A Goodness-of-fit test for marginal distribution of linear random fields with long memory[1]

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Abstract

This paper addresses the problem of fitting a known distribution function to the marginal distribution of a stationary long memory moving average random field observed on increasing ν-dimensional "cubic" domains when its mean μ and scale σ are known or unknown. Using two suitable estimators of μ and a classical estimate of σ, a modification of the Kolmogorov-Smirnov statistic is defined based on the residual empirical process and having a Cauchy-type limit distribution, independent of μ, σ and the long memory parameter d. Based on this result, a simple goodness-of-fit test for the marginal distribution is constructed, which does not require the estimation of d or any other underlying nuisance parameters. The result is new even for the case of time series, i.e., when $\nu = 1$. Findings of a simulation study investigating the finite sample behavior of size and power of the proposed test is also included in this paper.

1 Introduction

The last two decades have seen an increasing research activity in the areas of spatial statistics and random fields, see, e.g., the monographs of Ripley (1988), Ivanov and Leonenko (1989), Cressie (1993), Guyon (1995), and Stein (1999). While many of these works deal with rather simple autoregressive and point process models with short-range dependence, a number of empirical studies ranging from astrophysics to agriculture and atmospheric sciences indicate that spatial data may exhibit nonsummable correlations and strong dependence, see, e.g., Kashyap and Lapsa (1988), Gneiting (2000), Percival et al. (2008) and Carlos-Davila *et al.* (1985), among others.

Many of the applied works assume Gaussian random field model, which raises the question of goodness-of-fit testing. In the case of i.i.d. observations, the goodness-of-fit testing problem has been well studied, see, e.g., Durbin (1973, 1975), and D'Agostino and Stephens (1986), among others. Koul and Surgailis (2010) and Koul, Mimoto, and Surgailis (2013)

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discussed the problem of fitting a known distribution function (d.f.) to the marginal d.f. of a stationary long memory moving average time series when its mean μ is known and unknown. In particular, Koul et al. (2013) provided a class of weighted least squares estimators of μ for which the weak limit of the first order difference between the residual empirical and null distribution functions is a non-degenerate Gaussian distribution, yielding a simple Kolmogorov-type test for fitting a known distribution up to an unknown mean. In the same context, the latter paper also obtained the asymptotic chi-square distribution of test statistics based on integrated square difference between kernel type estimators of the marginal density of long memory moving averages with discrete time $t \in \mathbb{Z} := \{0, \pm 1, \cdots\}$ and the expected value of the density estimator based on errors.

The implementation of these tests involves the estimation of the long memory parameter d, and some asymptotic variances which in turn depend on d and some other underlying parameters. The task of estimating the long memory parameter d for random field data is prohibitive, if not impossible. To circumvent this, the aim of this paper is to propose a test for fitting a d.f. to the marginal distribution of a long memory random field without needing to estimate d. We note that the present paper as well as the above-mentioned goodness-of-fit studies under long memory deal exclusively with the case $d > 0$, since for $d \le 0$ the asymptotic behavior of the empirical process is very different (see Koul et al. (2013), p.207).

To be a bit more precise, consider a moving average random field

$$X_t = \sum_{s \in \mathbb{Z}^\nu} b_{t-s} \zeta_s, \quad t \in \mathbb{Z}^\nu, \tag{1.1}$$

indexed by points of ν-dimensional lattice $\mathbb{Z}^\nu := \{0, \pm 1, \pm 2, \ldots\}^\nu, \nu = 1, 2, \cdots$, where $\{\zeta_s, s \in \mathbb{Z}^\nu\}$ are i.i.d. r.v.'s with zero mean and unit variance. The moving-average coefficients $\{b_t, t \in \mathbb{Z}^\nu\}$ satisfy

$$b_t = (B_0(t/|t|) + o(1))|t|^{-(\nu - d)}, \quad t \in \mathbb{Z}^\nu \setminus \{0\}, \qquad \text{for some} \quad 0 < d < \nu/2, \tag{1.2}$$

where $B_0(x), x \in S_{\nu-1} := \{y \in \mathbb{R}^\nu : |y| = 1\}$ is a bounded piece-wise continuous function on the unit sphere $S_{\nu-1}$. Throughout the paper, for any $x \in \mathbb{R}^\nu$, $|x|$ denotes its Euclidean norm. The series in (1.1) converges in mean square and defines a stationary random field $\{X_t\}$ with $EX_0 = 0$ and

