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AN EXACT TEST AGAINST DECREASING MEAN TIME TO FAILURE CLASS ALTERNATIVES

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Abstract. The mean time to failure is a widely used concept to describe the reliability characteristic of a repairable system. In this paper, we develop a non-parametric method to test exponentiality against decreasing mean time to failure class. We derive the exact null distribution of the test statistic and then find the critical values for different sample sizes. Asymptotic properties of the proposed test statistic are studied. The test statistic is shown to be asymptotically normal and consistent against the alternatives. The Pitman's asymptotic efficacy shows that our test performs better than the other tests available in the literature. We also discuss how does the proposed method take the censoring information into consideration. Some numerical results are presented to demonstrate the performance of the testing method. Finally, we illustrate the test procedure using two real data sets.

Keywords: Exponential distribution; Mean time to failure; Pitman's asymptotic efficacy; Replacement model; U-statistics.

1. Introduction

Many results in life testing are based on the assumption that the lifetime of a product is described through exponential distribution. This assumption essentially implies that a used item is stochastically as good as a new one.

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That is, the unit in question does not age with time. Hence, there is no reason to replace a unit which is working. However, this is not always a realistic assumption and it is important to know which life distribution deserves membership benefits of an ageing class. In this scenario, tests for exponentiality designed to detect the appropriate alternative hypothesis have relevance in reliability theory. As constant failure rate characterizes the exponential distribution, the test is normally designed in such a way that our interest is to test the null hypothesis of constant failure rate against different ageing classes. The problem of testing exponentiality against a particular ageing class has been well studied in the literature; see Lai and Xie (2006) for an overview of such procedures.

The class which deals with failure with replacement distribution plays a key role in reliability theory in connection with maintenance strategies and renewal theory as reliability engineers can design appropriate maintenance policies for a particular task. Planned replacements are generally preferred to unscheduled maintenance to reduce in-service costs that are inherent in unexpected failures. In such cases, one strategy is to resort an age replacement policy in which an item is replaced either when it fails or at an age t whichever is earlier. In this context, Barlow and Proschan (1965) introduced the concept of mean time to failure (MTTF) to describe the reliability characteristics of a repairable system and studied the monotonic behaviour of MTTF. In engineering circles, MTTF is used extensively to understand expected product life cycles. Due to its potential applications in reliability engineering, the study of decreasing mean time to failure (DMTTF) classes of life distributions have been received much attention in recent time; see Knopik (2005, 2006), Li and Xu (2008), Asha and Nair (2010) and Kayid et al. (2013) and the references therein.

Barlow and Proschan (1965) showed that DMTTF class is situated between increasing failure rate (IFR) and new better than used in expectation

(NBUE) classes. In a further development Knopik (2006) proved that increasing failure rate average (IFRA) class belongs to DMTTF class, hence the DMTTF class is situated between IFRA and NBUE classes. Preservation of the DMTTF classes under various reliability operations have been studied by Knopik (2005, 2006). It includes the formation of parallel and series systems, weak convergence of distributions, mixture of distributions and convolution of distributions. In a different context, Li and Xu (2008) proposed a new class called new better than renewal used in reversed hazard order (NBURrh) which is equivalent to the DMTTF class, studied various properties and developed a test for exponentiality against DMTTF class. Asha and Nair (2010) considered the problem of stochastic orders in terms of MTTF to compare the life distributions and to understand the conditions on cumulative distribution function that render the monotone MTTFs. Kayid et al. (2013) presented new characterizations of the MTTF order in terms of the well-known hazard rate and reversed hazard rate orders. The problem of testing exponentiality against DMTTF class is also discussed in Kayid et al. (2013).

If the average waiting time between the consecutive failures is an important criterion in deciding whether to adopt an age replacement policy over the failure replacement policy for a given system, then a reasonable way to decide would be to test whether the life distribution of the given system is exponential. Rejection of the null hypothesis of exponentiality on the basis of the observed data would suggest as favoring the adoption of an age replacement plan. In this context, the problem of testing exponentiality against DMTTF class has great importance. As mentioned above, Li and Xu (2008) and Kayid et al. (2013) have initiated some work in this direction. Their test statistics are computationally complex. Motivated by these works, we develop a non-parametric test for testing exponentiality against DMTTF class. The test proposed here is simple and has very good efficiency.

