Preliminary Exam: Probability
9:00am – 2:00pm, August 27, 2004

Question 1. (10 points) Let \( \{X_n\} \) be a Cauchy sequence of random variables in \( L^p(\Omega) \) \( (p \geq 1) \), i.e., for any \( \varepsilon > 0 \), there exists an integer \( n_0 \) such that \( \|X_n - X_m\|_p \leq \varepsilon \) for all \( n, m \geq n_0 \). Prove the following:

(i) For any sequences \( \{a_k\} \) and \( \{b_k\} \) of positive numbers, there is a sequence of increasing positive integers \( \{n_k\} \) so that

\[
P\left( |X_{n_k+1} - X_{n_k}| > a_k \right) \leq \frac{b_k}{a_k};
\]

(ii) There is a sequence \( n_k \uparrow \infty \) of integers and a random variable \( X \in L^p(\Omega) \) such that \( X_{n_k} \) converges to \( X \) both in \( L^p \) and almost surely.

(iii) \( X_n \to X \) in \( L^p(\Omega) \) [that is, \( L^p(\Omega) \) is complete].

Question 2. (15 points) Let \( X_1, \ldots, X_n, \ldots \) be a sequence of i.i.d. random variables with \( \mathbb{E}|X_1| < \infty \) and \( \mathbb{E}(X_1) = 0 \). Let \( S_n = \sum_{i=1}^{n} X_i \). For any \( \varepsilon > 0 \), define

\[
N_\varepsilon = \sum_{n=1}^{\infty} \mathbb{1}_{\{|S_n| > \varepsilon n\}}.
\]

(i) Prove that, with probability 1, \( N_\varepsilon < \infty \) for all \( \varepsilon > 0 \), i.e. \( \mathbb{P}(N_\varepsilon < \infty \text{ for all } \varepsilon > 0) = 1 \).

(ii) We further assume \( \mathbb{E}(X_1^2) = \sigma^2 \), find

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \sum_{n=1}^{\infty} \mathbb{P}\{|X_1| > \varepsilon \sqrt{n}\}.
\]

[Hint: Start by calculating \( \int_{0}^{\infty} \mathbb{P}\{|X_1| > \varepsilon \sqrt{x}\} dx \).

(iii) Prove that if \( X_1 \sim N(0, 1) \), then

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \mathbb{E}(N_\varepsilon) = 1.
\]

Question 3. (10 points) Let \( X_1, \ldots, X_n, \ldots \) be a sequence of i.i.d. random variables such that

\[
P\{|X_1| > x\} \sim x^{-\alpha} \quad \text{as } x \to \infty,
\]

where \( \alpha \in (0, 1) \) is a constant. Prove the following statements:
(i) If the real sequence \( \{a_n\} \) satisfies \( \sum_{n=1}^{\infty} |a_n|^\alpha < \infty \), then \( \sum_{n=1}^{\infty} a_nX_n \) converges almost surely.

(ii) For any \( 0 < p < \alpha \), we have
\[
\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n^{1/p}} = 0 \quad \text{a.s.}
\]

**Question 4.** (15 points) Let \( X_1, \ldots, X_n, \ldots \) be a sequence of i.i.d. random variables such that \( \mathbb{P}\{X_1 > x\} = \mathbb{P}\{X_1 < -x\} \) and
\[
\mathbb{P}\{|X_1| > x\} = \begin{cases} 
1 & \text{if } 0 \leq x < e, \\
\frac{1}{x^2 \log x} & \text{if } x \geq e.
\end{cases}
\]
Prove the following statements:

(i) \( \mathbb{E}(X_1^2) = \infty. \)

(ii) Let \( Y_{n,m} = X_m \mathbf{1}_{\{|X_m| \leq \sqrt{n}\}}. \) As \( n \to \infty, \)
\[
\sum_{m=1}^{n} \mathbb{P}\{Y_{n,m} \neq X_m\} \to 0.
\]

(iii) As \( n \to \infty, \) \( \mathbb{E}(Y_{n,m}^2) \sim 2 \log \log n. \)

(iv) Let \( S'_n = \sum_{m=1}^{n} Y_{n,m}. \) Then
\[
\frac{S'_n}{\sqrt{2n \log \log n}} \Rightarrow \chi \quad \text{as } n \to \infty,
\]
where \( \chi \) is a standard normal random variable.

(v) Let \( S_n = \sum_{m=1}^{n} X_m. \) Then
\[
\frac{S_n}{\sqrt{2n \log \log n}} \Rightarrow \chi \quad \text{as } n \to \infty.
\]

**Question 5.** (15 points) Let \( X_1, \ldots, X_n, \ldots \) be i.i.d. r.v.'s with
\[
\mathbb{P}(X_1 = 1) = p > 1/2 \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p.
\]
Consider the asymmetric simple random walk \( \{S_n, n \geq 0\} \) on \( \mathbb{Z} \) defined by \( S_0 = 0 \) and \( S_n = X_1 + \cdots + X_n \) for \( n \geq 1. \) Given integers \( a < 0 < b, \) let \( T_a = \inf\{n > 0 : S_n = a\} \) and \( T_b = \inf\{n > 0 : S_n = b\}. \) It is known that
\[
\mathbb{P}\{T_a < T_b\} = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)},
\]
where \( \varphi(x) = [(1-p)/p]^x. \) Prove the following statements:
(i) If \( b > 0 \), then \( \mathbb{P}(T_b \leq \infty) = 1 \).