$$\mathrm{Cov}(X_0, X_t) \sim R_0(t/|t|)|t|^{-(\nu - 2d)}, \quad \text{as} \quad |t| \to \infty, \tag{1.3}$$

where R_0 is an even, strictly positive and continuous function on $S_{\nu-1}$, see, e.g., Surgailis (1982). Here, and in the sequel, for any two positive functions $h_1(t), h_2(t)$, $h_1(t) \sim h_2(t)$, $|t| \to \infty$ means $\lim_{|t| \to \infty} h_1(t)/h_2(t) = 1$.

Since $\sum_{t \in \mathbb{Z}^\nu \setminus \{0\}} |t|^{-(\nu - 2d)} = \infty$ for $0 < d < \nu/2$, the random field $\{X_t\}$ has long memory in the sense the sum of its autocovariances diverges.

Let F denote the marginal d.f. of X_0 having density f, and let F_0 be a known d.f. with density f_0. The problem of interest is to test the hypothesis

$$H_0 : F = F_0 \qquad \text{vs.} \qquad H_1 : F \neq F_0.$$

A motivation for this problem is that often in practice one uses inference procedures that are valid under the assumption of $\{X_t\}$ being a Gaussian field. The rejection of this hypothesis when F_0 is standard Gaussian d.f. would cast some doubt about the validity of such inference procedures.

Throughout the paper, Z denotes a $\mathcal{N}(0,1)$ r.v., $\to_D$ stands for the convergence in distribution, and $\to_p$ stands for the convergence in probability.

Now, define $A_n := [1,n]^\nu \cap \mathbb{Z}^\nu$, and

$$\widehat{F}_n(x) := n^{-\nu} \sum_{t \in A_n} I(X_t \leq x), \;\; x \in \mathbb{R}, \quad \theta := (v(1), d)', \quad \|f_0\|_\infty := \sup_{x \in \mathbb{R}} f_0(x),$$
$$\bar{X}_n := n^{-\nu} \sum_{t \in A_n} X_t, \quad v(1) := \int_{[0,1]^\nu} \int_{[0,1]^\nu} R_0\Big(\frac{u-v}{|u-v|}\Big) \frac{du dv}{|u-v|^{\nu-2d}}.$$

From Surgailis (1982) we obtain

$$\text{Var}(\bar{X}_n) \sim v(1) n^{2d-\nu} \quad \text{and} \quad \frac{n^{\nu/2-d} \bar{X}_n}{\sqrt{v(1)}} \to_D Z. \tag{1.4}$$

A test of H_0 is the Kolmogorov-Smirnov test based on $D_n := \sup_{x \in \mathbb{R}} |\widehat{F}_n(x) - F_0(x)|$. The limit distribution of the empirical process $\widehat{F}_n$ for long memory moving-average observations $\{X_t\}$ with one dimensional time $t \in \mathbb{Z}$ was studied in Giraitis, Koul, and Surgailis (1996), Ho and Hsing (1996), and Koul and Surgailis (2002). A similar problem for moving-average random fields in $\mathbb{Z}^\nu, \nu > 1$, was investigated in Doukhan, Lang, and Surgailis (2002) (DLS), where it was shown, under some conditions, that $n^{\nu/2-d} D_n / \big(\sqrt{v(1)} \|f_0\|_\infty\big) \to_D |Z|$.

Let $\hat{v}_n(1), \hat{d}_n$ be consistent and $\log(n)$ consistent estimators of $v(1)$ and d, under H_0, respectively. Let z_α be $100(1-\alpha)$th percentile of $\mathcal{N}(0,1)$ distribution. From the above result, we readily obtain that the test that rejects H_0 whenever

$$\frac{n^{\nu/2-\hat{d}} D_n}{\sqrt{\hat{v}_n(1)} \|f_0\|_\infty} \geq z_{\alpha/2} \tag{1.5}$$

is of the asymptotic size α.

