Preliminary Exam: Probability, Friday, August 22, 2008

The exam lasts from **9:00 am until 2:00 pm**. Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam consists of 7 problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.

On each page you turn in, write your assigned code number instead of your name. Separate and staple each problem and return it to its designated folder.

Problem 1. Let $\{X_k,\ k=1,2,\ldots\}$ be a sequence of symmetric and independent random variables such that there exist constant $c>0$ with $|X_k|\le c$ for all k. Let $S_n=\sum_{k=1}^{n}X_k$.

a. Assume that $Var(S_n)\xrightarrow[n\to\infty]{}\infty$. Prove that $\dfrac{S_n-E(S_n)}{\sqrt{Var(S_n)}}\xrightarrow[n\to\infty]{}Z$ in distribution where $Z\sim N(0,1)$.

b. Prove that if $\sum_{k=1}^{\infty}Var(X_k)<\infty$ then $\dfrac{S_n-E(S_n)}{\sqrt{Var(S_n)}}$ converges a.s.

c. Let $T_n=\sum_{k=2}^{n}X_k$. Is it possible, under the assumption of part b., that both (i) $\dfrac{S_n-E(S_n)}{\sqrt{Var(S_n)}}$ and (ii) $\dfrac{T_n-E(T_n)}{\sqrt{Var(T_n)}}$ converge in distribution to $N(0,1)$?

Hint: Let $\varphi_X(t)$ denote the c.f. of X. Compare $\lim_n \varphi_{S_n-E(S_n)}(t)$ and $\lim_n \varphi_{T_n-E(T_n)}(t)$ when both (i) and (ii) converge to $N(0,1)$.

Problem 2. Let $\{X_n,\ n=1,2,\ldots\}$ be a sequence of independent random variables. The distribution of $X_n,\ n\ge 1$ is given by :

$$X_n=\begin{cases}\pm 1 \ \text{ with probability } \dfrac{1-2^{-n}}{2}\\[2mm] \pm 2^n \ \text{ with probability } 2^{-n-1}\end{cases}$$

a. Does $\dfrac{X_n}{n} \to 0$, a.s. ? Justify.

b. Does the sequence $\{X_n\}$ satisfy the strong law of large numbers? Justify.

c. Does $\sum_{n=1}^{\infty} \dfrac{X_n}{n}$ converge a.s.? Justify.

d. Prove or produce a counter example. Let $\{Y_n,\ n=1,2,\ldots\}$ be a sequence of symmetric and independent random variables. If $\sum_{n=1}^{\infty} Y_n$ converge a.s. then $\sum_{n=1}^{\infty} Var(Y_n) < \infty$.

Problem 3. Let $\{F_n\}$ and $\{G_n\}$ be two increasing sequences of sub σ – algebras of $\overline{F}$ in a probability space $(\Omega, \overline{F}, P)$. Denote $F = \sigma(\bigcup_{n=1}^{\infty} F_n)$ and $G = \sigma(\bigcup_{n=1}^{\infty} G_n)$ and assume $F_n \supset G_n, n \geq 1$.

a. If $M_n \to M$ in L_1 then $E(M_n \mid G_n) \to E(M \mid G)$ in L_1.

b. Assume that $\{M_n, F_n,\ n=1,2,\ldots\}$ is a martingale. Prove that $\{E(M_n \mid G_n), G_n,\ n=1,2,\ldots\}$ is a martingale.

c. If $M_n \to M$ in L_1 and $\{M_n, F_n,\ n=1,2,\ldots\}$ is a martingale then $E(M_n \mid G_n) \to E(M \mid G)$ a.s.

d. Under the assumption of part c, is it true that $E(M_n \mid G_n) = E(M \mid G_n), n \geq 1$? If yes, prove it; if not give a counter example.

