \to L\left(\mathbf{R}^d, \mathcal{S}'\right)$ and $L : \mathcal{S}' \to \mathcal{S}'$ by

$$\begin{aligned} Au(x) &= -\sum_{i=1}^{d} (\partial_i u) x_i \\ Lu &= \frac{1}{2} \sum_{i=1}^{d} \partial_i^2 u, \end{aligned}$$

with $u \in \mathcal{S}'$, $x = (x_1 \cdots x_d) \in \mathbf{R}^d$.

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Growth Properties of the Coefficients

- For a bounded linear operator $T \in L\left(\mathbb{R}^d, H_p\right)$, its Hilbert–Schmidt norm is calculated as $\|T\|_{HS(p)} = \left(\sum_{i=1}^d \|Te_i\|_p^2\right)^{1/2}$, where $\{e_i\}_{i=1}^d$ is the canonical basis in $\mathbb{R}^d$.

Proposition

For the differential operators ∂_i, A, and L defined above the following properties hold true:

(a) *For any $p \geq q + 1/2$, and $1 \leq i \leq d$, $\partial_i : S_p \to S_q$ is continuous, and for $u \in S_p$,*

$$\|\partial_i u\|_q \leq C_q \|u\|_p,$$

where the constant C_q depends (only) on q.

(b) *For any $p \geq q + 1$, and $u \in S_p$,*

$$\|Lu\|_q \leq D_q \|u\|_p$$
$$\|Au\|_{HS(q)} \leq D_q \|u\|_p,$$

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Source of the Main Technical Problem

Why approximate solutions travel from space to space

Proof.

Part (b) follows from (a). Since $\partial_i^* = -\partial_i$, for $u \in S_p$, we have

$$\begin{aligned} \|\partial_i u\|_q^2 &= \sum_{|k|=0}^{\infty} (2|k| + d)^{2q} \langle \partial_i u, h_k \rangle^2 \\ &= \sum_{|k|=0}^{\infty} (2|k| + d)^{2q} \langle u, \partial_i h_k \rangle^2 \\ &\leq 2^{2q} \sum_{|k|=0}^{\infty} (2|k| + d)^{2\left(q+\frac{1}{2}\right)} \langle u, h_k \rangle^2 \leq C_q \|u\|_p^2. \end{aligned}$$

using the recurrence relation

$$h_l'(x) = \sqrt{\frac{l}{2}} h_{l-1}(x) - \sqrt{\frac{l+1}{2}} h_{l+1}(x).$$

□

The Monotonicity Condition

What keeps things within one space

Theorem

$$2\left\langle L(u-v),u-v\right\rangle_p + \|Au-Av\|^2_{HS(p)} \le \theta\|u-v\|^2_p,$$

holds true for $r \ge p+1$, $u, v \in S_r$.

Equation Coefficients are Differential Operators

- Operators

$$A(u)(h) = -\sum_{i=1}^{d} (\partial_i u) h_i \quad (A = -\nabla)$$

$$L(u) = \frac{1}{2} \sum_{i=1}^{d} \partial_i^2 u \quad (L = \frac{1}{2} \triangle)$$

$u \in \mathcal{S}'$, $h \in \mathbb{R}^d$, satisfy our conditions with $q \geq p + 1$.

- The unique solution of

$$\begin{cases} dX_t = \frac{1}{2} \triangle X_t dt - \nabla X_t \, dB_t \\ X_0 = \phi \in \mathcal{S}_{-p} \end{cases}$$

is $\phi(\cdot + B_t)$. If $\phi = \delta_0$, then $X_t = \delta_{B_t}$.

- Monotonicity holds true for more general differential operators

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Variational Method

On a tripple $V \hookrightarrow H \hookrightarrow V'$

This set–up arises in the study of SPDE's. Typical example is $H = L^2(\mathcal{O})$, $V = W^{1,2}$–Sobolev space

- Together with other regularity assumptions, the following coercivity condition is imposed

$$2\langle Lu, u\rangle + \|Au\|^2_{HS(H)} \leq -\delta\|u\|^2_V + \eta\|u\|^2_H$$

- This condition is violated in our case of differential operators! Let $X = S_{\frac{1}{2}}$, $X' = S_{-\frac{1}{2}}$, $H = L^2$. Then (X, H, X') is a normal triple with canonical bilinear form given by the L^2 inner product. Then for $\xi \in \mathcal{S} \subset X$,

$$2\langle \xi, L\xi\rangle_0 + |A\xi|^2_{HS(0)} + \delta\|\xi\|^2_{\frac{1}{2}} = \delta\|\xi\|^2_{\frac{1}{2}}$$

which cannot be dominated by using the L^2 norm. Note that the equality $2\langle \xi, L\xi\rangle_0 = -|A\xi|^2_{HS(0)}$ follows from integration by parts.