Next, consider the problem of fitting F_0 to F up to an unknown location+scale parameters. In other words now F denotes the marginal d.f. of Y_t, which is obeying the model

$$Y_t = \mu + \sigma X_t, \quad t \in \mathbb{Z}^\nu, \quad \text{for some } \mu \in \mathbb{R}, \; \sigma > 0, \tag{1.6}$$

with $\mathrm{Var}(X_0) = \sum_{t\in\mathbb{Z}^\nu} b_t^2 = 1$, and the problem of interest is to test

$$\mathcal{H}_0 : F(x) = F_0(\frac{x-\mu}{\sigma}), \quad \forall\, x \in \mathbb{R}, \text{ for some } \mu \in \mathbb{R}, \sigma > 0, \text{ vs.}$$
$$\mathcal{H}_1 : \mathcal{H}_0 \quad \text{is not true.}$$

Let $\widehat{\sigma}_n$ be a consistent estimator of σ under $\mathcal{H}_0$, $\bar{F}_n$ denote the empirical d.f. based on $(Y_t - \bar{Y}_n)/\widehat{\sigma}_n$, $t \in A_n$, and let $\bar{D}_n := \sup_x |\bar{F}_n(x) - F_0(x)|$. It follows from DSL, similarly as in the case $\nu = 1$ studied by Koul and Surgailis (2010) and Koul et al. (2013), that under $\mathcal{H}_0$, $n^{\nu/2-d}\bar{D}_n \to_p 0$, and hence $n^{\nu/2-d}\bar{D}_n$ cannot be used asymptotically to test for $\mathcal{H}_0$.

To circumvent this difficulty, in Section 2 we provide a class of weighted least squares estimators $\widetilde{Y}_{n\varphi}$ of μ for which the normalized weak limit of the spatial empirical process $\widetilde{F}_{n\varphi}$ based on residuals $(Y_t - \widetilde{Y}_{n\varphi})/\widehat{\sigma}_n, t \in A_n$, has a non-degenerate Gaussian distribution under $\mathcal{H}_0$ (Theorem 2.1), implying

$$\frac{n^{\nu/2-d}\widetilde{\mathcal{D}}_{n\varphi}}{\sqrt{v(\varphi)}\|f_0\|_\infty} \to_D |Z|, \tag{1.7}$$

where $\widetilde{\mathcal{D}}_{n\varphi} := \sup_{x\in\mathbb{R}} |\widetilde{D}_{n\varphi}(x)|$, $\widetilde{D}_{n\varphi}(x) := \widetilde{F}_{n\varphi}(x) - F_0(x)$. Here φ is a real valued function on $[0,1]^\nu$, and

$$v(\varphi) \;:=\; \int_{[0,1]^\nu}\int_{[0,1]^\nu} \varphi(u)\varphi(v) R_0\Big(\frac{u-v}{|u-v|}\Big)\frac{du\,dv}{|u-v|^{\nu-2d}}. \tag{1.8}$$

It follows from (1.7) that the test that rejects $\mathcal{H}_0$ whenever

$$\big(\sqrt{\widetilde{v}(\varphi)}\|f_0\|_\infty\big)^{-1} n^{\nu/2-\widetilde{d}}\widetilde{D}_{n\varphi} > z_{\alpha/2} \tag{1.9}$$

is of the asymptotic level α, where $\widetilde{v}(\varphi)$, $\widetilde{d}$ are, respectively, consistent and $\log(n)$-consistent estimators of $v(\varphi)$, d, under $\mathcal{H}_0$. This test is an extension of the test proposed in Koul et al. (2013) from the time series case to the random fields.

As indicated earlier, the implementation of the tests (1.5) and (1.9) requires $\log(n)$-consistent estimators of d and consistent estimators of $v(1)$ and $v(\varphi)$. Several approaches for estimating the underlying parameters in long memory random fields have been suggested in the literature. Frias et al. (2008) suggested an averaged periodogram estimator of d for long memory time series in the two-dimensional spatial case. Wang (2009) investigated the Geweke and Porter-Hudak estimator of d for long memory random fields. Under some general conditions these estimators are all $\log(n)$-consistent for d. One can use the HAC estimator of the long run variance $v(1)$ (Abadir et al. (2009)). An estimator of $v(\varphi)$ is presented in Sec.4 below. However, because of the slow rate of convergence of the estimators of d, the finite sample properties of the significance level of the above tests based on these estimators is not desirable.

The other papers that discuss the estimation of the long memory intensity d for some fully observable random field models include Boissy et al. (2005), Leonenko and Sakhno (2006), Frias et al. (2008), and Guo et al. (2009). However, most of these results do not apply to the model in (1.1) - (1.2). Examples of the model (1.1) - (1.2) to which our results apply are given in section 5 below.

