Preliminary Exam: Probability

9:00am – 2:00pm, Friday, January 6, 2012

The exam lasts from 9:00am until 2:00pm, with a walking break every hour.

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.

The exam consists of six main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step. On you work, label the steps this way: (i), (ii),...

On each page you turn in, write your assigned code number instead of your name. Separate and staple each main part and return each in its designated folder.

Question 1. Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{P}\{X_1 > x\} = e^{-x}$ $(x > 0)$ and let $M_n = \max_{1 \le j \le n} X_j$.

(i). (6 points) Show that $\limsup_{n\to\infty} \frac{X_n}{\ln n} = 1$ a.s.

(ii). (6 points) Show that $\lim_{n\to\infty} \frac{M_n}{\ln n} = 1$ a.s.

Question 2. Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. symmetric random variables such that

$$\mathbb{P}\{|X_1| > x\} = \frac{1}{x^p \ln x} \qquad \text{for all } x \ge e$$

and

$$\mathbb{P}\{X_1 = 0\} = 1 - e^{-p},$$

where $p \in (1, 2)$ is a constant. Prove the following statements.

(i). (3 points) $\mathbb{E}(|X_1|^p) = \infty$.

(ii). (3 points) Let $Y_k = X_k \mathbb{1}_{\{|X_k| \le (k \ln k)^{1/p}\}}$. Prove that $\mathbb{P}(X_k \ne Y_k \text{ i.o.}) = 0$.

(iii). (5 points) Verify that $\mathbb{E}(Y_k) = 0$ and

$$\sum_{k=3}^{\infty} \operatorname{Var}\left(\frac{Y_k}{(k \ln k)^{1/p}}\right) < \infty.$$

[Hint: You may use the following inequality: For all $n \ge 3$ and

$$\big[(n-1)\ln(n-1)\big]^{1/p} \le y \le \big[n \ln n\big]^{1/p},$$

we have

$$\sum_{m=n}^{\infty} \frac{1}{(m \ln m)^{2/p}} \le C\, \frac{y^{p-2}}{\ln y},$$

where $0 < C < \infty$ is a constant.]

(iv). (3 points) Let $S_n = X_1 + \cdots + X_n$ for $n \ge 1$. Prove that

$$\lim_{n\to\infty} \frac{S_n}{(n \ln n)^{1/p}} = 0, \qquad \text{a.s.}$$

Question 3. Let $\{\xi_n, \eta_n,\ n \geq 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) (5 points) Prove that for every $t \in \mathbb{R} = (-\infty, \infty)$, the random series

$$\sum_{n=1}^{\infty} \left(\xi_n \frac{\sin(nt)}{n} + \eta_n \frac{\cos(nt)}{n} \right) \tag{2.1}$$

converges almost surely.

(ii) (5 points) Denote the limit in (2.1) by $X(t)$. Show that $X(t)$ is a normal random variable. [Hint: You can either use characteristic functions or show that the series (2.1) also converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.]

(iii). (3 points) In fact $\{X(t), t \in \mathbb{R}\}$ is a Gaussian process [you do not need to prove this fact]. Verify that for every $s, t \in [0, 1]$,

$$\mathbb{E}\big[(X(t) - X(s))^2\big] = \sum_{n=1}^{\infty} \frac{2}{n^2} \Big(1 - \cos(n(t-s))\Big). \tag{2.2}$$

(iv). (4 points) For any $s, t \in [0, 1]$ such that $|t - s| \leq 1$, let $k \geq 1$ be the integer such that

$$\frac{1}{k+1} < |t-s| \leq \frac{1}{k}.$$

By breaking the series in (2.2) into two sums $\sum_{n=1}^{k}(\cdots) + \sum_{n=k+1}^{\infty}(\cdots)$, show that

$$\mathbb{E}\big[(X(t) - X(s))^2\big] \leq C|s-t| \quad \text{for all } s, t \in [0,1], \tag{2.3}$$

where $0 < C < \infty$ is a constant. [Hint: You may use the inequalities that $1 - \cos x \leq x^2$ and $1 - \cos x \leq 2$ for all $x \in \mathbb{R}$, as well as the fact that $\sum_{n=m}^{\infty} \frac{1}{n^2} \sim \frac{1}{m}$ as $m \to \infty$.]