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Existence of Weak Variational Solutions

Theorem

Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triplet (Unbounded Operator)with compact inclusions. Let the coefficients A, B of Equation

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

satisfy conditions [JC], [G-A], [G-B], and [C]. Let the initial condition ξ_0 be an H–valued random variable satisfying [IC]. Then there exists a weak solution $X(t)$ in $C([0,T],H)$, such that

$$E\left(\sup_{0\le t\le T}\|X(t)\|_H^2\right) < \infty, \quad \text{and} \quad E\int_0^T \|X(t)\|_V^2\, dt < \infty.$$

Conditions

(JC) (Joint Continuity) The mappings are continuous

$$\begin{aligned}(t, v) &\to A(t, v) \in V^* \\ (t, v) &\to B(t, v) Q B^*(t, v) \in \mathcal{L}_1(H)\end{aligned}$$

For some constant $\theta \geq 0$,

(G-A) (Growth on A - (Unbounded Operator))

$$\|A(t, v)\|_{V^*}^2 \leq \theta \left(1 + \|v\|_H^2\right), \; v \in V.$$

(G-B) (Growth on B)

$$\|B(t, v)\|_{\mathcal{L}_2(K_Q, H)}^2 \leq \theta \left(1 + \|v\|_H^2\right), \; v \in V.$$

Coercivity condition on A and B

(C) There exist constants $\alpha > 0$, $\gamma, \lambda \in \mathbb{R}$ such that for $v \in V$,

$$2\langle A(t,v), v\rangle + \|B(t,v)\|^2_{\mathcal{L}_2(K_Q,H)} \le \lambda\|v\|^2_H - \alpha\|v\|^2_V + \gamma.$$

Initial condition

(IC) For some constant c_0.

$$E\left\{\|\xi_0\|^2_H \left(\ln\left(3 + \|\xi_0\|^2_H\right)\right)^2\right\} < c_0,$$

Existence for Wiener noise (cPrm - similar)

Result on the existence of a weak solution.

Theorem

Let B_t^n be a standard Brownian motion in $\mathbb{R}^n$. There exists a weak solution to the following finite dimensional SDE,

$$dX(t) = a(t, X(t))dt + b(t, X(t))\, dB_t^n,$$

with an $\mathbb{R}^n$–valued $\mathcal{F}_0$–measurable initial condition ξ_0^n, if $a : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^n$, $b : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are continuous and satisfy the following growth condition

$$\begin{aligned} \|b(t,x)\|^2_{\mathcal{L}(\mathbb{R}^n)} &\leq K\left(1 + \|x\|^2_{\mathbb{R}^n}\right) \\ \langle x, a(t,x)\rangle_{\mathbb{R}^n} &\leq K\left(1 + \|x\|^2_{\mathbb{R}^n}\right) \end{aligned}$$

for $t \geq 0$ and $x \in \mathbb{R}^n$ and some constant K.

Approximation Problem

Lemma

The growth conditions [G-A] and [G-B] assumed for the coefficients A and B imply the following growth conditions on a^n and b^n,

$$\|a^n(t,x)\|^2_{\mathbb{R}^n} \leq \theta_n \left(1 + \|x\|^2_{\mathbb{R}^n}\right), \ \theta_n \to \infty$$

(Unbounded Operator)

$$\text{tr } (\sigma^n(t,x)) = \text{tr}\left(b^n(t,x)\left(b^n(t,x)\right)^T\right) \leq \theta \left(1 + \|x\|^2_{\mathbb{R}^n}\right).$$

The coercivity condition [C] implies that for a large enough value of θ,

$$2\left\langle a^n(t,x), x\right\rangle_{\mathbb{R}^n} + \text{tr}\left(b^n(t,x)\left(b^n(t,x)\right)^T\right) \leq \theta \left(1 + \|x\|^2_{\mathbb{R}^n}\right).$$

The constant θ_n depends on n, but θ does not.

