Problem 1.

Let \( \{X_k\}_{k \geq 1} \) be a sequence of i.i.d. random variables.

a. Prove that the following are equivalent:

(i) \( n \cdot P(|X_1| > \varepsilon \cdot \sqrt{n}) \xrightarrow[n \to \infty]{} 0 , \forall \varepsilon > 0 \)

(ii) \( [1 - P(|X_1| > \varepsilon \cdot \sqrt{n})]^n \xrightarrow[n \to \infty]{} 1 , \forall \varepsilon > 0 \)

(iii) \( \max \{ |X_k| \} \sqrt{n} \xrightarrow[n \to \infty]{} 0 \) in probability.

b. Assume that \( E(X_1^2) < \infty \). Do (i), (ii) and (iii) from part a. hold? Prove or give a counter example.

Problem 2.

Let \( X \geq 0 \) be a random variable. Assume \( \sum_{n=1}^{\infty} P(X > a_n) < \infty \) where \( (a_n)_{n \geq 0} \) denote a sequence of numbers so that \( a_0 = 0, a_{n+1} > a_n \) and \( \frac{a_n}{n} \uparrow \infty \). Let \( Y_n = X \cdot 1_{\{X < a_n\}}, n \geq 1 \). Prove the following

a. \( \sum_{m=1}^{\infty} m \cdot P(a_{m-1} \leq X < a_m) < \infty \)

b. For every \( N < n \) we have

\[
\sum_{m=1}^{n} \frac{E(Y_m)}{a_n} < \frac{n \cdot E(Y_n)}{a_n} + \sum_{m=N+1}^{n} \frac{m}{a_m} \cdot E(X; a_{m-1} \leq X < a_m)
\]

Hint: Observe that \( \sum_{m=1}^{n} E(Y_m) < n \cdot E(Y_n) \). Also use: \( \frac{n}{a_n} \leq \frac{m}{a_m} \) if \( m \leq n \).

\[
\sum_{m=1}^{n} \frac{E(Y_m)}{a_n} \xrightarrow[n \to \infty]{} 0
\]
Problem 3.

Let \( \{X_n\}_{1 \leq n} \) be a sequence of independent random variables. The distribution of \( X_n \), \( n \geq 1 \) is given by:

\[
X_n = \begin{cases} 
\pm 1 & \text{with probability } \frac{1}{2} - \frac{c}{2 \cdot n^2} \\
\pm n \cdot k^3 & \text{with probability } \frac{1}{2 \cdot n^2 k^3}, \ n \geq 1 
\end{cases}
\]

with \( c = \sum_{k=2}^{\infty} 1/k^3 < 1 \). Let \( S_n = \sum_{k=1}^{n} X_k \), \( n \geq 1 \).

a. Prove that \( \frac{S_n}{n} \rightarrow 0 \), a.s. (Hint: think about random series)

b. Let \( \{Y_n\}_{n \geq 1} \) be i.i.d. random variables with \( Y_1 = X_1 \) in distribution, namely

\[
Y_1 = \begin{cases} 
\pm 1 & \text{with probability } \frac{1}{2} - \frac{c}{2} \\
\pm k^3 & \text{with probability } \frac{1}{2 \cdot k^3}, \ k \geq 2
\end{cases}
\]

Let \( T_n = \sum_{k=1}^{n} Y_k \), \( n \geq 1 \). Prove \( \frac{T_n}{n^3} \rightarrow 0 \), a.s.

Problem 4.

Let \( \{X, X_k\}_{k \geq 1} \) be a sequence of i.i.d random variables. Denote by \( \varphi \) the c.f. of \( X \).

Let \( S_n = \sum_{k=1}^{n} X_k \). Prove the following:

a. If \( \varphi'(0) = \lim_{h \rightarrow 0} \frac{\varphi(h) - 1}{h} = 0 \) then \( \frac{S_n}{n} \rightarrow 0 \) in distribution.

b. Use the well known fact \( \frac{\log(1 + z)}{z} \rightarrow 1 \) (\( z \) denote a complex number) to get the converse of part a.: If \( \frac{S_n}{n} \rightarrow 0 \) in distribution then \( \varphi'(0) = 0 \).

c. Can the results of parts a. and b. be extended to the case \( \varphi'(0) = i \cdot a \) where \( a \) is any real-valued number? Either provide a proof or provide a counter example.
Problem 5.

Let \( \{X_k\}_{k \geq 1} \) be a sequence of independent random variables and let 
\[
S_n = \sum_{k=1}^{n} X_k, \quad \mu_n = E(S_n) \quad \text{and} \quad \sigma_n = s.d.(S_n). \]
In what follows you are asked to 
prove that \( \frac{S_n - \mu_n}{\sigma_n} \) converges in distribution as \( n \to \infty \) and identify the limit 
distribution.

a. \( X_k = Z_k \cdot 1_{\{Z_k \leq 1\}} \) where \( Z_k \sim \text{Poisson}(1/k), \ k \geq 1 \)

b. \( X_k \sim \text{Poisson}(1/k), \ k \geq 1 \)

c. \( X_k \sim \text{Poisson}(1/k^2), \ k \geq 1 \)

Problem 6.