The rest of the paper is organized as follows. In Section 2, based on U-statistics we propose a simple non-parametric test for testing exponentiality against DMTTF class. In Section 3, we derive the exact null distribution of the test statistic and then calculated the critical values for different sample sizes. The asymptotic normality and the consistency of the proposed test statistic are proved in Section 4. The comparison of our test with some other tests in terms of Pitman's asymptotic efficacy is also given. In Section 5, we report the result of the simulation study carried out to assess the performance of the proposed test. We illustrate our test procedure using two real data sets. Finally, in Section 6 we give conclusions of our study.

2. **Test statistic**

Let X be a non-negative random variable with an absolutely continuous distribution function $F(.)$. Suppose $\bar{F}(x) = P(X > x)$ denotes the survival function of X at x. Also let $\mu = E(X) = \int_0^\infty \bar{F}(t)dt < \infty$.

Consider an age replacement policy in which a unit is replaced by a new one (whose lifetime distribution is the same as F) at failure or t time units after installation, whichever occurs first and let $X_{[t]}$ be the associated random variable of interest. The survival function of $X_{[t]}$ is given by (Barlow and Proschan, 1965)

$$S_t(x) = \sum_{n=0}^{\infty} \bar{F}^n(t)\bar{F}(x - nt)I_{[nt,(n+1)t]}(x), \quad x > 0,$$

where I denotes the indicator function. The effectiveness of the age replacement policy is evaluated by studying the properties of $S_t(x)$. One characteristic that is extensively discussed in the context of age replacement policies is the mean of $X_{[t]}$.

The expected value of $X_{[t]}$, denoted by $M(t)$ is given by

$$M(t) = \frac{\int_0^t \bar{F}(x)dx}{F(t)}, \quad \text{for all} \quad t > 0.$$

The function $M(.)$ is known as MTTF. The above result enables us to calculate the MTTF using the distribution function of the random variable X. By observing the behaviour of MTTF one may realize the optimal time for which the replacement has to be done. Hence the purpose of this paper is to develop a criterion based on MTTF that helps the reliability engineers to devise a maintenance strategy that spells out schemes of replacement before failure occurs. In this direction, next we define decreasing mean time to failure class.

Definition 2.1. *Random variable X belongs to the DMTTF class if the function $M(.)$ is non-increasing for all $t > 0$.*

Next, we develop a simple method for testing exponentiality against DMTTF class. In fact this test procedure enables engineers to develop a better replace policy for efficient running of several systems under consideration.

We are interested to test the hypothesis

$$H_0 : F \quad \text{is exponential}$$

against

$$H_1 : F \quad \text{is DMTTF (and not exponential)},$$

on the basis of a random sample $X_1, X_2, ..., X_n$; from F.

First, we propose a measure of departure from the null hypothesis towards the alternative hypothesis. Make use of this measure, we develop a new non-parametric test. Our approach is based on U-statistics. The following lemma is useful in this direction and the proof is simple and hence omitted.

Lemma 2.1. *Let X be a non-negative random variable with an absolutely continuous distribution function $F(.)$, then X belongs to $DMTTF$ class if and only if*

$$\delta(x) = f(x)\int_0^x \bar{F}(t)dt - F(x)\bar{F}(x) \geq 0, \quad \text{for} \quad \text{all} \quad x > 0. \tag{1}$$

Note that H_0 holds if and only if equality attains in (1), whereas H_1 holds if and only if inequality in (1) is strict for some $x > 0$. Accordingly, the quantity $\delta(x)$ is a good measure of departure from the null hypothesis H_0 towards the alternative hypothesis H_1.

We define

$$\Delta(F) = \int_0^\infty \Big\{ f(x) \int_0^x \bar{F}(t)dt - F(x)\bar{F}(x) \Big\} dx.$$

From Lemma 2.1, it is clear that $\Delta(F)$ is zero under H_0 and is positive under H_1. The more $\Delta(F)$ differs from zero, the more there is evidence in favor of an F belongs to H_1.