Tightness in $C([0,T],V^*)$ - similar in $\mathcal{D}([0,T],V^*)$.

Theorem

Let the coefficients A, B of Equation

$$dX(t) \quad = A(t,X(t))dt + B(t,X(t))\,dW_t$$

satisfy conditions [JC], [G-A], [G-B], and [C]. Consider the family of measures μ_*^n on $C([0,T],V^*)$, with support in $C([0,T],H)$, defined by

$$\mu_*^n(Y) = \mu^n \left\{ x \in C([0,T],\mathbb{R}^n) \,:\, \sum_{i=1}^{n} x_i(t)\varphi_i \in Y \right\}; \quad Y \subset C([0,T],V^*),$$

where μ^n are distributions of finite dimensional solutions, $\varphi_i, i = 1, \ldots$ is an ONB in H, consisting of elements from V.
If the embedding $H \hookrightarrow V^*$ is compact (H-S in $\mathcal{D}$). Then the family of measures $\{\mu_*^n\}_{n=1}^{\infty}$ is tight on $C([0,T],V^*)$ ($\mathcal{D}([0,T],V^*)$).

Tightness in $C([0,T],V^*)$ and $\mathcal{D}([0,T],V^*)$

Theorem

(Mitoma: $C([0,T],\mathcal{S}'), \mathcal{D}([0,T],\mathcal{S}')$). Let $V \hookrightarrow H \hookrightarrow V^*$ be Gelfand triplet with Hilbert–Schmidt embeddings. Given $\{\mu_n\}_{n=1}^\infty$ Borel probability measures on $C([0,T],V^*)$ ($\mathcal{D}([0,T],V^*)$), s.t.

1. $\{\mu_n \circ \pi_V^{-1}\}_{n=1}^\infty$ *is tight on* $C([0,T],\mathbb{R})$ ($\mathcal{D}([0,T],\mathbb{R})$)
2. $\forall \varepsilon \; \exists M \; \forall n$

$$
\begin{aligned}
\mu_n\{f \in C([0,T],V^*) \; : \; \sup_t \|f(t)\|_H > M\} \quad &< \quad \varepsilon \\
(\mu_n\{f \in \mathcal{D}([0,T],V^*) \; : \; \sup_t \|f(t)\|_H > M\} \quad &< \quad \varepsilon)
\end{aligned}
$$

then $\{\mu_n\}_{n=1}^\infty$ *is tight on* $C([0,T],V^*)$ ($\mathcal{D}([0,T],V^*)$).

Thus $\|a^n\|_H \to \infty$ is not a problem as we take one dimensional projections only, but the price is H–S embedding.
This can be improved in $C([0,T],V^*)$.

Coming back from V^* to H

By the Skorokhod theorem, $X_n \to X$ a.s. in $C([0,T], V^*)$. Consider

$$\alpha_H : V^* \to \mathbb{R}, \quad \alpha_H(u) = \sup\left\{\langle u, v\rangle,\ v \in V,\ \|v\|_H \leq 1\right\}.$$

$\alpha_H(u) = \|u\|_H$ if $u \in H$, and is lower semicontinuous as a sup of continuous functions. For $u \in V^* \setminus H$, $\alpha_H(u) = +\infty$ Thus, we can extend the norm $\|\cdot\|_H$ to a lower semicontinuous function on V^*.

By the Fatou lemma,

$$\begin{aligned}
\int_{C([0,T],V^*)} \sup_{0\leq t\leq T} \|x(t)\|_H^2 \, \mu_*(dx) &= E\left(\sup_{0\leq t\leq T} \|X(t)\|_H^2\right) \\
&\leq E \liminf_{n\to\infty} \left(\sup_{0\leq t\leq T} \|X^n(t)\|_H^2\right) \\
&\leq \liminf_{n\to\infty} E\left(\sup_{0\leq t\leq T} \|X^n(t)\|_H^2\right) \\
&= \liminf_{n\to\infty} \int_{C([0,T],V^*)} \sup_{0\leq t\leq T} \|x(t)\|_H^2 \, \mu_*^n(dx) < C.
\end{aligned}$$