Because of the difficulty with the estimation of d for $\nu > 1$, it is highly desirable to avoid the need to estimate d and the variances $v(1)$ and $v(\varphi)$. With this goal in mind, we propose the following modification of the test (1.9). Let φ_1, φ_2 be two functions like φ and define

$$\Delta_n(\varphi_1, \varphi_2) \;:=\; \frac{\widehat{\sigma}_n \widetilde{D}_{n\varphi_1}}{|\widetilde{Y}_{n,\varphi_2} - \bar{Y}_n|}, \tag{1.10}$$

where $\widetilde{D}_{n\varphi_1}$ is similar as in (1.9) with $\varphi = \varphi_1$, and $\widetilde{Y}_{n,\varphi_2} - \bar{Y}_n$ is the difference of the two estimators of μ, see Sec.2. Note that $\Delta_n(\varphi_1, \varphi_2)$ does not involve any normalization depending on d or $v(\varphi_i), i = 1, 2$. Under some conditions on weight functions φ_1, φ_2, and under $\mathcal{H}_0$, we prove that

$$\Delta_n(\varphi_1, \varphi_2) \;\to_D\; \|f_0\|_\infty \sqrt{\frac{v(\varphi_1)}{v(\varphi_2)}} |\mathcal{C}(\rho(\varphi_1, \varphi_2))|, \tag{1.11}$$

where $\rho(\varphi_1, \varphi_2)$ is as in (2.5) below, $v(\varphi_i)$ is as in (1.8) with $\varphi = \varphi_i, i = 1, 2$, and where

$$\mathcal{C}(\rho) := Z_1/Z_2, \tag{1.12}$$

with (Z_1, Z_2) being a normal random vector with zero means, unit variances and a correlation coefficient ρ. The distribution of the r.v. $\mathcal{C}(\rho)$ is the Cauchy with probability density

$$p(x) := \frac{\sqrt{1-\rho^2}}{\pi((x-\rho)^2 + 1 - \rho^2)}, \quad x \in \mathbb{R}, \quad |\rho| < 1, \tag{1.13}$$

see, e.g., Johnson, Kotz, and Balakrishnan (1994). A consistent estimator $\widehat{\rho}_n(\varphi_1, \varphi_2)$ of $\rho(\varphi_1, \varphi_2)$, which avoids the estimation of d and $v(\varphi_1)$, $v(\varphi_2)$, is discussed in Sec.4.

Thus if $v(\varphi_1) = v(\varphi_2)$, then the corresponding GOF test of the asymptotic level α based on (1.11) rejects $\mathcal{H}_0$ whenever

$$\frac{\Delta_n(\varphi_1, \varphi_2)}{\|f_0\|_\infty} > c_\alpha(\widehat{\rho}_n(\varphi_1, \varphi_2)), \tag{1.14}$$

where $c_\alpha(\rho)$ is the upper α-quantile of the r.v. $|\mathcal{C}(\rho)|$, or the unique solution of

$$\alpha \;=\; 1 - \frac{1}{\pi}\Big[\arctan\Big(\frac{c_\alpha(\rho) - \rho}{\sqrt{1-\rho^2}}\Big) + \arctan\Big(\frac{c_\alpha(\rho) + \rho}{\sqrt{1-\rho^2}}\Big)\Big]. \tag{1.15}$$

In general, where $v(\varphi_1) \neq v(\varphi_2)$, the test that rejects $\mathcal{H}_0$ whenever

$$\widetilde{\Delta}_n := \|f_0\|_\infty^{-1} \Delta_n(\varphi_1, \varphi_2) \sqrt{\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1)} > c_\alpha(\widehat{\rho}_n(\varphi_1, \varphi_2)) \tag{1.16}$$

is of the asymptotic size α, where $\widetilde{v}_n(\varphi_i), i = 1, 2$ are defined in (3.4) below, which does not require the estimation of d.

