Probability

OLDER

6. For each of the following experiments and events determine the probability of the events.

(a) (4 pts.) Experiment: Four 6-sided fair dice are thrown

Event: All four numbers appearing are different

(b) (6 pts.) Experiment: Three balls are chosen at random without replacement from an urn containing 20 red and 40 white balls.

Event A: Exactly one red ball in the sample

Event B: 3^{rd} ball is red

Are A and B independent? Support your answer with analysis.

(c) (4 pts.) Experiment: Balls are drawn at random without replacement from the urn described in (b) until a red ball is selected.

Event: A red ball is first selected on draw number 7.

(d) (6 pts.) Experiment: X and Y are independent, each uniformly distributed on the unit interval [0, 1]

Event : $[X > Y] \cap [X + Y > \pi/4]$

(e) (6 pts.) Experiment: Four observations are taken independently from the normal distribution that has mean 50 and standard deviation 10.

Event A: All four observations exceed 55.

Event B: The first observation is less than the average of the last three observations.

7. (8 pts.) Let X, Y have the following joint probability mass function:

		y = 0	y = 1	y = 2
x	0	.3	.2	0
	1	0	.1	.1
	2	.2	0	.1

Obtain the following:

(a) The marginal probability mass functions of X and of Y

(b) $P[(X > 0) \cup (Y > 0)]$

(c) $P[Y \text{ is odd} \mid X \text{ is even}]$

(d) $E(XY)$

8. Let X have the density $f(x) = 1$ for $0 < x < 1$

$= 0$ otherwise

(a) (4 pts.) Find the c.d. f. $F(y)$ for $Y = 5X - 1$

(b) (4 pts.) Find $Var(Y)$.

(c) (6 pts.) Find a function h such that $Y = h(X)$ has density $g(y) = 3y^2$ for $0 < y < 1$

$= 0$ otherwise

9. (6 pts.) Let X_n have a uniform distribution on the interval $[0, 1/n)$ for $n = 1, 2, \ldots$ Prove that $\{X_n\}$ converges in probability to 0.

10. Consider the experiment where X distributed Poisson (λ) is observed and, conditional on $X = x$, a coin with probability π of Heads is given x independent tosses. Let Y denote the number of Heads among these tosses. *In solving (a) and (b), you may use the fact that a Poisson (λ) has mean λ and variance λ and a Binomial (n, p) has mean np and variance np(1 - p). You may choose to (c) first and deduce (a) and (b) after having established (c).*

(a) (2 pts.) Find E(Y).

(b) (4 pts.) Find V(Y).

(c) (6 pts.) Show that Y is distributed Poisson ($\lambda\pi$).

Statistics

11) (6 pts) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ, variance σ^2. Let $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)$. Prove that $E(S^2) = \sigma^2$.

12) Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density

$$f(x\,;\theta) = \theta x^{\theta - 1} \quad \text{for } 0 < x < 1, \text{ where } \theta > 1.$$

a) (5 pts.) Find the method of moments estimator of θ.

b) (6 pts.) Find the maximum likelihood estimator of θ.

c) (8 pts.) Use the fact (you need not prove it) that $Y = -\log(X_1)$ has the exponential distribution with mean $1/\theta$ to find an approximate 95% confidence interval on θ for $n = 100$. Hint: The exponential distribution with mean μ has variance μ^2.

d) (8 pts.) Let H_0: $\theta = 2$ and H_a: $\theta = 3$. Give the $\alpha = 0.05$ level most powerful test of H_0 vs H_a.

e) (2 pts.) Is this test uniformly most powerful for H_0: $\theta = 2$ vs H_a: $\theta > 2$? Why?

13) X have the binomial distribution with parameters n and p, with n known, p unknown. $0 < p < 1$.

(a) (6 pts.) Let $\hat{p} = X/n$. Find an approximation for the smallest n for which $P(|\hat{p} - p| \le 0.02) \ge .90$.

b) (5 pts.) Suppose you wish to test of H_0: $p \le 0.30$ vs H_a: $p > 0.30$. Give an approximate $\alpha = 0.05$ level test of H_0 vs H_a for $n = 100$.

c) (5 pts.) Let Φ be the cdf for the standard normal distribution. Express the power function $\pi(p)$ of the test in b) in terms of Φ.