Coming back from H to V

Apply the Itô formula and Coercivity

$$\begin{aligned} E\|X^n(t)\|_H^2 &= E\|\xi_0^n\|_H^2 + 2E\int_0^t \langle a^n(s,X^n(s)), X^n(s)\rangle_{\mathbb{R}^n}\, ds \\ &+E\int_0^t \operatorname{tr}\left(b^n(s,X^n(s))\,(b^n(s,X^n))^T\right)\, ds \\ &\le E\|\xi_0\|_H^2 + \lambda\int_0^t E\|X^n(s)\|_H^2\, ds - \alpha\int_0^t E\|X^n(s)\|_V^2\, ds + \gamma. \end{aligned}$$

Conclude that

$$\sup_n \int_0^T E\|X^n(t)\|_V^2\, dt < \infty.$$

Extend the norm $\|\cdot\|_V$ to a lower semicontinuous function on V^*

$$\alpha_V(u) = \sup\{\langle u, v\rangle,\ v\in V,\ \|v\|_V \le 1\},$$

since $\alpha_V(u) = \|u\|_V$ if $u \in V$, and for $u \in V^*\setminus V$, $\alpha_V(u) = +\infty$. By the Fatou lemma

$$\int_{C([0,T],V^*)}\int_0^T \|x(t)\|_V^2\, dt\, \mu_*(dx) < \infty.$$

Characterization of the limit

$$M_t(x) = x(t) - x(0) - \int_0^t A(s, x(s))\, ds,$$

is, in either case (Wiener, cPrm), a martingale. Three steps:

- Proving that M_t is a martingale by evaluating

$$\int (\langle M_t(x) - M_s(x), v\rangle\, g_s(x))\ \mu_*(dx) = 0$$

  for a bounded function g_s on $C([0,T], V^*)$, which is measurable with respect to the cylindrical σ–field generated by the cylinders with bases over $[0, s]$,
- Finding its increasing process $< M >_t$
- Using Martingale Representation Theorem

First two steps use uniform integrability. Wiener - usually of $X_n^2(t)$, cPrm - more delicate.

Lions' Theorem (extension of)- H–continuous version:

Theorem

Let $X(0) \in L^2(\Omega, H)$, $Y \in L^2([0,T] \times \Omega, V^*)$, $Z \in L^2([0,T] \times \Omega, \mathcal{L}_2(K_Q, H))$ *be both progressively measurable. Define a continuous* V^**–valued process*

$$X(t) = X(0) + \int_0^t Y(s)\, ds + \int_0^t Z(s)\, dW_s, \quad t \in [0,T].$$

If for its $dt \otimes P$*–equivalence class* $\hat{X}$ *we have* $\hat{X} \in L^2([0,T] \times \Omega, V)$*, then* X *is an* H*–valued continuous* $\mathcal{F}_t$*–adapted process,*

$$E \sup_{t \in [0,T]} \|X(t)\|_H^2 < \infty$$

Pathwise Uniqueness

Definition

If for any two H–valued weak solutions (X_1, W) and (X_2, W) of Equation

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

defined on the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ and with the same Q–Wiener process W, such that $X_1(0) = X_2(0)$, P–a.s., we have that

$$P\left(X_1(t) = X_2(t),\ 0 \leq t \leq T\right) = 1,$$

then we say that this Equation has pathwise uniqueness property.

The *weak monotonicity* condition

(WM) There exists $c \in \mathbb{R}$, such that for all $u, v \in V$, $t \in [0, T]$,

$$2\langle u - v, A(t,u) - A(t,v)\rangle + \|B(t,u) - B(t,v)\|^2_{\mathcal{L}_2(K_Q,H)} \leq c\|u - v\|^2_H.$$

Weak monotonicity is crucial in proving uniqueness of weak and strong solutions. In addition, it allows to construct strong solutions in the absence of the compact embedding $V \hookrightarrow H$.

Theorem

Let the conditions [JC], [GB], [C], [IC] hold true and assume the weak monotonicity condition [WM] and

(G-A) *(Growth on A)*

$$\|A(t,v)\|^2_{V^*} \leq \theta\left(1 + \|v\|^2_V\right), \; v \in V.$$

Then the solution to the variational SDE

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

is pathwise unique.

Proof.