A major advantage of the above class of tests, as φ varies, is that they and their null limit distributions *do not depend on the memory parameter* d of the underlying random field. This observation, together with the fact that the limit in (1.11) has a tractable classical distribution (1.13), represents a major novelty of the present paper. These tests are new even in the case $\nu = 1$. We also note that our results essentially use only the Uniform Reduction Principle for the residual empirical process (see Theorem 2.1 and Remark 2.1 below) and therefore are expected to hold under broader conditions on the random field than in (1.1)–(1.2), including in particularly anisotropic fractionally integrated fields discussed in Sec. 5 (5.13) and in Boissy et al. (2005) or Guo et al. (2009).

The rest of the paper is organized as follows. The main results about the spatial residual empirical process along with the limiting null distributions of the proposed test statistics are presented in section 2. Section 3 describes some consistent estimators of the parameter $\rho(\varphi_1, \varphi_2)$ based on a certain smoothing parameter q. Section 4 discusses the consistency of the proposed tests (1.14) and (1.16) against a class of fixed alternatives and asymptotic power of these tests against a sequence of local alternatives. Some examples of random fields to which the results of this paper are applicable are discussed in section 5. A simulation study was conducted to assess the effect of various underlying entities like q and φ on the finite sample level and power of these tests. These findings are presented in section 6.

2 Asymptotics of the spatial residual empirical process and test statistics

This section discusses the asymptotic behavior of a class of residual empirical processes and the above proposed test statistics. To define the residuals, we need to first introduce a class of estimators of μ. Recall the model (1.6) that includes the assumption $EX_0^2 = 1$. Let φ be a piece-wise continuously differentiable function on $[0, 1]^\nu$ and let

$$\varphi_{nt} := n^\nu \int_{\prod_{j=1}^\nu \left((t_j-1)/n, t_j/n\right]} \varphi(u) du, \qquad t = (t_1, \cdots, t_\nu) \in A_n$$

be its average value on cube $\prod_{j=1}^\nu \left((t_j - 1)/n, t_j/n\right] \subset [0, 1]^\nu$. Thus, $\bar{\varphi}_n := n^{-\nu} \sum_{t \in A_n} \varphi_{nt} = \int_{[0,1]^\nu} \varphi(u) du =: \bar{\varphi}$, $\forall\, n \geq 1$. Next, define

$$\widetilde{Y}_{n\varphi} \;:=\; n^{-\nu} \sum_{t \in A_n} Y_t[1 + \varphi_{nt}] = \mu\big(1 + \bar{\varphi}\big) + \sigma(\bar{X}_n + \bar{W}_{n\varphi}), \tag{2.1}$$

where $\{Y_t\}$ is as in (1.6), $\{X_t\}$ is a zero-mean moving-average random field in (1.1), and

$$\bar{W}_{n\varphi} \ := \ n^{-\nu} \sum_{t \in A_n} X_t \varphi_{nt}. \tag{2.2}$$

A slightly different version of $\widetilde{Y}_{n\varphi}$ was used in Koul et al. (2013), where φ_{nt} is replaced by $\varphi(t/n)$. For that version, Theorem 2.1 requires the additional condition that the bias $\mu\bar{\varphi}_n = \mu n^{-\nu} \sum_{t \in A_n} \varphi(\frac{t}{n}) = o(n^{\nu/2-d})$. Therefore the definition in (2.1) is preferable.

Note $\widetilde{Y}_{n0} = \bar{Y}_n$, $\bar{W}_{n1} = \bar{X}_n$, $E\bar{W}_{n\varphi} = 0$, and $\bar{\varphi} = 0$ implies that $\widetilde{Y}_{n\varphi}$ is a consistent and unbiased estimator of μ, i.e., $\widetilde{Y}_{n\varphi} \to_p \mu$, $E\widetilde{Y}_{n\varphi} = \mu$. Also note that when $\varphi(u) \geq -1$, $u \in [0,1]^\nu$ and $\bar{\varphi} = 0$, $\widetilde{Y}_{n\varphi}$ is a weighted least squares estimator since it minimizes the weighted sum of squares: $\widetilde{Y}_{n\varphi} = \operatorname{argmin}_{\mu \in \mathbb{R}} \sum_{t \in A_n} (Y_t - \mu)^2 [1 + \varphi_{nt}]$.

The following lemma discusses the asymptotic normality of the joint distributions of $(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2})$.

