1. Consider the set up in which our data are \((x_i, Y_i, w_i), 1 \leq i \leq n\), obeying the model

\[ Y_i = \beta_1 + w_i \beta_2 + x_i \beta_3 + \varepsilon_i, \quad i = 1, 2, \cdots, n, \]

where \(w_1, w_2, \cdots, w_n\) and \(x_1, x_2, \cdots, x_n\) are known constants; \(\beta_1, \beta_2, \text{ and } \beta_3\) are real parameters; and \(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\) are i.i.d. \(N(0; \sigma^2)\) random errors. Assume that \(\sum^n w_i = 0 = \sum^n x_i\).

For notation, let \(S_{ww} = \sum w_i^2, S_{xx} = \sum x_i^2, S_{wx} = \sum w_i x_i\) and so on.

a. Write the model in matrix form as \(Y = X\beta + \varepsilon\) describing entries in the matrix \(X\). [5]

b. If \(n > 3\), show that \(X\) will be full rank iff \(D = S_{xx}S_{ww} - S_{wx}^2 \neq 0\). [5]

c. Assuming \(X\) is of full rank, give an explicit formula for the least squares estimator \(\hat{\beta}\) of \(\beta = (\beta_1, \beta_2, \beta_3)\) (It will involve terms such as \(S_{xx}, S_{xy}\), etc.). You may use the following fact. [5]

\[
(X'X)^{-1} = \begin{pmatrix}
\frac{1}{n} & 0 & 0 \\
0 & S_{xx}/D & -S_{wx}/D \\
0 & -S_{wx}/D & S_{ww}/D
\end{pmatrix}.
\]

2. Let \(X, X_1, \ldots, X_n\) be i.i.d. r.v.'s such that for \(\theta > 0\), they have common density (with respect to Lebesgue measure),

\[
f_{\theta}(x) = \begin{cases} x \theta^2 e^{-\theta x}, & x > 0; \\ 0, & x \leq 0. \end{cases}\]

Let \(p = g(\theta) = (1 + \theta)e^{-\theta} = P_\theta(X > 1)\). The two natural estimators of \(p\) are

\[
\hat{p}_n = n^{-1} \sum_{i=1}^n I(X_i > 1), \quad \text{and} \quad \hat{p}_n = g(\hat{\theta}_n),
\]

where \(\hat{\theta}_n\) is the maximum likelihood estimator of \(\theta\).

a. Find the limiting distribution of \(\sqrt{n}(\hat{p}_n - p)\). [5]

b. Find the limiting distribution of \(\sqrt{n}(\hat{\theta}_n - \theta)\). [5]

c. Derive the asymptotic relative efficiency of \(\hat{p}_n\) with respect to \(\hat{p}_n\). [5]

3. Let \(\theta > 0\) and \(X_1, X_2, \ldots, \) be i.i.d. having uniform distribution on \((0, \theta)\). Let \(P_n\) and \(Q_n\) denote the joint distributions of \(X_1, X_2, \cdots, X_n\), when \(\theta = 1\), and when \(\theta = 1 - 1/n^p\),
respectively, where \( p \) is a fixed positive constant.

a. For which values of \( p \) are \( \{P_n\} \) and \( \{Q_n\} \) mutually contiguous? \[5\]
b. When \( \{P_n\} \) and \( \{Q_n\} \) mutually contiguous, identify the limit points of the distribution of \( dQ_n/dP_n \), under \( P_n \). \[5\]

4. Let \( X \) be a \( N(\theta, 1) \) r.v., with \( \theta \) in the set of integers \( \mathbb{N} = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \). Consider the problem of estimating of \( \theta \) with the loss function \( L(\theta, a) \) as the 0-1 loss.

a. Suppose an estimator \( T \) is equivariant, i.e., satisfies \( T(x + k) = T(x) + k \), for all \( x \in \mathbb{R} \) and all \( k \in \mathbb{N} \). Show that the risk function of \( T \) is constant in \( \theta \). \[5\]
b. Let \( S(X) = X - [X] \), where \([X]\) is the integer nearest to \( X \). Show that every equivariant estimate is of the form \( [X] - v(S(X)) \), for some measurable function \( v \) of \( S(X) \). \[5\]
c. Find the minimum risk equivariant estimate of \( \theta \). \[5\]
d. Which of the three estimates \( X, [X], \) and \( S \), are (i) sufficient for \( \theta \), (ii) complete sufficient for \( \theta \). \[5\]

