1. Let $X_1, \ldots, X_n$ be iid from a Geometric distribution with a parameter $p \in (0, 1)$, i.e.

$$P(X_i = x) = p(1-p)^x, \ x = 0, 1, 2, \ldots$$

(a) Find a complete and sufficient statistics for $p$ based on $X_1, \ldots, X_n$.

(b) Find the moment generating function of $X_1$ and the first two moments of $X_1$.

(c) Let $g(p) = p^2$. Formulate the two methods of finding UMVUE and use each to find it for $g(p)$ when $n = 2$.

(d) Show that $\frac{n}{\sum_{i=1}^{n} X_i}$ is the MLE of $p$. Compute its asymptotical variance, $n > 1$.

(e) Find the MLE of $1/p$. Using delta-method find its asymptotical variance, $n > 1$.

(f) Let $g(p) = 1/p$ and let the loss function be $\mathbb{E}(p\delta(X_1, \ldots, X_n) - 1)^2$. Using a $\beta(a, b)$ prior find a Bayes estimator of $g(p)$, $n > 1$.

(g) Calculate the loss of the Bayes estimator in (f). Find conditions on $a, b$ that would make the estimator in (f) minimax. Are they satisfied? $n > 1$.

(h) Calculate the asymptotical efficiency of the MLE of $1/p$ and the Bayes estimator of $1/p$ with the prior $\beta(\sqrt{n}, n - \sqrt{n})$, $n > 1$.

(i) Under the loss $\mathbb{E}(p\delta(X_1, \ldots, X_n) - 1)^2$, show that the MLE of $1/p$ is inadmissible when $n$ is large enough.

(j) Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : 1/3 \leq p \leq 2/3$ vs $K : p < 1/3$ or $p > 2/3$ in the fullest possible detail, $n > 1$.

2. Let $X_1, \ldots, X_n$ be iid from the density $\lambda x^{-2}I(x > \lambda), \lambda > 0$.

(a) Construct the MRE estimator of $\lambda$ under the loss $\mathbb{E}(\delta(X_1, \ldots, X_n)/\lambda - 1)^2$, $n > 2$.

(b) Is the estimator in (a) consistent in probability for $\lambda$?

(c) Derive a UMP test of size $\alpha \in (0, 1)$ for testing $H : \lambda = 1$ vs $K : \lambda > 1$ in the fullest possible detail.

3. An experiment is designed to compare moisture content in three types of pigment pastes. For each type of pigment paste, 12 observations are obtained. Let $Y_{ij}$ be the moisture level of the $j$-th observation in the $i$-th type of pigment paste. The main interest is to compare the average moisture level among three types of pigment pastes. A fixed-effect model may be used to fit the data as follows:

$$Y_{ij} = \mu + \theta_i + e_{ij}, e_{ij} \sim iid N(0, \sigma^2)$$

for $i = 1, 2, 3$ and $j = 1, \cdots, 12$. To avoid identifiability issues, we set $\sum_{i=1}^{3} \theta_i = 0$ and remove $\theta_3$ from the above formulation, that is, the parameters in our model are $\mu, \theta_1$, and $\theta_2$.

(a) Let $\beta = (\mu, \theta_1, \theta_2)^T$. Specify the design matrix $X$ in the linear model of matrix form $Y = X\beta + e$ and compute the correlation between the least squares estimates (LSE) for $\theta_1$ and $\theta_2$.

(b) Construct $(1 - \alpha)100\%$ confidence intervals for $\theta_1 - \theta_2$ based on the LSEs $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\sigma}^2$ for $\theta_1, \theta_2$ and $\sigma^2$ respectively, and $t$ distribution.

(c) Show that a $(1 - \alpha)$ joint confidence region for $\theta_1$ and $\theta_2$ can be specified by

$$(\theta - \hat{\theta})^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (\theta - \hat{\theta}) \leq \frac{2\hat{\sigma}^2}{12} F_{2,33,\alpha},$$
where \( \mathbf{\theta} = (\theta_1, \theta_2)^T \), \( \hat{\mathbf{\theta}} \) is the LSE and \( F_{2,33;\alpha} \) is upper \( \alpha \) quantile of \( F_{2,33} \).

4. In the previous problem, the samples were considered to be independent and identically distributed. In fact, the data set was actually obtained through the following nested study design:

(a) sample 3 barrels of pigment paste;
(b) take 2 samples from the content of each barrel;
(c) each sample is mixed evenly and divided into 2 parts. Then the measurement of the moisture content is obtained through each part. Let \( Y_{ijkl} \) be the moisture content for the \( l \)-th part of the \( k \)-th sample from the \( j \)-th barrel of the \( i \)-th type.

Consider the following mixed effects model

\[
Y_{ijkl} = \mu + \theta_i + \beta_{ij} + \delta_{ijk} + e_{ijkl} \quad (1)
\]

for \( i, j = 1, \cdots, 3 \), \( k = 1, 2 \) and \( l = 1, 2 \), where \( \mu \) is the fixed effect part, \( \theta_i \) is the fixed type effect, \( \beta_{ij} \) is the random barrel effect and \( \delta_{ijk} \) is the random sample effect and \( e_{ijkl} \) is the measurement error. Assume that \( \beta_{ij} \) are iid \( N(0, \sigma^2_{\beta}) \), \( \delta_{ijk} \) are iid \( N(0, \sigma^2_{\delta}) \) and \( e_{ijkl} \) are iid \( N(0, \sigma^2) \). In addition, \( \beta_{ij}, \delta_{ijk} \) and \( e_{ijkl} \) are independent.

(a) Complete the ANOVA table for the above nested random effects model. Provide the formula for sum of squares in terms of \( Y_{ijkl} \).

<table>
<thead>
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<th>source</th>
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</tr>
<tr>
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<td>MSE</td>
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</tr>
</tbody>
</table>

Given the model (1), obtain the expectation of the mean squares.

\[
E(MSE) = , E(MSB) = , E(MSA) = .
\]

(b) Assume that \( MSE, MSA, MAB \) are known. Using the ANOVA table in part (a), obtain the unbiased moment estimates for \( \sigma^2, \sigma^2_{\beta} \) and \( \sigma^2_{\delta} \).

(c) Suppose we would like to obtain a new observation from the type 1 pigment paste through the same procedure described in the problem. Please give the best linear unbiased prediction for the moisture content of the new sample \( Y_1^* = \mu + \theta_1 + \beta_{11} + \delta_{111} \).

5. Let \( g \) be a function with \( \int g(x)dx = 1 \), and let \( f \) be a density with support \( S \). Define \( g^* = gI_S / \int_S g(x)dx \). Prove that \( \int |g^*(x) - f(x)|dx \leq \int |g(x) - f(x)|dx \).