December 2009,

Preliminary Exam: Probability, Friday, January 8, 2010

The exam lasts from **9:00 am until 2:00 pm**. Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam consists of **6** problems. The several steps that the problems are made of are designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.

On each page you turn in, write your assigned code number instead of your name. Separate and staple each problem and return it to its designated folder.

Problem 1. Let X and Y be two positive and independent random variables.

(a) (5 points) Prove $E[(\min\{X,Y\})^2] = \int_{z=0}^{\infty} 2zP(X > z)P(Y > z)dz$

(b) (5 points) Prove or disprove: If $E(X) = \infty$ and $E(Y) = \infty$ then $E[(\min\{X,Y\})^2] = \infty$

Problem 2. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$. Denote $S_n = X_1 + \ldots + X_n$. Assume that for any $t > 0$, N_t is a positive integer-valued random variable. Prove the following statements:

(a) (8 points) If for some positive integer-valued function $a(t)$ which satisfies $\lim_{t\to\infty} a(t) = \infty$ and $\frac{N_t}{a(t)} \to 1$ in probability then as $t \to \infty$

$$\frac{S_{N_t} - S_{a(t)}}{\sqrt{a(t)}} \to 0 \text{ in probability.}$$

[Hint: You may want to consider applying an appropriate maximal inequality on the event: $a(t)(1-\delta) \le N_t \le a(t)(1+\delta)$.]

(b) (6 points) Show by using (a) that $\frac{S_{N_t}}{\sqrt{a(t)}} \Rightarrow N(0,1)$ as $t \to \infty$.

(c) (6 points) Let $Y_1, Y_2, \ldots$ be i.i.d. positive random variables with $E(Y_i) = \mu$ and $Var(Y_i) = 1$. Let $T_n = Y_1 + \ldots + Y_n$ and $N_t = \sup\{k : T_k \le t\}$. Apply (b) to prove

$$\frac{\mu \cdot N_t - T_{N_t}}{\sqrt{t/\mu}} \Rightarrow N(0,1) \text{ as } t \to \infty.$$

[Hint: You may use the LLN for N_t without proof. It says that $\frac{N_t}{t} \to \frac{1}{\mu}$ almost surely as $t \to \infty$.]

(d) (6 points) We continue with the setup of part (c).

(i) Prove that $\max\limits_{1 \le m \le 2t/\mu} \{\frac{Y_m}{\sqrt{t}}\} \to 0$ in probability as $t \to \infty$.

(ii) Show how to use (i) in order to get $\frac{T_{N_t} - t}{\sqrt{t}} \to 0$ in probability as $t \to \infty$.

[Hint: $0 \le t - T_{N_t} \le Y_{N_t + 1}$].

(iii) Prove $\frac{\mu \cdot N_t - t}{\sqrt{t/\mu}} \Rightarrow N(0,1)$ as $t \to \infty$.

Problem 3. (15 points) Let $F_k, k = 0, \ldots, n$ be an increasing sequence of $\sigma-$algebras. Let $H_k \in F_k, k = 0, \ldots, n-1$, $\xi \in F_n$ be integrable random variables, and $H_n \equiv 0$. Find $Y_k \in F_k, Z_k \in F_k, k = 0, \ldots, n$ (in terms of $H_k, k = 0, \ldots, n$ and ξ) so that the following holds

(a) $\{Z_k\}$ is a martingale with respect to $\{F_k\}_{\mathrm{k=0}}^{\mathrm{n}}$.

(b) $Y_0 = Z_0$ and $Y_n = \xi$.

(c) $Y_{k+1} - Y_k = H_k + Z_{k+1} - Z_k$, $k = 0, \ldots, n-1$

Problem 4. Let $\{X_k, F_k\}, k = 0, 1, \ldots$ be a SUPERMG with $X_0 = 0$ and $X_{k+1} \ge X_k - 1$. Let $b < 0 < a$ and let $\tau = \min\{0 \le k:\ X_k > a \text{ or } X_k < b\}$ ($\tau = \infty$ if no such k exists).

(a) (6 points) Prove $X_\tau = \lim_{n\to\infty} X_{\tau\wedge n}$ exists a.s. (Hint: $X_{\tau\wedge n}$ is bounded from below. Warning: we may have $P(\tau = \infty) > 0$.)

(b) (5 points) Prove that $\liminf_{n\to\infty} E(X_{\tau\wedge n}) \ge E(X_\tau)$. Can you conclude that $E(X_\tau) \le 0$?

(c) (6 points) Prove: $P(X_\tau \ge a) \le \dfrac{1-b}{a+1-b}$

Problem 5. Let $X_n \Rightarrow X$ where the symbol $\Rightarrow$ means convergence in distribution. In what follows the symbol $\overset{D}{=}$ means equal in distribution.

(a) (6 points) Let k be a positive integer. Assume that for each n $X_n \overset{D}{=} \sum_{i=1}^{k} X_{n,i}$, where $\{X_{n,1}, \ldots, X_{n,k}\}$ are i.i.d. Prove that $\{X_{n,1}, n = 1, 2, \ldots\}$ is tight.

(b) (5 points) We continue with the setup of part a. Prove that the following representation holds: $X \overset{D}{=} \sum_{i=1}^{k} X_i$ with $\{X_1, \ldots, X_k\}$ i.i.d.

(c) (6 points) Prove that if for each n $X_n \overset{D}{=} \sum_{i=1}^{n} X_{n,i}$, where $\{X_{n,1}, \ldots, X_{n,n}\}$ are i.i.d. , then for each k we have the representation $X \overset{D}{=} \sum_{i=1}^{k} Y_{k,i}$ with $\{Y_{k,1}, \ldots, Y_{k,k}\}$ i.i.d.

Problem 6. In what follows $\varphi_X(t) = E(e^{itX})$, $-\infty < t < \infty$ represent characteristic function of the r.v. *X*.

(a) (5 points) Prove: If $\varphi_X(t_0) = 1$, $|t_0| > 0$ then $\sum_{k=-\infty}^{\infty} P(X = \frac{2\pi k}{t_0}) = 1$.

(b) (5 points) Prove that if there is a sequence of numbers $t_n \to 0$ so that $\varphi_X(t_n) = 1$ for each *n*, then $P(X = 0) = 1$.

(c) (5 points) Assume that $X_n - Y_n \Rightarrow 0$ where for each *n* $\{X_n, Y_n\}$ are i.i.d. Assume also that there is $t_n \to 0, |t_n| > 0$ with $\varphi_{X_n}(t_n) = 1$ for each *n*. Is it necessarily true that $X_n \Rightarrow 0$? Prove or produce a counter example.