5. Suppose that \( X_1, X_2, \cdots, X_n \) are independent r.v.’s, with \( X_i \) having \( N(\mu_i, 1) \) distribution. Consider the following hypotheses:

\[
H_0 : \mu_i = 0 \quad \text{for all } i = 1, \cdots, n, \quad \text{vs.} \quad H_1 : \mu_i = ab_i \text{ with } b_i \text{ i.i.d. Bernoulli(}p\text{)}, \text{ independent of all } X_j, 1 \leq j \leq n, \]

where \( a \in \mathbb{R} \) and \( 0 < p < 1 \) are known constants. Let \( \mu = (\mu_1, \cdots, \mu_n)^T \). Note that under the null hypothesis, \( \mu = 0_{n \times 1} \).

a. Find the marginal distributions of \( X_1, X_2, \cdots, X_n \) under \( H_1 \). \[5\]
b. Show that the likelihood ratio statistic for testing (1) is

\[
W = \prod_{i=1}^{n} \left\{ 1 + p \left( \exp\left( -\frac{a^2}{2} \right) \exp(aX_i) - 1 \right) \right\}.
\]

c. For a given \( c > 0 \), let \( \varphi_c = I\{W > c\} \) be a test function corresponding to the hypotheses (1). Namely, the test \( \varphi_c \) rejects \( H_0 \) if \( W > c \) and accept \( H_0 \), if \( W \leq c \). Define the risk of \( \varphi_c \) to be

\[
\text{Risk}_\pi(\varphi_c) = P_0(W > c) + E_\pi[P_\mu(W \leq c|\mu)].
\]

where \( P_0 \) is the probability measure under the null hypothesis and \( P_\mu \) is the probability measure under the alternative conditional on \( \mu \), and the expectation is taken with respect to the distribution \( \pi \) of \( \mu = (ab_1, ab_2, \ldots, ab_n) \). Show that the test \( \varphi_1 = I\{W > 1\} \) minimizes the risk \( \text{Risk}_\pi(\varphi_c) \) w.r.t. \( c > 0 \). \[5\]
d. Show that the risk of $\varphi_1$ has a lower bound
\[
\text{Risk}_\pi(\varphi_1) \geq 1 - \frac{1}{2} \sqrt{E_0(W^2)} - 1,
\]
where the expectation $E_0$ is taken with respect to $P_0$. \[5\]

6. Let $\mathcal{X} = \{1, 2, \cdots, k\}$, with $k < \infty$ and $\{P_\theta, \theta \in \mathbb{R}\}$ be a family of probabilities on $\mathcal{X}$ such that $P_\theta(x) > 0$, for all $\theta \in \mathbb{R}$ and $x \in \mathcal{X}$.

a. Suppose that $T_n$ is a sequence of estimates such that $\sup_n E_\theta T_n^2 < \infty$. Show that there is a subsequence $T_{n_i}$ and $T$ such that, for all $\theta$, $E_\theta T_{n_i} \rightarrow E_\theta T$. \[5\]

b. Suppose that there is no unbiased estimate of the function $g(\theta)$. Let $T_n$ is a sequence of estimates which are asymptotically unbiased, i.e. for all $\theta$, $E_\theta T_n \rightarrow g(\theta)$. Show that for all $\theta$, $\text{Var}_\theta(T_n) \rightarrow \infty$. \[5\]

7. Suppose $\theta = (\theta_1, \theta_2)$ is a bivariate parameter and the parameter space is $\Theta = \Theta_1 \times \Theta_2$. Suppose that $\{f(x|\theta) : \theta \in \Theta\}$ is family of densities such that $f(x|\theta) > 0$ for all $x, \theta$. Suppose $T_1$ is sufficient for $\theta_1$, whenever $\theta_2$ is fixed and known and $T_2$ is sufficient for $\theta_2$, whenever $\theta_1$ is fixed and known. Show that $(T_1, T_2)$ is sufficient for $(\theta_1, \theta_2)$. \[5\]

8. Let $X_1, X_2, \ldots, X_n$ be i.i.d observations from $U(\theta - 1, \theta + 1)$ with $\theta$ in the set of integers $N = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

a. Find a MLE $\hat{\theta}_n$ such that under 0-1 loss $\hat{\theta}_n$ has constant risk. \[5\]

b. Is it consistent? \[5\]

c. Show that $\hat{\theta}_n$ is minimax. \[5\]

d. Show that $\hat{\theta}_n$ is not admissible by constructing an estimate that has 0 risk at $\theta = 0$. \[5\]