14) A statistics professor gave his class an assignment: Drop two thumb tacks on a hard surface, and count the number of the tacks for which the point is upward. Repeat this experiment 1000 times.

a) (8 pts.) The professor suspected that student XYZ, being lazy, would just make up the results, so he decided to perform an $\alpha = 0.05$ level goodness-of-fit chisquare test. XYZ's results were:

$X = 0$ occurred 80 times, $X = 1$ 440 times, $X = 2$ 480 times

State a null hypothesis and then carry out the test. Hint: Under the null hypothesis the MLE for the probability that the tack point is upward is the proportion of occurrences of "up" for all tacks thrown.

b) (4 pts.) The professor suspected that others in his class of 30 had cheated, so he performed the test for each student. He rejected the null hypothesis for two of the 30 students and therefore failed those two students. If you were the lawyer for those two students, what argument would you use to convince a jury that failing them was unfair?

15) In order to determine whether a three week course designed to improve students' scores on the verbal part of the graduate record exam (GRE) was effective 20 students were chosen randomly. These students were then given a one hour verbal test, then paired according to their scores, the two with the highest scores being in pair one, those with the next highest in pair 2, and so on. Then one member of each pair was chose randomly using a coin flip and placed in the treatment group. The others were placed in the control group. The 10 students in the treatment group were given the three week course. The others did not take a course. After the course ended all 20 students took the GRE verbal test. Their scores were 10 times the follows. Use these numbers for any analysis.

Pair	1	2	3	4	5	6	7	8	9	10
Treatment	73	67	65	60	61	54	53	45	46	43
Control	71	64	66	57	56	54	52	44	47	39

a) (10 pts.) State a model, then test H_0: no treatment effect vs H_a treatment improves scores for $\alpha = 0.05$.

b) (5 pts.) Give a 95% confidence interval on the mean treatment effect δ.

16) Suppose that (x_{1i}, x_{2i}, Y_i) are observed for $i = 1, 2, \ldots, 6$, where the x_{1i} and x_{2i} are constants and are constants and Y_i is a random variable, where $Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$. Suppose that the ε_i are independent, each with the $N(0, \sigma^2)$ distribution. The observations are:

i	1	2	3	4	5	6
x_{1i}	0	0	1	1	2	2
x_{2i}	1	-1	0	0	-1	1
Y_i	0	0	2	2	6	2

a) (8 pts.) Find the least squares estimates of β_1 and β_2.

b) (5 pts.) Find the residuals $e_i = Y_i - \hat{Y}_i$, and use these to estimate σ^2.

c) (5 pts.) Find a 95% confidence interval on β_2.

Probability

6. (5 pts) Show that if $P(E_i) = 1$ for all $i = 1, 2, \ldots$, then $P(\cap_{i=1}^{\infty} E_i) = 1$.

7. (5 pts) You roll a pair of fair dice. One of them is red and the other is green. Consider the following three events:

   A = {an even number shows on red die},
   B = {an odd number shows on green die},
   C = {the product of the two dice is even}.

   Are these three events independent? You must give the right reason.

8. (6 pts) Consider independent Bernoulli trials, each resulting in a success with probability p, are performed. What is the probability of r successes occurring before m failures?

9. (12 pts) Find the density function of X + Y where (X,Y) has the joint density

$$f(x,y) = \frac{1}{2} y e^{-xy} \qquad 0 < x < \infty,\ 0 < y < 2.$$

10. (10 pts) Let X and Y be independent normal $N(\mu, \sigma^2)$ random variables. Show that X + Y is independent of X − Y.

11. (12 pts) Let X and Y be two independent exponential random variables with parameter $\lambda > 0$. Find the density function of X + Y.

12. (5 pts) Urn A has 6 white and 4 black balls. Urn B has 5 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is randomly selected, whereas if the outcome is tails, then a ball from urn B is selected. Suppose a black ball is selected. What is the probability that the coin landed tails?

13. (10 pts) The lifetimes of interactive computer chips produced by a certain company are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma^2 = 9 \times 10^{10}$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6 hours?