Problem 4. An individual possesses r umbrellas which he employs in going from his home to office and vice versa (from his office to his home). If he is at home (or office) at the beginning (or end) of the day and it is raining then he will take an umbrella with him to the office (or home) provided there is one at the place. Assume that independent of the past, it rains at the beginning (or the end) of a day with probability $p > 0$.

a. Let the state i be the number of umbrellas at his present location(the number of umbrellas at the other location is $r - i$). Find the transition matrix $\lfloor P_{i,j} \rfloor, 0 \le i, j \le r$ and show that the following is a stationary distribution for the Markov chain defined by $\lfloor P_{i,j} \rfloor$:

$$\pi_i = \begin{cases} \dfrac{q}{r+q} & \text{if } \ i = 0 \\ \dfrac{1}{r+q} & \text{if } \ i = 1,\dots,r \end{cases}$$

where $q = 1 - p$.

b. Let $\{X_n, \ n = 0, 1, 2, \dots\}$ be a Markov chain associated with $\lfloor P_{i,j} \rfloor$. Assume that $P(X_0 = i) = \pi_i, \ 0 \le i \le r$. Prove that $\dfrac{\sum_{k=0}^{n-1} \sqrt{X_k}}{n}$ converge a.s. and identify the limit.

Problem 5. a. Let $\{X_k, \ k = 1,2,\dots\}$ be a sequence of i.i.d. random variables and assume that there is $c \in R$ so that $\dfrac{\sum_{k=1}^{n} X_k}{n} \xrightarrow[n\to\infty]{} c$, a.s.

Show that $E|X_1| < \infty$ and $c = E(X_1)$.

b. Let $\underline{X} = \{X_k, \ k = 1,2,\dots\}$ and let $\underline{Y} = \{Y_k, \ k = 1,2,\dots\}$ be two ergodic stationary sequences and denote by $P^{\underline{X}}$ and $P^{\underline{Y}}$ the probability measures induced on R^∞. Assume that $E|X_1| < \infty$, $E|Y_1| < \infty$ and $E(X_1) \ne E(Y_1)$. Show that $P^{\underline{X}}$ and $P^{\underline{Y}}$ are singular. Justify your answer. (Hint: $P^{\underline{X}}$ and $P^{\underline{Y}}$ are singular if there exist two disjoint sets $A, C \in B(R^\infty)$ such that $P^{\underline{X}}(A) = P^{\underline{Y}}(C) = 1$.)

Problem 6. Let $g : R \to R$ be a bounded function with bounded derivative.

a. Let $X \sim N(0, \sigma^2)$. Prove: $E(g(X)X) = \sigma^2 E(g'(X))$.
(Hint: integration by parts.)

Assume further that

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

where $Z_1,\ Z_2$ are independent N(0,1) and $a,b,c,d \in R$.

b. If $COV(X_1,X_2)=0$ then

$$E(e^{i\cdot(t_1X_1+t_2X_2)}) = E(e^{i\cdot t_1 X_1})E(e^{i\cdot t_2 X_2}),\ \text{for } (t_1,t_2)\in R^2$$

.

c. Prove that $X_2 - \dfrac{E(X_1X_2)}{Var(X_1)} \cdot X_1$ is independent of $g(X_1)$.

(Hint: use relationship between $X_2 - \dfrac{E(X_1X_2)}{Var(X_1)} \cdot X_1$ and X_1)

d. Prove: $COV(g(X_1),X_2) = COV(X_1,X_2)\cdot E(g'(X_1))$

Problem 7. Let $\{M(t),\ t\geq 0\}$ be a real-valued process whose all finite dimensional distributions are Gaussian and such that $E(M(t))=0$ for all *t*.

Assume there exists a function $h: R_+ \to R$ such that

$$E(M(t)M(s)) = h(s) \text{ for all } 0\leq s\leq t.$$

Prove:

a. The function $h(t),\ t\geq 0$ is non-negative and non-decreasing.

b. $\{M(t),\ t\geq 0\}$ has the same finite dimensional distributions as $\{B(h(t)),\ t\geq 0\}$ where $\{B(t),\ t\geq 0\}$ is a real-valued standard Brownian motion.

c. Assume that *h* is continuous and *h*(0) = 0. Assume also that $\max\limits_{0\leq t\leq 1} M(t)$ is a random variable. Find

$$\lim_{u\to\infty}\frac{1}{u^2}\cdot \log P(\max_{0\leq t\leq 1} M(t) > u).$$