(ii) For every integer \( a < 0 \),

\[
\mathbb{P}\left( \min_n S_n \leq a \right) = \mathbb{P}\left( T_a < \infty \right) = \left( \frac{p}{1 - p} \right)^a.
\]

Show that \( \mathbb{E}\left( \min_n S_n \right) > -\infty \).

(iii) For every integer \( b > 0 \), \( \mathbb{E}(T_b) = b/(2p - 1) \).

[Hint: Use the fact that \( \{S_n - (2p - 1)n\}_{n \geq 0} \) is a martingale.]

**Question 6.** (15 points) Let \( Y_n \) \((n \geq 1)\) be i.i.d. normal random variables with mean 0 and variance \( \sigma^2 \), and let \( S_n = Y_1 + \cdots + Y_n \). For each \( u \in \mathbb{R} \), define

\[
X_n^u = \exp\left( uS_n - \frac{1}{2}nu^2\sigma^2 \right).
\]

(i) Show that for every \( u \in \mathbb{R} \), \( \{X_n^u\} \) is a martingale. What is \( \mathbb{E}(X_\infty^u) \)?

(ii) Show that for every \( u \in \mathbb{R} \), \( \{X_n^u\} \) converges a.s. to a random variable \( X_\infty^u \) and \( X_\infty^u < \infty \) a.s.

(iii) Show that \( \sum_{n=1}^{\infty} \mathbb{E} \left( \sqrt{X_n^u} \right) < \infty \).

(iv) For each \( u \in \mathbb{R} \), what is the distribution of \( X_\infty^u \)?

(v) For each \( u \neq 0 \), is the sequence \( \{X_n^u\} \) uniformly integrable?

**Question 7.** (20 points) Let \( W = \{W(t), t \geq 0\} \) be a standard Brownian motion in \( \mathbb{R} \). For any integer \( n \geq 1 \), let \( I_{n,k} = [k2^{-n}, (k+1)2^{-n}] \) \((k = 0, 1, \ldots, 2^n - 1)\) be dyadic intervals of order \( n \) in \([0, 1] \). Define

\[
\Delta_{n,k} = \max_{t \in I_{n,k}} |W(t) - W(k2^{-n})|.
\]

(i) Use the reflection principle \( \mathbb{P}\left\{ \max_{[0,t]} |W(s)| \geq a \right\} = 2\mathbb{P}\{|W(t)| \geq a\} \) to show that for any \( a > 1 \),

\[
\mathbb{P}\{\Delta_{n,k} \geq a n^{-n/2}\} \leq 4 \exp\left(-a^2/2 \right).
\]

(ii) For any \( \varepsilon > 0 \), let \( \beta = 2(1 + \varepsilon) \log 2 \) and \( a_n = \sqrt{\beta n} \). Then

\[
\mathbb{P}\{\Delta_{n,k} \geq a_n n^{-n/2} \text{ for some } 0 \leq k \leq 2^n - 1\} \leq 4 \cdot 2^{-n \varepsilon}.
\]
(iii) Let $\text{osc}(\delta) = \sup\{|W(s) - W(t)| : s, t \in [0,1], |s - t| \leq \delta\}$ be the modulus of continuity of $W$ on $[0,1]$. Prove that

$$\limsup_{\delta \to 0} \frac{\text{osc}(\delta)}{\sqrt{\delta \log(1/\delta)}} \leq 6 \quad \text{a.s.}$$

[Hint: Use the Borel-Cantelli Lemma and triangle's inequality.]

**Question 8.** (Optional) Let $W_i = \{W_i(t), t \geq 0\}$ ($i = 1, 2, \ldots, d$) be $d$ independent standard Brownian motions in $\mathbb{R}$ ($d \geq 3$). For $t \geq 0$, let $W(t) = (W_1(t), \ldots, W_d(t))$. Then $W = \{W(t), t \geq 0\}$ is called a Brownian motion in $\mathbb{R}^d$. For any $T > 0$ and $\varepsilon > 0$, define

$$S_T = \int_T^\infty 1_{\{\|W(s)\| \leq \varepsilon\}}ds,$$

which is the total time after $T$ spent by $W$ in the ball $B(0, \varepsilon)$. Assume $\varepsilon < \sqrt{T}$. Prove the following statements:

(i) There exists some constant $c_1 > 0$ such that $\mathbb{E}(S_T) \geq c_1 \varepsilon^d T^{1-\frac{d}{2}}$.

(ii) For some finite constant $c_2 > 0$,

$$\mathbb{E}(S_T^2) \leq c_2 \varepsilon^{d+2} T^{1-\frac{d}{2}}.$$

(iii) Apply the Paley-Zygmund inequality: for all non-negative random variable $Y$ and $0 \leq \lambda < 1$,

$$\mathbb{P}\{Y \geq \lambda \mathbb{E}(Y)\} \geq (1 - \lambda)^2 \frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)}$$

to show that

$$\mathbb{P}\{\exists t > T \text{ such that } \|W(t)\| \leq \varepsilon\} \geq \left(\frac{\varepsilon}{\sqrt{T}}\right)^{d-2}.$$