To find the test statistic, we express $\Delta(F)$, in a more convenient way. The following observation is useful in this direction. The survival function of the random variable $X_{(1:n)} = min(X_1, X_2, ..., X_n)$ is given by

$$\bar{F}_{X_{(1:n)}}(x) = (\bar{F}(x))^n.$$

Hence

$$E(X_{(1:n)}) = \int_0^\infty (\bar{F}(x))^n dx.$$

In particular, when $n = 2$

$$E(X_{(1:2)}) = \int_0^\infty (\bar{F}(x))^2 dx. \tag{2}$$

We use Fubini's theorem to obtain

$$\begin{aligned}
\Delta(F) &= \int_0^\infty f(x) \int_0^x \bar{F}(t)dtdx - \int_0^\infty F(x)\bar{F}(x)dx \\
&= \int_0^\infty \bar{F}(t) \int_t^\infty f(x)dxdt - \int_0^\infty (1 - \bar{F}(x))\bar{F}(x)dx \\
&= \int_0^\infty \bar{F}^2(t)dt - \int_0^\infty \bar{F}(x)dx + \int_0^\infty \bar{F}^2(x)dx \\
&= 2\int_0^\infty \bar{F}^2(x)dx - \int_0^\infty \bar{F}(x)dx \\
&= 2E(X_{(1:2)}) - \mu.
\end{aligned}$$

To obtain the test statistic, note that $X_{(1:2)} = X_1 I(X_1 < X_2) + X_2 I(X_2 < X_1)$. Hence for $h^*(X_1, X_2) = 2X_1 I(X_1 < X_2) + 2X_2 I(X_2 < X_1) - X_1$, we have $E(h^*(X_1, X_2)) = \Delta(F)$. A U-statistic based on a symmetric kernel $h(.)$ is given by

$$\hat{\Delta}(F) = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} h(X_i, X_j), \tag{3}$$

where $h(X_1, X_2) = \frac{1}{2}(4X_1 I(X_1 < X_2) + 4X_2 I(X_2 < X_1) - X_1 - X_2)$. Clearly $\hat{\Delta}(F)$ is an unbiased estimator of $\Delta(F)$. After simplification, we can rewrite the above expression as

$$\begin{aligned} \hat{\Delta}(F) &= \frac{4}{n(n-1)} \sum_{i=1}^{n} (n-i) X_{(i)} - \frac{1}{n} \sum_{i=1}^{n} X_i \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} (3n - 4i + 1) X_{(i)}, \end{aligned} \tag{4}$$

where $X_{(i)}$, $i = 1, 2, ..., n$, is the i-th order statistics based on the random sample $X_1, X_2, ...X_n$; from F. To make the test scale invariant, we consider

$$\Delta^*(F) = \frac{\Delta(F)}{\mu}, \tag{5}$$

which can be estimated by

$$\hat{\Delta}^*(F) = \frac{1}{(n-1)} \frac{\sum_{i=1}^{n} (3n - 4i + 1) X_{(i)}}{\sum_{i=1}^{n} X_i}. \tag{6}$$

Note that, under the null hypothesis H_0, the test $\hat{\Delta}^*(F)$ is asymptotically distribution free which we will prove in Section 4. Hence the test procedure is to reject the null hypothesis H_0 in favour of the alternative hypothesis H_1 for large values of $\hat{\Delta}^*(F)$.

3. Exact null distribution

In this section, we derive the exact null distribution of the test statistic. Then we calculate the critical values for different sample size. We use Theorem 3.1 of Box (1954) to find the exact null distribution of the test statistic.

Theorem 3.1. *Let X be continuous non-negative random variable with $\bar{F}(x) = e^{-\frac{x}{2}}$. Let $X_1, X_2, ..., X_n$ be independent and identical samples from F. Then for fixed n*

$$P(\Delta^*(F) > x) = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \Big(\frac{d_{i,n} - x}{d_{i,n} - d_{j,n}}\Big) I(x, d_{i,n}),$$

provided $d_{i,n} \neq d_{j,n}$ for $i \neq j$, where

$$I(x,y) = \begin{cases} 1 & if \quad x \leq y \\ 0 & if \quad x > y \end{cases} \quad and \quad d_{i,n} = \frac{(n-2i+1)}{(n-1)}.$$

Proof: Rewrite the test statistic given in equation (4) as

$$\hat{\Delta}(F) = \sum_{i=1}^{n} 2X_{(i)} \Big[\frac{(n-i+1)^2}{n(n-1)} - \frac{(n-i)^2}{n(n-1)} - \frac{(n+1)}{2n(n-1)}\Big].$$

