STT 871-872 Preliminary Examination, January 2008
Saturday, January 12, 2008, 10am - 3pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used.

A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

Throughout, n is a known positive integer denoting the sample size and $\mathbb{R}$ is the real line.

1. Let $Y_1, Y_2, \cdots, Y_n$ be i.i.d. from the uniform distribution $U(0, \theta)$ with an unknown $\theta > 1$. Suppose that one observes only $X_1, X_2, \ldots, X_n$ with

$$X_i = \begin{cases} Y_i & Y_i \geq 1 \\ 1 & Y_i < 1 \end{cases}, 1 \leq i \leq n.$$

Derive a UMVUE of θ based on $X_1, X_2, \ldots, X_n$. **(12)**

2. For a $\theta \in \mathbb{R}$ let the random observation X have distribution P_θ. Assume $E_\theta X^2 < \infty$, for all $\theta \in \mathbb{R}$. Consider the problem of estimation $q(\theta) := E_\theta X$ with square error loss. Let a, b be some real numbers. Show that the estimator $aX + b$ is inadmissible if either $a < 0$ or $a > 2$. **(10)**

3. (a) Let a and b be two known positive numbers. Let $\rho(u) := -auI(u < 0) + buI(u \geq 0)$. Let Y be a r.v. with d.f. F having derivative f. Show that $E\rho(Y - t)$ is minimized by any value of t satisfying $F(t) = b/(a + b)$. **(6)**

(b) Recall that in a given estimation problem with the loss function L, an estimator $\delta(X)$ of a real parametric function $q(\theta)$ is said to be risk unbiased if

$$E_\theta L(q(\theta), T(X)) \leq E_\theta L(q(\vartheta), T(X)), \quad \forall \vartheta \neq \theta.$$

Show that an estimator $\delta(X)$ of $q(\theta)$ is risk unbiased with respect to the loss function $L(q(\theta), \delta(X)) = \rho(\delta(X) - q(\theta))$ if $F_\theta(q(\theta)) = a/(a + b)$, where $F_\theta(t) := P_\theta(\delta(X) \leq t)$. **(6)**

4. For any event A, let A^c denote its complement and $\pi = P(A)$. Prove the following statements.

(a) For any r.v. X, and for any event A with $0 < \pi < 1$, **(4)**

$$EX^2 \geq \frac{(EXI_A)^2}{\pi} + \frac{(EXI_{A^c})^2}{1-\pi}.$$

(b) Let Y be a r.v. with $EY = 0$, $EY^2 = 1$. Then, **(7)**

$$P(Y < y) \geq \frac{y^2}{y^2+1}, \qquad \text{for all } y \geq 0 \text{ such that } 0 < P(Y < y) < 1.$$

(c) Let Y be a r.v. $EY = 0$, $EY^2 = 1$, ξ_p be its pth percentile such that $0 < p < 1$, $\xi_p \geq 0$ and $0 < P(Y < \xi_p) < 1$, and let $q = 1 - p$. Then, **(5)**

$$|\xi_p| \leq \sqrt{p/q}.$$

5. let $U_1, U_2, \cdots,$ be i.i.d. r.v.'s uniformly distributed on $(0,1)$, and let

$$Y_n := (U_1 \times U_2 \times \cdots \times U_n)^{1/n}.$$

Show that $n^{1/2}(Y_n - 1/e)$ converges in distribution and identify this weak limit. **(12)**

6. Let

$$K(s) := \int_{-\infty}^{\infty} \frac{e^{-|x-s|}}{(1+e^{-|x-s|})^2} e^{-|x|} dx, \quad s \in \mathbb{R}.$$

(a) Is $\sup_{s \in \mathbb{R}} K(s) < \infty$. **(2)**

(b) Obtain the $\lim_{s \to 0+} [K(s) - K(0)]/s$. **(8)**

7. Let f be a Lebesgue density on $(0, \infty)$, $0 < \alpha < 1$ and let **(12)**

$$f_1(x) := e^{-x}, \qquad f_2(x) := xe^{-x}, \qquad x > 0.$$

Consider the problem of testing $H_0 : f = f_1$, vs. $H_1 : f = f_2$ based on one observation X. Show that the power of any test of size α of the hypothesis H_0 vs. H_1 is at most $(1 - \ell n(\alpha))\alpha$.

8. Suppose the data consists of n i.i.d. observations (Z_i, Y_i), $1 \leq i \leq n$ from the model $Y = \beta X + \varepsilon$, $X = Z + \eta$, for some $\beta \in \mathbb{R}$, where Z, ε, and η are mutually independent mean zero finite variance r.v.'s. Also assume that $Var(\varepsilon) = 1 = Var(\eta)$.

(a) Show that conditional mean and variance of Y, given Z are $E(Y|Z) = \beta Z$ and $Var(Y|Z) = \beta^2 + 1$. **(3)**

(b) Let $\hat{\beta} := \sum_{i=1}^n Z_i Y_i / \sum_{i=1}^n Z_i^2$. Show that $\hat{\beta}$ is unbiased for β. **(3)**

(c) Assume Z_i, ε_i, η_i, $i = 1, \cdots, n$ are independent i.i.d. $N(0,1)$ r.v.'s. Let $\tau := \sqrt{\sum_{i=1}^n Z_i^2}$. Obtain the exact distribution of $\tau(\hat{\beta} - \beta)$. **(4)**

(d) Let $0 < \alpha < 1$. Under the normality assumptions in part (c), derive $(1 - \alpha)$-confidence level confidence set for β in the fullest possible detail. **(6)**