1. Suppose an observation $X$ takes values $-1, 0, 1$, with respective probabilities $\theta/2, 1-\theta, \theta/2$, for some $0 \leq \theta \leq 1$, i.e., it has the following density:

$$f_\theta(x) := \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1, \ 0 \leq \theta \leq 1.$$ 

(a) Determine the class $U$ of all unbiased estimators of zero based on $X$.  
(b) Obtain maximum likelihood estimator $\hat{\theta}(X)$ of $\theta$.  
(c) Prove or disprove: $\hat{\theta}(X)$ is UMVU estimator for $\theta$. 

2. Suppose $k \geq 1$ is a known integer, $X \sim P_\theta$, $\theta \in \mathbb{R}^k$. 

(a) Let $T_1(X)$ and $T_2(X)$ be two UMVU estimators of $q(\theta)$. Show that

$$P_\theta(T_1(X) = T_2(X)) = 1, \ \forall \theta \in \mathbb{R}^k.$$ 

(b) Let $X \sim N(\theta, 1), \theta \in \mathbb{R}$. Show that $T(X) := X^2 - 1$ is UMVU estimator of $\theta^2$. 

(c) Let $S(X) := T(X)I(|X| < 1)$. Show that $E_\theta S^2(X) > 0$ and $T(X) = X^2 - 1$ is an inadmissible estimator of $\theta^2$ w.r.t. the square error loss.

3. Let $X$ be a Binomial$(n, p)$ random variable, for some $0 \leq p \leq 1$. Define a class of estimators

$$T_\alpha(X) := \begin{cases} X/n, & \text{with prob. } 1 - \alpha, \\ 1/2, & \text{with prob. } \alpha, \quad 0 \leq \alpha < 1. \end{cases}$$

Let $R(p, T_\alpha)$ denote the risk of the estimator $T_\alpha$ under the square error loss. Compare $R(p, T_\alpha)$ for $0 < \alpha < 1$ with $R(p, T_0)$. Find an $\alpha \in (0, 1)$ such that $\sup_{0 \leq p \leq 1} R(p, T_\alpha) < \sup_{0 \leq p \leq 1} R(p, T_0)$. Is the estimator $T_0$ minimax? 

4. Let $a$ and $k$ be unknown positive integers. Let $f_\theta$ denote the density of uniform distribution on $(\theta - 1/2, \theta + 1/2), \theta \in \mathbb{R}$. For $\theta_1 = a, \theta_2 = a + 1, \ldots, \theta_k = a + k - 1$, let

$$f_k(x) := \frac{1}{k} \sum_{j=1}^{k} f_{\theta_j}(x), \quad k = 1, 2, \ldots.$$ 

Let $X_1, X_2, \ldots, X_m$ be i.i.d. observations from $f_k$, $k = 1, 2, \ldots, m$. 

(a) Find the MLE $\hat{k}$ of $k$.  
(b) Show that $\hat{k}$ is consistent for $k$, as $m \to \infty$. 

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.’s from the parametric density

$$f_\theta(x) := 2\theta x e^{-\theta x^2}, \quad x > 0, \quad \theta > 0.$$  

(a) Derive UMP unbiased test of size $0 < \alpha < 1$, for testing $H_0 : \theta = 1$, vs. $H_1 : \theta \neq 1$, in the fullest possible detail.  

(b) Discuss the corresponding confidence interval for $\theta$. What optimality properties does it have, if any.  

6. Let $Y_1, \ldots, Y_m$ and $Z_1, \ldots, Z_m$ be mutually independent observable r.v.’s with $Y_i \sim N(\xi_i, \sigma^2)$, $Z_i \sim N(\xi_i + c, \sigma^2)$, $i = 1, 2, \ldots, m$. Here, $\xi_1, \xi_2, \ldots, \xi_m, c$ and $\sigma^2$ are all unknown parameters. Consider the problem of testing

$$H : \xi_1 = \xi_2 = \cdots = \xi_m \quad \text{vs.} \quad K : \text{not } H.$$  

(a) Obtain the explicit expression for the residual sum of squares, $SSE$, under the full model $H \cup K$ and show that it is non-negative, and describe the $F$-test for the above problem in complete detail.  

(b) Is the estimator $SSE/(m - 1)$ consistent for $\sigma^2$, as $m \to \infty$?  

(c) Consider the alternatives where $m^{-1} \sum_{i=1}^{m} (\xi_i - \bar{\xi})^2 \to \psi > 0$, as $m \to \infty$. Show that the power of the test in (a) for these alternatives converges to 1, as $m \to \infty$.  

7. Recall that in the regression model $Y_i = c_i \beta + \tau_i \eta_i$, where the r.v.’s $\eta_i$’s have zero mean and unit variance, and $c_i, \tau_i > 0$ are known numbers, the weighted least square estimator of $\beta$ is obtained by minimizing the sum $\sum_{i=1}^{n} \tau_i^{-2} (Y_i - c_i b)^2$ w.r.t. $b$.  

Let $0 < x_1 < x_2 < \cdots < x_n$ be known positive numbers. Suppose that the observations $Y_i$’s are generated by the following model:

$$Y_i = x_i \beta + U_i, \quad U_i = e_1 + e_2 + \cdots + e_i, \quad i = 1, 2, \ldots, n,$$

where $e_i, 1 \leq i \leq n$ are i.i.d. r.v.’s with mean 0 and $\text{Var}(e_i) = \sigma^2(x_i - x_{i-1})$, $i = 1, \ldots, n$, where $x_0 = 0$.  

(a) Obtain the weighted least square estimator $\hat{\beta}$ of $\beta$ and show that it depends only on $x_n$ and $Y_n$. Prove the consistency of $\hat{\beta}$ for $\beta$. Explicitly state the assumptions needed for consistency, if any.  

(b) Derive an expression for the test statistic for testing $H_0 : \beta = 1$ vs $H_1 : \beta > 1$ and describe the distribution of the test statistics under $H_0$.  

8. (a). Let $X, X_i, i \geq 1$ be i.i.d. r.v.’s with $EX = 0$, $\sigma^2 := EX^2$, $\tau := EX^4 < \infty$. Let

$$T_n := \left( n^{-1} \sum_{i=1}^{n} X_i^2 \right)^{1/2}.$$  

Derive the asymptotic distribution of $n^{1/2}(T_n - \sigma)$.  

(b). Suppose a sequence of r.v.’s $Y_n$ is such that for some constants $a$ and $b > 0$, $n^{1/2}(Y_n - a)$ converges in distribution to $\mathcal{N}(0, b^2)$. Then are the following statements true or false, in general.

(i) $\lim_{n \to \infty} EY_n = a$. (ii) $\lim_{n \to \infty} \text{Var}(Y_n) = b$.  

2