Let \( \{X_k\}_{k \geq 0} \) be a positive supermartingale with respect to the increasing sequence of 
\( \sigma \) - algebras \( \{F_k\}_{k \geq 0} \).

a. Let \( \{Y_k\}_{k \geq 0} \) be another \( \{F_k\}_{k \geq 0} \) - supermartingale. Let \( T \geq 0 \) be a stopping time. 
Assume \( X_T \geq Y_T \), a.s. Prove that \( \{W_k\}_{k \geq 0} \) is \( \{F_k\}_{k \geq 0} \) - supermartingale, where 
\[
W_k = \begin{cases} 
X_k & \text{if } 0 \leq k < T \\
Y_k & \text{if } k \geq T 
\end{cases}
\]

b. Let \( b > a > 0 \) and assume that \( X_0 > a \). Define 
\[
S = \inf\{k : X_k \leq a\} \\
T = \inf\{k > S : X_k \geq b\}
\]
( both \( S \) and \( T \) can get the value \( \infty \)). Let 
\[
Z_k = \begin{cases} 
1 & \text{if } 0 \leq k < S \\
X_k/a & \text{if } S \leq k < T \\
b/a & \text{if } T \leq k
\end{cases}
\]
Prove that \( \{Z_k\}_{k \geq 0} \) is a \( \{F_k\}_{k \geq 0} \) - supermartingale.

c. We continue with the setup of part b. Let \( U \) be the number of up-crossings of 
\([a,b]\) by \( \{X_k\}_{k \geq 0} \). Prove that \( E(Z_T) \leq 1 \) and \( P(U \geq 1) \leq a/b \).
Problem 7.

Let $X, X_k, k \geq 0$ be a sequence of $L^1$ random variables defined on $(\Omega, G, \mathbb{P})$ and let $F_k \subset G$ be a decreasing sequence of $\sigma$-algebras, i.e. $F_k \downarrow F$. In what follows we denote $M_k = \sup \{|X_{k_2} - X_{k_1}|, k \geq 0\}$. Prove the following

a. If $E |X_k - X| \xrightarrow{k \to \infty} 0$ then $E |E_{F_k}(X_k) - E_F(X)| \xrightarrow{k \to \infty} 0$

b. If $E(M_1) < \infty$ then there is an integrable random variable $M$, so that:

$E |E_F(M_k) - E_F(M)| \xrightarrow{k \to \infty} 0$ and $E_F(M_k) \xrightarrow{k \to \infty} E_F(M)$ almost surely.

c. If $X_k \xrightarrow{k \to \infty} X$ almost surely and $E(M_1) < \infty$ then

$E_{F_k}(|X_k - X|) \xrightarrow{k \to \infty} 0$ almost surely.

Also, prove that: $E_{F_k}(X_k) \xrightarrow{k \to \infty} E_F(X)$ almost surely.

Remark. The dominated convergence theorem for conditional expectations in the textbook deals with the case $F_k \uparrow F_\infty$.

Problem 8.

Let $W(t)$, $0 \leq t \leq 1$ be a standard Brownian motion. Let $\{t_k\}_{k \geq 1}$ be a sequence of numbers in $(0, 1)$. For each $n \geq 1$ we denote by

$0 = t_{n,0} < t_{n,1} < t_{n,2} < \ldots < t_{n,n} < t_{n,n+1} = 1$ the order statistics of $\{0, 1, t_1, \ldots, t_n\}$. We assume that $\lambda_n = \max_{0 \leq k \leq n} \{|t_{n,k+1} - t_{n,k}|\} \xrightarrow{n \to \infty} 0$

Finally define $Q_n = \sum_{k=0}^{n} [W(t_{n,k+1}) - W(t_{n,k})]^2$.

a. Prove that $Var(Q_n) \xrightarrow{n \to \infty} 0$. What can you say about the convergence in probability of $\{Q_n\}$? Explain.

b. Let $0 < s < t < u < 1$. Let $F$ be a $\sigma$-algebra defined by:

$F = \sigma(\{W(u) - W(t), |W(t) - W(s)|\})$.

Find the conditional distribution of $(W(u) - W(t)) \cdot (W(t) - W(s))$ given $F$. Use it to calculate: $E_F(W(u) - W(s))^2$.

c. Define a decreasing sequence of $\sigma$-algebras by $F_n = \sigma(H_n), n \geq 1$, where we let $H_n = \{|W(t_{m,k+1}) - W(t_{m,k})|: 0 \leq k \leq m, m \geq n\}$. How many random variables are in $H_n$ but not in $H_{n+1}$ (i.e., $H_n \cap (H_{n+1})^c$)? What is the relationship to $t_{m+1}$?

d. Prove that $(Q_n, F_n)_{n \geq 1}$ is a Backwards Martingale. What can you say about the convergence of $\{Q_n\}$ in almost sure sense? Explain.