Problem 1 Let $\theta > 0$ and $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\theta, \theta)$. Obtain the M.L.E. $\hat{\theta}_n$ of $\theta$ and derive the asymptotic distribution of $n^{1/2}(\hat{\theta}_n - \theta)$. (12 points)

Problem 2 Let $f(x)$ be a Lebesgue pdf on $(0, +\infty)$ and $X$ a single observation from $f$. Let $f_0(x) = e^{-x}I_{(0, +\infty)}(x)$, $f_1(x) = 2^{-1}x^2e^{-x}I_{(0, +\infty)}(x)$ and consider testing of the hypotheses $H_0: f = f_0$ vs $H_1: f = f_1$. Show that any test of size $\alpha$ ($0 < \alpha < 1$) has power at most $\alpha \left\{ (\ln \alpha)^2 / 2 - \ln \alpha + 1 \right\}$. (12 points)

Problem 3 Suppose $X_1, X_2, \ldots, X_n$ are i.i.d $N(\theta, 1)$
(a) Show that a UMP test for $H_0: \theta = 0$ vs $H_1: \theta \neq 0$ does not exist; (4 points)
(b) Consider the group of transformations $\{g, e\}$ where $g(x) = -x, e(x) = x$. Find the UMPI test under this group of transformations; (4 points)
(c) Let $\phi(X_1, X_2, \ldots, X_n)$ be the UMPI test. As $n \to \infty$, what is the behavior of $E_{\theta} \phi(X_1, X_2, \ldots, X_n)$? (4 points)

Problem 4 Suppose one observes $X \in \mathcal{X}$ having density $f_{\theta}(x)$ w.r.t. a $\sigma$-finite measure $\nu$ on $\mathcal{X}$, for $\theta \in \Theta \subset \mathbb{R}$. One is interested in estimating $h(\theta)$, a real valued function of $\theta$, under the loss function $L(\delta, h(\theta))$ which is strictly increasing in $\delta$ for $\delta > h(\theta)$, and strictly decreasing in $\delta$ for $\delta < h(\theta)$.
Suppose that $h(\theta)$ is nonconstant and has a global minimum at a point $\theta^*$. Assume that $f_{\theta^*}(x) > 0$, a.e. $\nu$. Let $T(X)$ be an unbiased estimator of $h(\theta)$ with finite risk under $L$.
(a) Let $A_\varepsilon := \{T(X) < h(\theta^*) - \varepsilon\}, \varepsilon > 0$. Show that there exists an $\varepsilon > 0$ such that $P_{\theta^*}(A_\varepsilon) > 0$; (6 points)
(b) Prove that $T(X)$ is inadmissible for $h(\theta)$. (6 points)

Problem 5 Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, 1)$. Construct an asymptotically pivotal quantity and a $1 - \alpha$ asymptotically correct confidence set for $\mu^2$. (12 points)
Problem 6  Let a parameter $\theta \in \Theta = (1, 2)$ and $Y_1, \cdots, Y_n$ a random sample from $U(\theta, 2\theta)$. Suppose that instead of $Y_1, \cdots, Y_n$ one observes $X_1, \cdots, X_n$ which are

$$
X_i = \begin{cases} 
2 & Y_i > 2 \\
Y_i & Y_i \leq 2 
\end{cases}.
$$

(a) Denote the $\sigma$-finite measure $\nu = \delta + m$ in which $\delta$ is the point mass measure at $\{2\}$ and $m$ the Lebesgue measure. Show that the pdf of $X_1$ with respect to $\nu$ is $f_{\theta}(x) = \left(\frac{2\theta - 2}{\theta}\right) I_{\{2\}}(x) + \left(\frac{1}{\theta}\right) I_{(\theta, 2)}(x)$; (8 points)

(b) Let $R = \sum_{i=1}^{n} I(X_i = 2)$, show that the UMVUE of $1 - \theta^{-1}$ based on $X_1, \cdots, X_n$ is $n^{-1}R/2$. (6 points)

Problem 7  Let $(x_i, Y_i), i = 1, 2, \ldots, 2n$ be bivariate data with $x_i = i/2n, i = 1, 2, \ldots, 2n$ and

$$
Y_i = \beta_{10} + \beta_{11}x_i + \epsilon_i, i = 1, 2, \ldots, n, \\
Y_i = \beta_{20} + \beta_{21}x_i + \epsilon_i, i = n + 1, 2, \ldots, 2n
$$

where $\epsilon_i, i = 1, \ldots, 2n.$ are iid from $N(0, 1)$.

(a) Give the explicit formulae of the LSE $\hat{\beta} = \left(\hat{\beta}_{10}, \hat{\beta}_{11}, \hat{\beta}_{20}, \hat{\beta}_{21}\right)$ of the vector of parameters $\beta = (\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21})$; (6 points)

(b) Show that $\hat{\beta}_{10}$ and $\hat{\beta}_{21}$ are independent. (6 points)

Problem 8  Let $X$ be a single observation from $\Gamma(\alpha, \gamma)$ with $\alpha > 0$ known and one wishes to estimate $\gamma > 0$ under the loss $L\{\gamma, T(X)\} = \{1 - T(X) \gamma^{-1}\}^2$.

(a) Show that the generalized Bayes estimator with the improper prior $\pi(\gamma) = \gamma^{-2}$ is $T_s(X) = X/ (\alpha + 2)$; (6 points)

(b) Show that $T_s(X)$ is inadmissible. (8 points)