Or

$$\hat{\Delta}(F) = \frac{2n}{(n-1)} \sum_{i=1}^{n} X_{(i)} \Big[\frac{(n-i+1)^2}{n^2} - \frac{(n-i)^2}{n^2} - \frac{(n+1)}{2n^2}\Big]. \qquad (7)$$

Hence, in terms of the normalized spacings, $D_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, with $X_0 = 0$, we obtain

$$\hat{\Delta}^*(F) = \frac{\sum_{i=1}^{n} d_{i,n} D_i}{\sum_{i=1}^{n} D_i},$$

where $d_{i,n}$'s are given by

$$\begin{aligned} d_{i,n} &= \frac{2}{(n-1)} \big[(n-i+1) - \frac{(n+1)}{2}\big] \\ &= \frac{(n-2i+1)}{(n-1)}. \end{aligned}$$

Note that the exponential random variable with rate $\frac{1}{2}$ is distritbuted same as χ^2 random variable with 2 degrees of freedom. Hence the result follows from Theorem 3.1 of Box (1954) by taking $s = g_i = 1$.

The critical values of the test for different n under the null distribution is tabulated in Table 1.

TABLE 1. Critical Values

n	90% level	95% level	97.5% level	99% level
3	0.5264	0.5919	0.6382	0.6792
4	0.3855	0.4435	0.4896	0.5361
5	0.3134	0.3633	0.4050	0.4501
6	0.2679	0.3131	0.3507	0.3925
7	0.2362	0.2777	0.3126	0.3516
8	0.2127	0.2513	0.2840	0.3205
9	0.1945	0.2307	0.2615	0.2962
10	0.1798	0.2141	0.2432	0.2763
15	0.1351	0.1628	0.1865	0.2135
20	0.1115	0.1353	0.1558	0.1795
25	0.0966	0.1178	0.1361	0.1573
30	0.0861	0.1054	0.1221	0.1415
40	0.0722	0.0889	0.1033	0.1201
50	0.0631	0.0781	0.0909	0.1059
75	0.0498	0.0620	0.0725	0.0847
100	0.0423	0.0528	0.0619	0.0725

4. **Asymptotic properties**

In this section, we investigate the asymptotic properties of the proposed test statistic. The test statistic is asymptotically normal and consistent against the alternatives. The null variance of the test statistic is shown to be free from the parameter. Making use of asymptotic distribution we calculate the Pitman's asymptotic efficacy and then compare our test with the other tests available in the literature.

4.1. **Consistency and asymptotic normality.** As the proposed test is based on U-statistics, we use the asymptotic theory of U-statistics to discuss the limiting behaviour of $\hat{\Delta}(F)$. The consistency of the test statistic is due to Lehmann(1951) and we state it as next result.

Theorem 4.1. *The $\hat{\Delta}(F)$ is a consistent estimator of $\Delta(F)$.*

Corollary 4.1. *The $\hat{\Delta}^*(F)$ is a consistent estimator of $\Delta^*(F)$.*

Proof: Note that $\bar{X}$ is consistent estimator of μ. As we can write

$$\hat{\Delta}^*(F) = \frac{\hat{\Delta}(F)}{\Delta(F)} . \frac{\Delta(F)}{\mu} . \frac{\mu}{\bar{X}},$$

the proof is an immediate consequence of Theorem 4.1.

Next we find the asymptotic distribution of the test statistic.

Theorem 4.2. *The distribution of $\sqrt{n}(\hat{\Delta}(F)-\Delta(F))$, as $n\to\infty$, is Gaussian with mean zero and variance $4\sigma_1^2$, where σ_1^2 is the asymptotic variance of $\hat{\Delta}(F)$ and is given by*

$$\sigma_1^2=\frac{1}{4}Var\Big(4X\bar{F}(X)+4\int_0^X ydF(y)-X\Big). \tag{8}$$

Proof: Since the kernel has degree 2, using the central limit theorem for U-statistics (Hoeffding, 1948), $\sqrt{n}(\hat{\Delta}(F)-\Delta(F))$ has limiting distribution

$$N\big(0,4\sigma_1^2\big),\quad \text{as} \quad n\to\infty,$$

where the value of σ_1^2 is specified in the theorem. For finding σ_1^2, consider

$$\begin{aligned} E(h(x,X_2)) &= \frac{1}{2}E\Big(4xI(x<X_2)+4X_2I(X_2<x)-x-X_2\Big)\\ &= \frac{1}{2}\Big(4x\bar{F}(x)+4\int_0^x ydF(y)-x-\mu\Big) \end{aligned}$$

Hence

$$\sigma_1^2 = \frac{1}{4}V\Big(4X\bar{F}(X)-4\int_0^X ydF(y)-X\Big),$$

which completes the proof.