(v). (3 points) Apply (ii) and (2.3) to show that for every integer $m \geq 1$, there is a constant C_m such that

$$\mathbb{E}\big(|X(t) - X(s)|^{2m}\big) \leq C_m |t-s|^m \quad \text{for all } s, t \in [0,1].$$

Further, show that, for any $\gamma \in (0, 1/2)$, for almost every $\omega \in \Omega$ there is a constant $C(\omega)$ such that

$$\big|X(q) - X(r)\big| \leq C(\omega)|q - r|^{\gamma} \quad \text{for all } q, r \in \mathbb{Q}_2 \cap [0,1],$$

where $\mathbb{Q}_2 = \{k2^{-n} : k, n \geq 0\}$ denotes the set of dyadic rationals. [This implies that $X(t)$ has a modification which is uniformly Hölder continuous on [0, 1] of any order $\gamma < 1/2$. But you do not need to prove this last statement.]

Question 4. Let $\{X_n, n \geq 0\}$ be a sequence of i.i.d. $N(\mu, \sigma^2)$ random variables with $\mu < 0$. Let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and

$$W = \sup_{n \geq 0} S_n.$$

(i). (4 points) Prove that $\mathbb{P}\big\{W < \infty\big\} = 1$.

(ii). (5 points) Compute $\mathbb{E}\big(e^{\lambda S_{n+1}} | \mathcal{F}_n\big)$, where $\lambda \in \mathbb{R}$.

(iii). (5 points) Show that there exists a unique $\lambda_0 > 0$ such that $\{e^{\lambda_0 S_n}, n \geq 0\}$ is a positive martingale.

(iv). (5 points) Show that for every $a > 1$,

$$\mathbb{P}\Big(e^{\lambda_0 W} > a\Big) \leq \frac{1}{a}$$

and thus, for every $t > 0$, $\mathbb{P}\big(W > t\big) \leq e^{-\lambda_0 t}$.

(v). (4 points) Show that for every $0 < \lambda < \lambda_0$, $\mathbb{E}(e^{\lambda W}) < \infty$.

Question 5. Let $\{X_n, n \geq 1\}$ be a martingale sequence w.r.t. the filtration $\{\mathcal{F}_n\}$.

(i). (5 points) Let $n_k \uparrow \infty$ and assume that $\{X_{n_k}, k \geq 1\}$ is uniformly integrable. Prove that $\{X_n, n \geq 1\}$ is also uniformly integrable.

(ii). (5 points) Let N be a stopping time relative to the filtration $\{\mathcal{F}_n\}$ and define

$$Y_k = X_{N \wedge k} = X_N \mathbb{1}_{\{N \leq k\}} + X_k \mathbb{1}_{\{N > k\}}.$$

Prove that if $\mathbb{E}(|X_N|) < \infty$ and $\lim_{k \to \infty} \mathbb{E}\big(|X_k| \mathbb{1}_{\{N > k\}}\big) = 0$, then $\{Y_k, k \geq 1\}$ is uniformly integrable.

(iii). (3 points) Use Part (i) to show that the 2nd condition in Part (ii) can be weakened to $\liminf_{k \to \infty} \mathbb{E}\big(|X_k| \mathbb{1}_{\{N > k\}}\big) = 0$. You may use the fact that $\{Y_k, k \geq 1\}$ is a martingale w.r.t. $\{\mathcal{F}_k\}$ without proof.

Question 6. Let $\{B(t), t \geq 0\}$ be a real-valued standard Brownian motion with respect to the canonical filtration $\{\mathcal{F}_t, t \geq 0\}$.

(i). (5 points) For each $t > 0$ we define the process $\{\widetilde{B}(s) : 0 \leq s \leq t\}$ by $\widetilde{B}(s) = B(t-s) - B(t)$. Prove that $\{\widetilde{B}(s) : 0 \leq s \leq t\} = \{B(s) : 0 \leq s \leq t\}$ in distribution. [That is, the two processes have the same finite-dimensional distributions.]

(ii). (4 points) For $t > 0$, denote $M_t^B = \max_{0 \leq s \leq t} B(s)$ and $M_t^{\widetilde{B}} = \max_{0 \leq s \leq t} \widetilde{B}(s)$. Prove that $M_t^{\widetilde{B}} = M_t^B - B(t)$ almost surely.

(iii). (4 points) Show how to conclude that $M_t^B - B(t) = M_t^B$ in distribution. Is it true that $M_t^B - B(t) = |B(t)|$ in distribution?

(iv). (5 points) Let $0 < t < T$ be fixed. Define

$$\tau = \inf\{u \geq t : B(u) = M_t^B\} \quad \text{and} \quad \rho = \inf\{u \geq t : B(u) = 0\}.$$

Prove $\mathbb{P}(\tau \leq T) = \mathbb{P}(\rho \leq T)$ by using (iii) and the reflection principle.