Let X_1, X_2 be two weak solutions, $Y(t) = X_1(t) - X_2(t)$, and denote its V–valued progressively measurable version by $\bar{Y}$. Applying the Itô formula and the monotonicity condition [WM] yields

$$\begin{aligned} e^{-\theta t}\|Y(t)\|_H^2 &= -\theta\int_0^t e^{-\theta s}\|Y(s)\|_H^2\,ds \\ &\quad + \int_0^t e^{-\theta s}\Big(2\left\langle \bar{Y}(s), A(s,X_1(s)) - A(s,X_2(s))\right\rangle \\ &\quad + \|B(s,X_1(s)) - B(s,X_2(s))\|^2_{\mathcal{L}_2(K_Q,H)}\Big)\,ds \\ &\quad +2\int_0^t e^{-\theta s}\left\langle Y_s, (B(s,X_1(s)) - B(s,X_2(s)))\,dW_s\right\rangle_H \\ &\le M_t, \end{aligned}$$

where M_t is a real–valued continuous local martingale represented by the stochastic integral above. The inequality above also shows that $M_t \geq 0$. Hence by the Doob maximal inequality, $M_t = 0$. □

As a consequence of an infinite dimensional version of the result of Yamada and Watanabe we have the following corollary.

Corollary

Under conditions of the Existence Theorem and assuming [WM] (weak monotonicity), the variational SDE

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

has unique strong solution.

Yamada-Watanabe argument does not go through in general for cPrm, but works if U is separable.

Exponential Ultimate Boundedness

Definition

We say that the variational solution of the variational SDE

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))\, dW_t$$

is exponentially ultimately bounded in the mean square sense (m.s.s.), if there exist positive constants c, β, M, such that

$$E\,\|X^x(t)\|_H^2 \le c\, e^{-\beta t}\|x\|_H^2 + M, \qquad \text{for all } x \in H.$$

Theorem

The strong solution $\{X^x(t),\ t \geq 0\}$ *of equation*

$$\begin{cases} dX(t) & = A(X(t))\ dt + B(X(t))\ dW_t \\ X(0) & = x \in H \end{cases}$$

where A and B are in general non–linear mappings, is exponentially ultimately bounded in the m.s.s. if there exists a function $\Psi : H \to \mathbb{R}$ *to which Itô's formula can be applied and, in addition, such that*

(1) $c_1\|x\|_H^2 - k_1 \leq \Psi(x) \leq c_2\|x\|_H^2 + k_2$, *for some positive constants* $c1, c_2, k_1, k_2$ *and for all* $x \in H$,

(2) $\mathcal{L}\Psi(x) \leq -c_3\Psi(x) + k_3$, *for some positive constants* c_3, k_3 *and for all* $x \in V$.

where

$$\mathcal{L}\Psi(u) = \langle \Psi'(u), A(u)\rangle + \operatorname{tr}\left(\Psi^{''}(u)B(u)QB^*(u)\right).$$

If A and B are linear and satisfy coercivity condition, the Lyapunov function can be written explicitly

$$\Psi_0(x) = \int_0^T \int_0^t E \left\| X_0^x(s) \right\|_V^2 \, ds\, dt,$$

for T large enough.

- Our example SPDE,

$$\begin{cases} d_t u(t,x) & = \left(\alpha^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} + \beta \dfrac{\partial u(t,x)}{\partial x} + \gamma u(t,x) + g(x)\right) dt \\ & \quad + \left(\sigma_1 \dfrac{\partial u(t,x)}{\partial^x} + \sigma_2 u(t,x)\right) dW_t, \\ u(0,x) & = \varphi(x) \in L^2\left((-\infty,\infty)\right) \cap L^1\left((-\infty,+\infty)\right), \end{cases}$$

- If $-2\alpha^2 + \sigma_1^2 < 0$, then the coercivity and weak monotonicity conditions hold true. The growth [G-B] holds and

$$\|A(t,v)\|_{V^*}^2 \leq \theta\left(1 + \|v\|_V^2\right),\ v \in V.$$

and that there exists a unique strong solution $u^\varphi(t)$ in $L^2\left(\Omega, C([0,T],H)\right) \cap L^2(\Omega \times [0,T], V)$. Then we can conclude that the solution is exponentially ultimately bounded in the m.s.s. by reducing the case to a liner equation (dropping g).

L. Gawarecki, V. Mandrekar.
Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations. Springer (2011).