Corollary 4.2. *Let X be continuous non-negative random variable with $\bar{F}(x)=e^{-\frac{x}{\lambda}}$, then under H_0, as $n\to\infty$, $\sqrt{n}(\widehat{\Delta}(F)-\Delta(F))$ is Gaussian random variable with mean zero and variance $\sigma_0^2=\frac{\lambda^2}{3}$.*

Proof: Under H_0, we have

$$E(X-x|X>x)=\lambda. \tag{9}$$

That is

$$\frac{1}{\bar{F}(x)}\int_x^\infty ydF(y)-x=\lambda$$

or

$$\int_x^\infty ydF(y) = (x+\lambda)\bar{F}(x).$$

Since

$$\int_0^x ydF(y) + \int_x^\infty ydF(y) = \lambda,$$

we have

$$\int_0^x ydF(y) \quad = \quad \lambda - (x+\lambda)\bar{F}(x).$$

Hence using (8) we obtain

$$\begin{aligned}
\sigma_0^2 &= 4\sigma_1^2 \\
&= V\Big(4X\bar{F}(X) - 4\lambda - 4(X+\lambda)\bar{F}(X) - X\Big) \\
&= V\Big(- 4\lambda\bar{F}(X) - X\Big) = \frac{\lambda^2}{3}.
\end{aligned}$$

Using Slutsky's theorem, the following result can be easily obtained from Corollary 4.2.

Corollary 4.3. *Let X be continuous non-negative random variable with $\bar{F}(x) = e^{-\frac{x}{\lambda}}$, then under H_0, as $n \to \infty$, $\sqrt{n}(\widehat{\Delta}^*(F) - \Delta^*(F))$ is Gaussian random variable with mean zero and variance $\sigma_0^2 = \frac{1}{3}$.*

Hence in case of the asymptotic test, for large values of n, we reject the null hypothesis H_0 in favour of the alternative hypothesis H_1, if

$$\sqrt{3n}(\widehat{\Delta}^*(F)) > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 4.1. *One can also look at the problem of testing exponentiality against the dual concept increasing mean time to failure (IMTTF) class. We reject the null hypothesis H_0 in favour of IMTTF class, if*

$$\sqrt{3n}(\widehat{\Delta}^*(F)) < -Z_\alpha.$$

4.2. **Pitman's asymptotic efficacy.** The Pitman efficiency is the most frequently used index to make a quantitative comparison of two distinct asymptotic tests for a certain statistical hypothesis. The efficacy value of a test statistic can be interpreted as a power measure of the corresponding test. The Pitman's asymptotic efficacy (PAE) is defined as

$$PAE(\Delta^*(F)) = \frac{|\frac{d}{d\lambda}\Delta^*(F)|_{\lambda\to\lambda_0}}{\sigma_0}, \tag{10}$$

where λ_0 is the value of λ under H_0 and σ_0^2 is the asymptotic variance of $\Delta^*(F)$ under the null hypothesis. In our case, the PAE is given by

$$\begin{aligned} PAE(\Delta^*(F)) &= \frac{|\frac{d}{d\lambda}\Delta^*(F)|_{\lambda\to\lambda_0}}{\sigma_0} \\ &= \sqrt{3}(W'(\lambda_0) - W(\lambda_0)\mu_a'(\lambda_0)), \end{aligned}$$

where $W = 2E(X_{(1:2)})$ and μ_a is the mean of X under the alternative hypothesis and the prime denotes the differentiation with respect to λ. We calculate the PAE value for three commonly used alternatives which are the members of DMTTF class

(i) the Weibull family: $\bar{F}(x) = e^{-x^\lambda}$ for $\lambda > 1$, $x \geq 0$

(ii) the linear failure rate family: $\bar{F}(x) = e^{(-x-\frac{\lambda}{2}x^2)}$ for $\lambda > 0$, $x \geq 0$

(iii) the Makeham family: $\bar{F}(x) = e^{-x-\lambda(e^{-x}+x-1)}$ for $\lambda > 0$, $x \geq 0$.