Statistics

14. (15 pts) Suppose that $X \sim$ Exponential (θ) and, conditional on $X = x$, $Y \sim$ Exponential (θx), that is,

$$f_\theta(x) = \theta e^{-\theta x}, x > 0 \quad and \quad f_\theta(y \mid x) = \theta x e^{-\theta xy}, y > 0 \quad \text{where } \theta > 0.$$

Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be *iid* with the same distribution as (X, Y).

(a) Show that $T = \frac{1}{2n}\sum_1^n X_i(1+Y_i)$ is sufficient for θ.

(b) Find the mean and variance of T. Recall that the Exponential (θ) has mean $1/\theta$ and variance $1/\theta^2$.

15. (15 pts) Let X_i and Y_i, $i = 1, 2, \ldots, n$ be independent Bernoulli random variables with $X_i \sim$ B(1, π) and $Y_i \sim$ B(1, 1 - π) with $0 \le \pi \le 1$.

(a) Show that $Z = \sum_1^n X_i - \sum_1^n Y_i$ is sufficient for π.

(b) Determine the *mle* of π based on Z, and its mean and variance.

(c) Is the *mle* consistent estimator of π in this case? Explain your answer.

16. (15 pts) Suppose that X_i, $i = 1, 2, \ldots, n$ are *iid* Poisson (λ). You will have to use the Poisson tables for Part (a) the normal tables for Part (b).

(a) With $n = 5$, determine the rejection region of the level .005 UMP test of H_0: $\lambda = 1$ vs. H_1: $\lambda > 1$ and the power of the test at $\lambda = 1.4$.

(b) With $n = 25$, determine the approximate rejection region of the level .005 UMP test of H_0: $\lambda = 1$ vs. H_1: $\lambda > 1$ and the power of the test at $\lambda = 1.4$.

17. (20 pts) Consider the linear regression model $Y_i = \beta x_i + e_i$, $i = 1, 2, \ldots, n$ where the x_i are constants and the random errors e_i are independent with $E(e_i) = 0$, $V(e_i) = \sigma^2$.

(a) Derive the least squares estimate of the slope β.

(b) Derive the mean and the variance of the estimate.

(c) Determine the 95% confidence interval estimate of β under the usual normal theory assumption on the random errors and evaluate it for the $n = 6$ data: (1, 1), (1,2), (2, 3), (2, 5), (3, 6), (3, 7).

TABLE A–4 *(continued)* Percentiles of the *F* distribution

Den. df	*A*	Numerator df 1	2	3	4	5	6	7	8	9
1	.50	1.00	1.50	1.71	1.82	1.89	1.94	1.98	2.00	2.03
	.90	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9
	.95	161	200	216	225	230	234	237	239	241
	.975	648	800	864	900	922	937	948	957	963
	.99	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,981	6,022
	.995	16,211	20,000	21,615	22,500	23,056	23,437	23,715	23,925	24,091
	.999	405,280	500,000	540,380	562,500	576,400	585,940	592,870	598,140	602,280
2	.50	0.667	1.00	1.13	1.21	1.25	1.28	1.30	1.32	1.33
	.90	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	.95	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
	.975	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4
	.99	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4
	.995	199	199	199	199	199	199	199	199	199
	.999	998.5	999.0	999.2	999.2	999.3	999.3	999.4	999.4	999.4
3	.50	0.585	0.881	1.00	1.06	1.10	1.13	1.15	1.16	1.17
	.90	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	.95	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	.975	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5
	.99	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3
	.995	55.6	49.8	47.5	46.2	45.4	44.8	44.4	44.1	43.9
	.999	167.0	148.5	141.1	137.1	134.6	132.8	131.6	130.6	129.9
4	.50	0.549	0.828	0.941	1.00	1.04	1.06	1.08	1.09	1.10
	.90	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	.95	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	.975	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	.99	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
	.995	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1
	.999	74.1	61.2	56.2	53.4	51.7	50.5	49.7	49.0	48.5
5	.50	0.528	0.799	0.907	0.965	1.00	1.02	1.04	1.05	1.06
	.90	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	.95	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	.975	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	.99	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
	.995	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8
	.999	47.2	37.1	33.2	31.1	29.8	28.8	28.2	27.6	27.2
6	.50	0.515	0.780	0.886	0.942	0.977	1.00	1.02	1.03	1.04
	.90	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	.95	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	.975	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	.99	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	.995	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4
	.999	35.5	27.0	23.7	21.9	20.8	20.0	19.5	19.0	18.7
7	.50	0.506	0.767	0.871	0.926	0.960	0.983	1.00	1.01	1.02
	.90	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	.95	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	.975	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	.99	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	.995	16.2	12.4	10.9	10.1	9.52	9.16	8.89	8.68	8.51
	.999	29.2	21.7	18.8	17.2	16.2	15.5	15.0	14.6	14.3

