Preliminary Exam: Probability
Friday, August 28, 2009

The exam lasts from **9:00 am until 2:00 pm**. Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam consists of **7** problems. The several steps that problems 3-7 are made of are designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.

On each page you turn in, write your assigned code number instead of your name. Separate and staple each problem and return it to its designated folder.

Problem 1. (5 points) Let X and Y be independent random variables and let $f : R \to R$ and $g : R \to R$ be measurable functions. Prove that if $E| f(X) - g(Y)| < \infty$ then $E| f(X)| < \infty$ and $E|g(Y)| < \infty$.

Problem 2. (7 points) Let $\{ X_n, \ n \geq 0\}$ be a sequence of random variables that are integer-valued. Assume that for each $k \in Z$

$$P(X_n = k) \xrightarrow[n\to\infty]{} P(X_0 = k).$$

Prove that $\sum_{k \in Z} | P(X_n = k) - P(X_0 = k)| \xrightarrow[n\to\infty]{} 0$.

Problem 3. Let $F_k, k=0,...,n$ be an increasing sequence of $\sigma-$algebras. Let $\{S_k, k=0,...,n\}$ be an L_2 martingale with respect to $\{F_k\}$ with $S_0=0$. Let $\lambda>0$ and define: $\tau=\min\{0\le k\le n: |S_k|>\lambda\}$
($\tau=n$ if no k with $|S_k|>\lambda$ exists)

a. (5 points) Denote $X_k=S_k-S_{k-1},\ 1\le k\le n$. Prove that $Y_k=\sum_{i=1}^{k} S_{i-1}X_i$ is a MG. What is $E(Y_\tau)$?

b. (5 points) Prove that $E(S_{\tau-1}S_\tau)\le\lambda E(|S_n|)$.

c. (3 points) Assume the algebraic identity :

$$S_{k-1}^2+\sum_{i=1}^{k-1}X_i^2=2S_{k-1}S_k-2\sum_{i=1}^{k}S_{i-1}X_i,\quad 2\le k\le n\ ,$$

and prove that

$$E(S_{\tau-1}^2+\sum_{i=1}^{\tau-1}X_i^2)\le 2\lambda E(|S_n|).$$

Problem 4. Let $\{X_n, n\ge 0\}$ be a sequence of random variables taking values in [0, 1]. We set $F_n=\sigma\{X_k, k=0,1,...,n\}$. Assume that $X_0=a$ a.s., where $a\in[0,1]$ is a constant, and

$$P(X_{n+1}=\frac{X_n}{2}\,|\,F_n)=1-X_n,\quad P(X_{n+1}=\frac{1+X_n}{2}\,|\,F_n)=X_n$$

a. (7 points) Prove that $\{X_n, n\ge 0\}$ is a martingale which converges a.s. and in L^2 to a random variable Y.

b. (4 points) Show that $E[X_{n+1}-X_n)^2]=E[X_n(1-X_n)]/4$

c. (7 points) Compute $E[Y(1-Y)]$and determine the distribution of Y.

Problem 5. Let ε, U and W be independent random variables with:

(i) $P(\varepsilon = 1) = P(\varepsilon = -1) = 1/2$,

(ii) $0 < E|W|^{\alpha} < \infty$ where $\alpha > 0$ is a constant, and

(iii) U is uniformly distributed on (0, 1).

We define

$$Y = \varepsilon \cdot U^{-1/\alpha} \cdot W$$

a. (5 points) Prove that Y is symmetric, unbounded with $P(Y = y) = 0,\ |y| > 0$.

b. (5 points) Prove that $\lim_{\lambda \to \infty} \lambda^{\alpha} P(|Y| > \lambda) = E|W|^{\alpha}$

c. (5 points) Define $\{a_n\}$, a positive and increasing sequence of constants with $a_n \to \infty$, such that for each $\lambda > 0$ we have:

$$\lim_{n \to \infty} n \cdot P(|Y| > a_n \lambda) = \lambda^{-\alpha}$$

Problem 6. Let $W_t, t \geq 0$ be a standard Brownian motion. Let $\tau = \inf\{t : \mathrm{W_t} \notin (-1, 2)\}$.

a. (5 points) Prove directly that $X_t = W_t^4 - 6tW_t^2 + 3t^2$ is a martingale with respect to $\{F_t = \sigma(W_s, 0 \leq s \leq t)\}$, namely show by using the basic definition of Brownian motion that $E(X_t - X_s \mid F_s) = 0,\ \ 0 \leq s \leq t$.

b. (6 points) Find $E(\tau)$ and $E(\tau \cdot W_{\tau})$. Justify each step in your calculations.

   Hint: you may use without proof that $W_t^3 - 3tW_t$ is a martingale.

c. (5 points) Use the values of $E(\tau)$ and $E(\tau \cdot W_{\tau})$ to develop a system of 2 equations for

   $A = E(\tau \mid W_{\tau} = -1)$ and $B = E(\tau \mid W_{\tau} = 2)$. Find A and B by solving the system.

d. (5 points) Find $E(\tau \cdot W_{\tau}^2)$ and $E(\tau^2)$.

Problem 7. Let $\{ X_n,\ n \geq 1\}$ be a sequence of independent and symmetric random variables. Let $S_n = \sum_{k=1}^{n} X_k$, $Y_m = S_{2^{m+1}} - S_{2^m}$, and $Y_m^* = \max_{2^m < k \leq 2^{m+1}} | S_k - S_{2^m} |$.

a. (5 points) Prove that if $\frac{S_n}{n} \to 0$ a.s. then $\sum_{m=0}^{\infty} P(Y_m > 2^m \varepsilon) < \infty$ for all $\varepsilon > 0$.

b. (5 points) Prove that $\sum_{m=0}^{\infty} P(Y_m > 2^m \varepsilon) < \infty$ for all $\varepsilon > 0$ if and only if $\frac{Y_m^*}{2^m} \to 0$ a.s.

c. (7 points) Prove that if $\sum_{m=0}^{\infty} P(Y_m > 2^m \varepsilon) < \infty$ for all $\varepsilon > 0$ then $\frac{S_{2^m}}{2^m} \to 0$ a.s. (Hint: Try to express $\frac{S_{2^m}}{2^m}$ by using $\{ \frac{Y_k}{2^k} \}$)

d. (4 points) Show that if $\sum_{m=0}^{\infty} P(Y_m > 2^m \varepsilon) < \infty$ for all $\varepsilon > 0$ then $\frac{S_n}{n} \to 0$ a.s.