For the first case we obtain the exponential distribution when $\lambda = 1$ and the other two cases the distributions become exponential when $\lambda = 0$.

By direct calculations, we observe that the PAE for Weibull distribution is equal to 1.2005; while for linear failure rate distribution and the Makeham distribution these values are, 0.8660 and 0.2828, respectively.

Next we compare the performance of the proposed test with other tests available in the literature by evaluating the PAE of the respective tests. Li and Xu (2008) proposed a new class called NBURrh and developed a test for exponentiality against NBURrh class. Asha and Nair (2010) observed that the NBURrh class is equivalent to DMTTF class. Hence we compare our test with tests proposed by Li and Xu (2008) and Kayid et al. (2013). The Table 2 gives the PAE values for different test procedures.

Table 2. Pitman's asymptotic efficacy (PAE)

Distribution	Proposed test	Li and Xu (2008)	Kayid et al. (2013)
Weibull	1.2005	1.1215	0.4822
Linear failure rate	0.8660	0.5032	0.4564
Makeham	0.2828	0.2414	2.084

From the Table 2, it is clear that our test is quite efficient for the Weibull and linear failure rate alternatives. In this case, PAE values of our test are greater than the test proposed by Li and Xu (2008) and Kayid et al. (2013). Our test performs better than that of Li and Xu (2008) for Makeham alternative also. Note that the test proposed by Kayid et al. (2013) has good efficacy for Makeham alternative even though their test shows poor performance against the other two given alternatives.

Remark 4.2. *The PAE value reported here against Li and Xu (2008) is the square root of the values given in their paper as Li and Xu (2008) considered the squared values of the PAE values defined in equation (10). We noted that the claim made by Kayid et al. (2013) about the linear failure rate distribution is not correct as they compared with the original values appeared in Li and Xu (2008). Accordingly, Li and Xu (2008) test is better than that of Kayid et al. (2013) against the linear failure rate alternative.*

Next we discuss the case with censored observation which are very common in lifetime data. Note that the methods proposed by Li and Xu (2008) and Kayid et al. (2013) can not handle censoring situation.

5. **The case of censored observations**

Next we discuss the case with censored observations which are very common in lifetime data. Suppose we have randomly right-censored observations such that the censoring times are independent of the lifetimes. Under this set up the observed data are n independent and identical copies of (X^*, δ), with

$X^* = min(X, C)$, where C is the censoring time and $\delta = I(X \le C)$. Now we need to address the testing problem mentioned in Section 2 based on n independent and identical observation $\{(X_i, \delta_i), 1 \le i \le n\}$. Observe that $\delta_i = 1$ means i^{th} object is not censored, whereas $\delta_i = 0$ means that i^{th} object is censored by C, on the right. Usually we need to redefine the measure $\Delta(F)$ to incorporates the censored observations. We refer to Koul and Susarla (1980) for more details. The U-statistics formulations helps us to solve the problem using the $\Delta(F)$ in an easy way. Using the right-censored version of a U-statistic introduced by Datta et al. (2010) an estimator $\Delta(F)$ with censored observation is given by

$$\hat{\Delta}_c(F) = \frac{1}{\binom{n}{2}} \sum_{i \in P_{n,2}} \frac{h(X^*_{i_1}, X_{i^*_2}) \prod_{l \in \underline{i}} \delta_l}{\prod_{l \in \underline{i}} K_c(X^*_l)}, \tag{11}$$

where $h(X^*_1, X^*_2) = \frac{1}{2}(4X^*_1 I(X^*_1 < X^*_2) + 4X^*_2 I(X^*_2 < X^*_1) - X^*_1 - X^*_2)$, provided $K_c(X^*_l) > 0$, with probability 1, where $P_{n,m}$ denotes the number of permutation of m object from n, the notation $\underline{i}$ is used to indicate that l is one of the integers $\{i_1, ..., i_m\}$, and K_c is the survival function of the censoring variable C. As K_c is un-known one can find a Kaplan-Meier estimator $\hat{K}_c$ of K_c, where the role of censored and failed observations are reversed. Similarly an estimator of μ is given by

$$\hat{X}_c = \frac{1}{n} \sum_{i=1}^{n} \frac{X^*_i \delta_i}{K_c(X^*_i)}. \tag{12}$$