Table of Cumulative Poisson

														mu													
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358	0.6990	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932	0.4628	0.4337	0.4060	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197	0.9004	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572	0.7306	0.7037	0.6767	0.6496	0.6227	0.5960	0.5697	0.5438	0.5184	0.4936
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068	0.8913	0.8747	0.8571	0.8386	0.8194	0.7993	0.7787	0.7576	0.7360	0.7141
4	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704	0.9636	0.9559	0.9473	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8629
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920	0.9896	0.9868	0.9834	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981	0.9974	0.9966	0.9955	0.9941	0.9925	0.9906	0.9884	0.9858	0.9828	0.9794
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992	0.9989	0.9985	0.9980	0.9974	0.9967	0.9958	0.9947	0.9934
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9998	0.9997	0.9995	0.9994	0.9991	0.9989	0.9985	0.9981
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

For example, with $\lambda = 1.7$, $P(X \leq 3) = .9068$

Table of Cumulative Poisson

	mu																										
x	2.8	2.9	3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5	5.1	5.2	5.3	5.4
0	0.0608	0.0550	0.0498	0.0450	0.0408	0.0369	0.0334	0.0302	0.0273	0.0247	0.0224	0.0202	0.0183	0.0166	0.0150	0.0136	0.0123	0.0111	0.0101	0.0091	0.0082	0.0074	0.0067	0.0061	0.0055	0.0050	0.0045
1	0.2311	0.2146	0.1991	0.1847	0.1712	0.1586	0.1468	0.1359	0.1257	0.1162	0.1074	0.0992	0.0916	0.0845	0.0780	0.0719	0.0663	0.0611	0.0563	0.0518	0.0477	0.0439	0.0404	0.0372	0.0342	0.0314	0.0289
2	0.4695	0.4460	0.4232	0.4012	0.3799	0.3594	0.3397	0.3208	0.3027	0.2854	0.2689	0.2531	0.2381	0.2238	0.2102	0.1974	0.1851	0.1736	0.1626	0.1523	0.1425	0.1333	0.1247	0.1165	0.1088	0.1016	0.0948
3	0.6919	0.6696	0.6472	0.6248	0.6025	0.5803	0.5584	0.5366	0.5152	0.4942	0.4735	0.4532	0.4335	0.4142	0.3954	0.3772	0.3594	0.3423	0.3257	0.3097	0.2942	0.2793	0.2650	0.2513	0.2381	0.2254	0.2133
4	0.8477	0.8318	0.8153	0.7982	0.7806	0.7626	0.7442	0.7254	0.7064	0.6872	0.6678	0.6484	0.6288	0.6093	0.5898	0.5704	0.5512	0.5321	0.5132	0.4946	0.4763	0.4582	0.4405	0.4231	0.4061	0.3895	0.3733
5	0.9349	0.9258	0.9161	0.9057	0.8946	0.8829	0.8705	0.8576	0.8441	0.8301	0.8156	0.8006	0.7851	0.7693	0.7531	0.7367	0.7199	0.7029	0.6858	0.6684	0.6510	0.6335	0.6160	0.5984	0.5809	0.5635	0.5461
6	0.9756	0.9713	0.9665	0.9612	0.9554	0.9490	0.9421	0.9347	0.9267	0.9182	0.9091	0.8995	0.8893	0.8786	0.8675	0.8558	0.8436	0.8311	0.8180	0.8046	0.7908	0.7767	0.7622	0.7474	0.7324	0.7171	0.7017
7	0.9919	0.9901	0.9881	0.9858	0.9832	0.9802	0.9769	0.9733	0.9692	0.9648	0.9599	0.9546	0.9489	0.9427	0.9361	0.9290	0.9214	0.9134	0.9049	0.8960	0.8867	0.8769	0.8666	0.8560	0.8449	0.8335	0.8217
8	0.9976	0.9969	0.9962	0.9953	0.9943	0.9931	0.9917	0.9901	0.9883	0.9863	0.9840	0.9815	0.9786	0.9755	0.9721	0.9683	0.9642	0.9597	0.9549	0.9497	0.9442	0.9382	0.9319	0.9252	0.9181	0.9106	0.9027
9	0.9993	0.9991	0.9989	0.9986	0.9982	0.9978	0.9973	0.9967	0.9960	0.9952	0.9942	0.9931	0.9919	0.9905	0.9889	0.9871	0.9851	0.9829	0.9805	0.9778	0.9749	0.9717	0.9682	0.9644	0.9603	0.9559	0.9512
10	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992	0.9990	0.9987	0.9984	0.9981	0.9977	0.9972	0.9966	0.9959	0.9952	0.9943	0.9933	0.9922	0.9910	0.9896	0.9880	0.9863	0.9844	0.9823	0.9800	0.9775
11	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9991	0.9989	0.9986	0.9983	0.9980	0.9976	0.9971	0.9966	0.9960	0.9953	0.9945	0.9937	0.9927	0.9916	0.9904
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9995	0.9993	0.9992	0.9990	0.9988	0.9986	0.9983	0.9980	0.9976	0.9972	0.9967	0.9962
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9990	0.9988	0.9986
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9995
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

