STT 871-872 Preliminary Examination
Wednesday, August 24, 2011
12:30 p.m. - 5:30 p.m.

1. Let X denote one observation from an unknown distribution.

(a) Give a level 0 and power 1 test of $H_0 : X \sim \text{Ber}(1/2)$ vs. $H_1 : X \sim N(0, 1)$. **(5)**

(b) Give a test of level $\alpha = 0.031$ for $H_0 : X \sim \text{Bin}(5, 1/2)$ vs. $H_1 : X \sim \text{Poisson}(5)$. **(5)**

2. Let $0 < p < 1$ be unknown. Consider a sequence of independent Bernoulli random variables with success probability p. Let X_i denote the number of successes in the first i trials, $1 \le i \le n$.

(a) Compute $E(X_i \,|\, X_n)$, $i = 1, 2, \cdots, n-1$. **(5)**

(b) Consider the linear model given by

$$X_i = i\,p + \epsilon_i, \quad 1 \le i \le n$$

where ϵ_i, $i = 1, 2, \cdots, n$ are iid F with F unknown. Find the least squares estimator $\hat{p}$ of p. Derive explicitly the BLUE (best linear unbiased estimator) of p. **(5)**

3. Let $F(x)$, $x \in \mathbb{R}$ be a known cdf and let $f(x)$ be its corresponding density.

(a) Show that for each $\theta > 0$, $[F(x)]^\theta$ is a cdf on $\mathbb{R}$. **(4)**

(b) Let $X_1, X_2, \cdots, X_n$ be iid random variables from the cdf $[F(x)]^\theta$, for $x \in \mathbb{R}$ and unknown $\theta > 0$. Find a complete sufficient statistic and UMVUE of $1/\theta$. **(6)**

4. Let $X_1, X_2, \cdots, X_n$ be iid $U(-\theta, \theta)$, where $\theta > 0$ is unknown.

(a) Find the maximum likelihood estimate $\hat{\theta}_n$ of θ. Find constants a_n and b_n (possibly depending on θ) such that $a_n\hat{\theta}_n + b_n$ converges weakly to a non-degenerate distribution, as $n \to \infty$. **(4)**

(b) Find the constant c_0 and a group of transformations such that $c_0\,\hat{\theta}_n$ is the MRE (minimum risk equivariant) estimator under the loss function $L(\delta, \theta) = (\delta - \theta)^2/\theta^2$, for each $n \ge 2$. **(6)**

(c) Show that the estimator in part (b) is a minimax under the same loss function as in part (b), and for all $n \ge 2$. **(6)**

5. Let (Y_1, Y_2) be a bivariate random vector with the following distribution:

$$P(Y_1 > y_1, Y_2 > y_2) = \exp\{-\beta(y_1 + y_2) - \delta \max(y_1, y_2)\},$$

for $\beta > 0$ and $\delta > 0$. Let $M = 1$ if $Y_1 \ne Y_2$, and 0, otherwise, and let $W_1 = \min\{Y_1, Y_2\}$, $W_2 = \max\{Y_1, Y_2\}$, $T = Y_1 + Y_2$, and $V = W_2 - W_1$.

(a) Show that (M, W_2, T) is a *minimal sufficient* statistic under a suitable dominating measure. **(6)**

(b) Show that W_1 is independent of (M, V) and find the distribution of W_1. **(6)**

6. Let $\theta \in \mathbb{R}$ and let F be a cdf such that $F(0) = 1/2$. Let $X_1, X_2, \cdots, X_n$ be iid from $F(\cdot - \theta)$ that is $P_\theta(X_i \leq x) = F(x - \theta)$, $x \in \mathbb{R}$, for all $i = 1, 2, \cdots, n$. Define the n order statistics by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

(a) Construct a sequence $\{ k_n \}$ of positive integers $1 \leq k_n \leq n$ satisfying the following two properties:

$$(1) \qquad P_\theta(X_{(k_n)} \leq \theta < X_{(n-k_n+1)}) \geq 1 - \alpha, \quad \text{for all } \theta \in \mathbb{R}.$$

$$(2) \qquad P_\theta(X_{(k_n)} \leq \theta < X_{(n-k_n+1)}) \to 1 - \alpha, \quad \text{as } n \to \infty.$$ **(6)**

(b) Now assume F has derivative f at 0 with $f(0) > 0$. Find the constant w such that

$$n^{1/2} \left(X_{(n-k_n+1)} - X_{(k_n)} \right) \to w,$$

in probability as $n \to \infty$. **(6)**

7. Let $(X_1, X_2, X_3) \sim \text{Multinomial}(n, p_1, p_2, p_3)$. The Hardy-Weinberg equilibrium states that

$$H \,:\, p_1 = \theta^2, \quad p_2 = 2\,\theta\,(1-\theta) \quad \text{and} \quad p_3 = (1-\theta)^2$$

for some $0 < \theta < 1$.

(a) Show that $X_2 + 2X_1$ has a Binomial distribution under H. **(6)**

(b) Show that the level of UMP unbiased test for testing H versus K : not H, is determined by the conditional distribution of X_1, given $X_2 + 2X_1$. Find an expression for this conditional distribution. **(6)**

8. Consider the linear model

$$y_{ij} = \mu_i + \epsilon_{ij},$$

where ϵ_{ij} are iid $N(0, \sigma^2)$, $\sigma^2 > 0$ and unknown, for $j = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, p$. Consider the problem of testing the hypothesis $H : \mu_i = \beta_0 + \beta_1\, x_i$, for some unknown real numbers β_0 and β_1, where x_i's are known constants with $\sum_{i=1}^{p} x_i = 0$.

(a) Find the UMP invariant test at level α for testing H versus the alternative K : not H. **(6)**

(b) Find the asymptotic expression of the power for the following sequence of alternatives

$$\mu_i = \theta + h_i/\sqrt{n}, \qquad i = 1, 2, \cdots, p,$$

as $n \to \infty$, based on the constants h_i that satisfy $\sum_{i=1}^{p} h_i = \sum_{i=1}^{p} x_i\, h_i = 0$. **(6)**

(c) Does the test in part (a) still achieve level α, as $n \to \infty$, even if the distribution of ϵ_{ij}'s are not normal but still have mean 0 and unknown variance σ^2. Prove or disprove. **(6)**