Hence in right censoring situation, the test statistic is given by

$$\hat{\Delta}^*_c(F) = \frac{\hat{\Delta}_c(F)}{\hat{X}_c(F)}, \tag{13}$$

and the test procedure is to reject H_0 in favour of H_1 for large values of $\widehat{\Delta}^*_c(F)$.

Remark 5.1. *In the estimation problem discussed above we are forced to treat the largest observations as failures. If one of the largest observations is censored observation, estimating the tail of the survival distribution becomes* *an issue since the KaplanMeier does not drop to zero. The same issue is inherited here as we need to substitute the Kaplan-Meier estimator for the survival function of censoring variable C. Note that when the largest observations are indeed true failures, for such a sample, this assumption is irrelevant. Moreover, censoring occurs rarely in the age replace models.*

Next we obtain the limiting distribution of the test statistic. Let $N_i^c(t) = I(X_i^* \leq t, \delta_i = 0)$ be the counting process corresponds to the censoring variable for the i^{th} individual, $Y_i(u) = I(X_i^* \geq u)$. Also let λ_c be the hazard rate of C. The martingale associated with this counting process is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t Y_i(u)\lambda_c(u)du. \tag{14}$$

Let $G(x) = P(X_1 \leq x, \delta = 1)$ and

$$w(t) = \frac{1}{\bar{G}(t)} \int_X \frac{h_1(x, X_2^*)}{K_c(x)} I(x > t)dG(x) \tag{15}$$

where $h_1(x, X_2^*) = Eh(x, X_2^*)$.

Theorem 5.1. *If $Eh^2(X_1^*, X_2^*) < \infty$, $\int \frac{h_1(x,X_2^*)}{K_c^2(x)} dG(x) < \infty$ and $\int_0^\infty w^2(t)\lambda_c(t)dt < \infty$, then the distribution of $\sqrt{n}(\hat{\Delta}_c(F) - \Delta(F))$, as $n \to \infty$, is Gaussian with mean zero and variance $4\sigma_{1c}^2$, where σ_{1c}^2 is given by*

$$\sigma_{1c}^2 = Var\Big(\frac{h_1(X, X_2^*)\delta_1}{K_c(X^*)} + \int w(t)dM_1^c(t)\Big). \tag{16}$$

Corollary 5.1. *Under the assumptions of Theorem 5.1, if $E(X_1^2) < \infty$, the distribution of $\sqrt{n}(\widehat{\Delta}_c^*(F) - \Delta^*)$, as $n \to \infty$, is Gaussian with mean zero and variance σ_c^2, where*

$$\sigma_c^2 = \frac{\sigma_{1c}^2}{\mu^2}. \tag{17}$$

Proof. Note that $\hat{X}_c$ is a consistent estimator for μ (Zhao and Tsiatis, 2000). Hence the result follows from Theorem 5.1 by applying Slutsky's theorem.

6. Simulation and data analysis

Next, we report a simulation study done to evaluate the performance of our test against various alternatives. The simulation was done using R program. Finally, we illustrate our test procedure using two real data sets.

6.1. Monte carlo study.

First we find the empirical type 1 error of the proposed test. we simulate random sample from the exponential distribution with cumulative distribution function $F(x) = 1 - \exp(-x), x \geq 0$. Since the test is scale invariant, we can take the scale parameter to be unity, while performing the simulations. A random sample of different sample size is drawn from the exponential distribution specified above and the value of the test statistic is calculated. We check whether this particular realization of the test statistic accepts or rejects the null hypothesis of exponentiality. Then we repeat the whole procedure ten thousand times and observe the proportion of times the proposed test statistic takes the correct decision of rejecting the null hypothesis of exponentiality and this gives the empirical type I error. The procedure has been repeated for different values of n and is reported in Table 3. The Table 3 shows that the empirical type 1 error is a very good estimator of the size of the test.