5.5	5.6	5.7	5.8	5.9	6	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7
0.0041	0.0037	0.0033	0.0030	0.0027	0.0025	0.0022	0.0020	0.0018	0.0017	0.0015	0.0014	0.0012	0.0011	0.0010	0.0009
0.0266	0.0244	0.0224	0.0206	0.0189	0.0174	0.0159	0.0146	0.0134	0.0123	0.0113	0.0103	0.0095	0.0087	0.0080	0.0073
0.0884	0.0824	0.0768	0.0715	0.0666	0.0620	0.0577	0.0536	0.0498	0.0463	0.0430	0.0400	0.0371	0.0344	0.0320	0.0296
0.2017	0.1906	0.1800	0.1700	0.1604	0.1512	0.1425	0.1342	0.1264	0.1189	0.1118	0.1052	0.0988	0.0928	0.0871	0.0818
0.3575	0.3422	0.3272	0.3127	0.2987	0.2851	0.2719	0.2592	0.2469	0.2351	0.2237	0.2127	0.2022	0.1920	0.1823	0.1730
0.5289	0.5119	0.4950	0.4783	0.4619	0.4457	0.4298	0.4141	0.3988	0.3837	0.3690	0.3547	0.3406	0.3270	0.3137	0.3007
0.6860	0.6703	0.6544	0.6384	0.6224	0.6063	0.5902	0.5742	0.5582	0.5423	0.5265	0.5108	0.4953	0.4799	0.4647	0.4497
0.8095	0.7970	0.7841	0.7710	0.7576	0.7440	0.7301	0.7160	0.7017	0.6873	0.6728	0.6581	0.6433	0.6285	0.6136	0.5987
0.8944	0.8857	0.8766	0.8672	0.8574	0.8472	0.8367	0.8259	0.8148	0.8033	0.7916	0.7796	0.7673	0.7548	0.7420	0.7291
0.9462	0.9409	0.9352	0.9292	0.9228	0.9161	0.9090	0.9016	0.8939	0.8858	0.8774	0.8686	0.8596	0.8502	0.8405	0.8305
0.9747	0.9718	0.9686	0.9651	0.9614	0.9574	0.9531	0.9486	0.9437	0.9386	0.9332	0.9274	0.9214	0.9151	0.9084	0.9015
0.9890	0.9875	0.9859	0.9841	0.9821	0.9799	0.9776	0.9750	0.9723	0.9693	0.9661	0.9627	0.9591	0.9552	0.9510	0.9467
0.9955	0.9949	0.9941	0.9932	0.9922	0.9912	0.9900	0.9887	0.9873	0.9857	0.9840	0.9821	0.9801	0.9779	0.9755	0.9730
0.9983	0.9980	0.9977	0.9973	0.9969	0.9964	0.9958	0.9952	0.9945	0.9937	0.9929	0.9920	0.9909	0.9898	0.9885	0.9872
0.9994	0.9993	0.9991	0.9990	0.9988	0.9986	0.9984	0.9981	0.9978	0.9974	0.9970	0.9966	0.9961	0.9956	0.9950	0.9943
0.9998	0.9998	0.9997	0.9996	0.9996	0.9995	0.9994	0.9993	0.9992	0.9990	0.9988	0.9986	0.9984	0.9982	0.9979	0.9976
0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9996	0.9995	0.9994	0.9993	0.9992	0.9990
1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9998	0.9997	0.9997	0.9996
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Probability