For finding empirical power against different alternatives, we simulate observations from the Weibull, linear failure rate and Makeham distributions with various values of $\lambda > 1$ ($\lambda < 1$ corresponds to the distribution having increasing MTTF) where the distribution functions were given in the Section 4. As pointed out earlier these are typical members of the DMTTF class. The empirical powers for the above mentioned alternatives are given in Tables 4, 5 and 6. From these tables we can see that empirical powers of the test approaches to one when the θ values are going away from the null hypothesis value as well as when n takes large values.

Next we illustrate our test procedure using two real data sets.

TABLE 3. Empirical type 1 error of the test

n	5% level	1% level
10	0.0605	0.0195
20	0.0537	0.0162
30	0.0532	0.0134
40	0.0503	0.0142
50	0.0514	0.0116
60	0.0508	0.0126
70	0.0504	0.0103
80	0.0495	0.0098
90	0.0484	0.0097
100	0.0482	0.0097

TABLE 4. Empirical power: Weibull distribution

λ	(5% level)	(1%level)
1.2	0.6733	0.3783
1.4	0.9918	0.9409
1.6	1	0.999
1.8	1	1
2	1	1

TABLE 5. Empirical power: Linear failure rate distribution

λ	(5% level)	(1%level)
1.2	0.4379	0.2830
1.4	0.8618	0.7409
1.6	0.9946	0.9658
1.8	1	0.9997
2	1	1

TABLE 6. Empirical Power : Makeham distribution

λ	(5% level)	(1%level)
1.2	0.386	0.1793
1.4	0.8029	0.5895
1.6	0.9715	0.9035
1.8	0.9979	0.9897
2	0.9998	0.9992

6.2. **Data analysis.** To demonstrate our testing method, we first apply it to the data set consists of $n = 27$ observations of the intervals between successive failures (in hours) of the air-conditioning systems of 7913 jet air planes of a fleet of Boeing 720 jet air planes as reported in Proschan (1963). The data is given in Table 7. The value of the test statistic corresponds to

Table 7. The time of successive failures of the air-conditioning systems of 7913 jet air planes

97	51	11	4	141	18	142
68	77	80	1	16	106	206
82	54	31	216	46	111	39
63	18	191	18	163	24	

Table 8. Survival days of chronic granulocytic leukemia

7	47	58	74	177	232	273	285
317	429	440	445	455	468	495	497
532	571	579	581	650	702	715	779
881	930	900	968	1,077	1,109	1,314	1,334
1,367	1,534	1,712	1,784	1,877	1,886	2,045	2,056
2,260	2,429	2,509					

this particular data set is 0.06668645. The critical values of the exact test statistic corresponds to $n = 27$ is 0.1863, hence we can not reject the null hypothesis of exponentially in favour of DMTTF class at 0.05 level. This suggest that there is no advantage in adopting an age replacement policy in this example.

The second data set is taken from Bryson and Siddiqui (1969). It represents the survival times, in days from diagnosis, of 43 patients suffering from chronic granulocytic leukemia and given in Table 8. The calculated value of the test statistic is 0.1479901. The critical values of the exact test statistic correspond to $n = 43$ is 0.1466. Hence the test suggests to reject the null hypothesis gainst DMTTF alternatives at 0.05 level.

7. Conclusions

In order that a device or system is able to perform its intended functions without disruption due to failure, several types of maintenance strategies that spell out schemes of replacement before failure occurs, have been devised in reliability engineering. In this context, the MTTF gave an idea about the optimal time for the replacement to be done. Testing exponentiality against DMTTF class enables reliability engineers to decide whether to adopt a planned replacement policy over unscheduled one. To address this issue, a new testing procedures for exponentiality against DMTTF class was introduced and studied. It is simple to devise, calculate and have exceptionally high efficiency for some of the well-known alternatives relative to other more complicated tests.

We obtained the exact null distribution of the test statistic. We studied the asymptotic properties of the test statistic. Using asymptotic theory of U-statistics, we showed that the test statistic was unbiased, consistent and has limiting normal distribution. A comparison between the proposed test and two other related ones in the literature was conducted through evaluating the Pitman's asymptotic efficacy. We illustrated our test procedure using two real data sets. We also discussed how does the proposed method deals with the right censored observations which arise commonly in lifetime study.

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