4. For each of the following experiments and events determine the probability of the event. Assume all (the dice are equally likely to result in any of the numbers 1, 2, ... , 6.

(a) (4 pts.)	Experiment:	Five 6-sided dice are thrown
	Event:	At least two 6's occur.
(b) (7 pts.)	Experiment:	600 6-sided dice are thrown
	Event :	At least 120 of the tosses result in 6. *Use an approximation.*
(c) (7 pts.)	Experiment:	600 6-sided dice are thrown.
	Event :	The total of all the numbers obtained is less than 2000. *Use an approximation. Hint. The variance of the uniform distribution on* 1, 2, ... , 6 *is* 35/12.
(d) (4 pts.)	Experiment:	Two dice are thrown until 6-6 occurs.
	Event :	The first 6-6 occurs on throw number 5.
(e) (7 pts.)	Experiment:	Two dice are thrown 100 times.
	Event :	Fewer than two 6-6's occur. *Use an approximation.*
(f) (3 pts.)	Experiment:	A box has 10 eggs. 3 of these are rotten. 4 eggs are drawn randomly without replacement.
	Event:	The sample contains 3 rotten eggs

5. (8 pts.) A test for the HIV virus has probability 0.99 of being positive if the person really has the virus. It has probability 0.005 of being positive if the person does not have the virus. Among all people taking the test 0.02% (1/50 of 1%) actually have the virus. If one of these people is chosen randomly and the test is positive, what is the conditional probability that the person has the virus?

6. Let X have the density $f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ ½ & \text{for } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$

(a) (5 pts) Find the c.d.f. F(x) for X.

(b) (5 pts.) Find Var(X).

(c) (5 pts.) Find the density of $Y = X^2$.

7. Let X_n have cdf F_n for n = 1, 2, ...

(a) (5 pts.) Define: F_n converges in distribution to a cdf F.

(b) (5 pts.) Let X_n take the values 1/n and n with probabilities 1 – 1/n and 1/n. Does $\{X_n\}$ have a limiting distribution? If so, what is it? Show it.

(c) (5 pts.) Let Y_n be uniformly distributed on 1/n, 2/n , … , n/n. Does Y_n have a limiting distribution? If so, what is it? Show it.

Statistics

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2 = 100)$. (The normal distribution has variance 100.)

(a) (5 pts.) Find the rejection region of the uniformly most powerful $\alpha = .05$ test of H_0: $\mu = 0$ vs. H_1: $\mu > 0$.

(b) (5 pts.) What is the power of the test at $\mu = 2$ if $n = 25$?

(c) (5 pts.) What must n be so that the power at $\mu = 2$ is .80?

9. Let $Z = X + Y$ where $X \sim B(m, .8)$ and $Y \sim B(200 - m, .2)$ are independent binomial random variables and m is unknown, $m \in \{0, 1, \ldots, 200\}$.

(a) (4 pts.) Use the method of moments to find an estimator of m based on Z.

(b) (4 pts.) What are the expected value and the standard deviation of your estimate?

(c) (4 pts.) Construct an approximate 95% CI estimate of m assuming the realization $z = 124$.

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from

(*) $$f(x \mid \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0 \quad \text{where } \theta > 0.$$

(a) (4 pts.) Determine the expectation $E_\theta(X)$ and the standard deviation $SD_\theta(X)$ of the distribution with density (*).

(b) (4 pts.) Find $\hat{\theta}_{MLE}$, the mle of θ based on $X_1, X_2, \ldots, X_n$

(c) (6 pts.) For large n, determine an approximate 99% CI on θ based on $\hat{\theta}_{MLE}$.

11. Consider the sample space of four points and the two (discrete) distributions given below. Consider testing the simple versus simple hypotheses H_0: $X \sim P_0$ vs. H_1: $X \sim P_1$. The Neyman Pearson tests have Rejection Regions of the form $\{x \mid \Lambda(x) < c\}$ for some c where $\Lambda(x) = p_0(x)/p_1(x)$.

x	x_1	x_2	x_3	x_4
$p_0(x)$	0.1	0.1	0.7	0.1
$p_1(x)$	0.2	0.4	0.1	0.3
$\Lambda(x)$				

(a) (2 pts.) Compute the likelihood ratio $\Lambda(x)$ for each x and place the values in the table above.

(b) (6 pts.) Of all possible tests of size $\alpha = .2$, which is most powerful?

(c) (2 pts.) What is the power of the most powerful test of size $\alpha = .2$?

(d) (2 pts.) Find a test of the same size but with less power than that of the NP test.

12. Consider the standard normal theory regression model with regression function $E(Y \mid x) = \beta_0 + \beta_1 x + \beta_2 x^2$ and the data

x	-1	-1	0	0	1	1
y	3	9	15	13	8	12

(a) (6 pts.) Find the least squares estimates of the parameters β_0, β_1, and β_2.

(b) (4 pts.) Determine SSE.

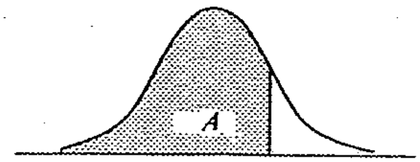
(c) (4 pts.) Find the estimated variance of $\hat{\beta}_2$

(d) (4 pts.) Give a 95% CI estimate of β_2.

(e) (4 pts.) Carry-out the .05 level F-test of H_0: $\beta_1 = \beta_2 = 0$.

TABLE A–2 Percentiles of the t distribution

Entry is $t(A;\ \nu)$ where $P\{t(\nu) \leq t(A;\nu)\} = A$



$t\ (A;\ \nu)$

	A						
ν	.60	.70	.80	.85	.90	.95	.975
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.537	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

9. (6 pts.) Let X_n have a uniform distribution on the interval $[0, 1/n)$ for $n = 1, 2, \ldots$ Prove that $\{X_n\}$ converges in probability to 0.

10. Consider the experiment where X distributed Poisson (λ) is observed and, conditional on $X = x$, a coin with probability π of Heads is given x independent tosses. Let Y denote the number of Heads among these tosses. *In solving (a) and (b), you may use the fact that a Poisson (λ) has mean λ and variance λ and a Binomial (n, p) has mean np and variance np(1 - p). You may choose to (c) first and deduce (a) and (b) after having established (c).*

(a) (2 pts.) Find E(Y).

(b) (4 pts.) Find V(Y).

(c) (6 pts.) Show that Y is distributed Poisson ($\lambda\pi$).

Statistics

11) (6 pts) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ, variance σ^2. Let $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. Prove that $E(S^2) = \sigma^2$.

12) Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density

$f(x ; \theta) = \theta x^{\theta - 1}$ for $0 < x < 1$, where $\theta > 1$.

a) (5 pts.) Find the method of moments estimator of θ.

b) (6 pts.) Find the maximum likelihood estimator of θ.

c) (8 pts.) Use the fact (you need not prove it) that $Y = -\log(X_1)$ has the exponential distribution with mean $1/\theta$ to find an approximate 95% confidence interval on θ for $n = 100$. Hint: The exponential distribution with mean μ has variance μ^2.

d) (8 pts.) Let H_0: $\theta = 2$ and H_a: $\theta = 3$. Give the $\alpha = 0.05$ level most powerful test of H_0 vs H_a.

e) (2 pts.) Is this test uniformly most powerful for H_0: $\theta = 2$ vs H_a: $\theta > 2$? Why?