Lemma 2.1 *Let $\varphi_i(x), x \in [0,1]^\nu, i = 1,2$ be two piecewise continuously differentiable functions and suppose $\{X_t\}$ satisfy (1.1) and (1.2). Then*

$$n^{\nu/2-d}(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2}) \to_D (W_1, W_2), \tag{2.3}$$

where (W_1, W_2) is a zero mean normal vector with covariance $EW_1W_2 = v(\varphi_1, \varphi_2)$, with

$$v(\varphi_1, \varphi_2) := \int_{[0,1]^\nu} \int_{[0,1]^\nu} \varphi_1(u)\varphi_2(v) R_0\Big(\frac{u-v}{|u-v|}\Big) \frac{du\, dv}{|u-v|^{\nu-2d}}, \tag{2.4}$$

and variances $EW_i^2 = v(\varphi_i, \varphi_i) \equiv v(\varphi_i), i = 1,2$ as in (1.8). In addition, if $v(\varphi_i) > 0, i = 1,2$, the limit vector in (2.3) can be represented as $(W_1, W_2) = (\sqrt{v(\varphi_1)}\, Z_1, \sqrt{v(\varphi_2)}\, Z_2)$, where (Z_1, Z_2) is a zero mean normal vector with unit variances $EZ_1^2 = EZ_2^2 = 1$ and the correlation coefficient $EZ_1Z_2 =: \rho(\varphi_1, \varphi_2)$ equal to

$$\rho(\varphi_1, \varphi_2) \ = \ \frac{v(\varphi_1, \varphi_2)}{\sqrt{v(\varphi_1)v(\varphi_2)}}. \tag{2.5}$$

Proof of Lemma 2.1. By (1.3) and the dominated convergence theorem,

$$\operatorname{Var}(\bar{W}_{n\varphi}) = \frac{1}{n^{2\nu}} \sum_{t,s \in A_n} \varphi_{nt}\varphi_{ns} E X_t X_s \sim v(\varphi) n^{2d-\nu}, \quad n \to \infty. \tag{2.6}$$

Similarly, (1.3), (2.6) and the dominated convergence theorem yield

$$\operatorname{Cov}(n^{\nu/2-d}\bar{W}_{n\varphi_1}, n^{\nu/2-d}\bar{W}_{n\varphi_2}) \ \to \ v(\varphi_1, \varphi_2).$$

The asymptotic normality of $(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2})$ can be established following the scheme of discrete stochastic integrals, see e.g. Surgailis (1982), Koul and Surgailis (2002, Lemma 2.4 (iii)), Giraitis et al. (2012, Prop.14.3.1). Details are omitted for the sake of brevity.

Next, we discuss the limit of the residual empirical process. Let

$$\widehat{\sigma}_n^2 := n^{-\nu} \sum_{t \in A_n} (Y_t - \bar{Y}_n)^2 \tag{2.7}$$

be an estimator of σ^2 in (1.6) and

$$\begin{aligned} \widetilde{F}_{n\varphi}(x) &:= n^{-\nu} \sum_{t \in A_n} I(Y_t - \widetilde{Y}_{n\varphi} \le \widehat{\sigma}_n x) \;=\; \widehat{F}_n(x + x\epsilon_n + \widetilde{\delta}_{n\varphi}), \\ \widetilde{\delta}_{n\varphi} &:= (\widetilde{Y}_{n\varphi} - \mu)/\sigma, \quad \epsilon_n := (\widehat{\sigma}_n - \sigma)/\sigma. \end{aligned} \tag{2.8}$$

Define

$$\widetilde{D}_{n\varphi}(x) \;:=\; \widetilde{F}_{n\varphi}(x) - F_0(x), \qquad \widetilde{D}_{n\varphi} := \sup_{x \in \mathbb{R}} |\widetilde{D}_{n\varphi}(x)| \tag{2.9}$$

and $\Delta_n(\varphi_1, \varphi_2)$ as in (1.10), where φ_1, φ_2 are as in the above lemma.

Let ζ be a copy of ζ_0. Assume that the innovation distribution satisfies

$$E\zeta^4 < \infty, \tag{2.10}$$

$$|Ee^{\mathbf{i}u\zeta}| \le C(1 + |u|)^{-\delta}, \qquad \text{for some } 0 < C < \infty,\ \delta > 0, \forall\, u \in \mathbb{R}. \tag{2.11}$$

Under (2.11), it is shown in DSL that the d.f. F of X_0 is infinitely differentiable and for some universal positive constant C,

$$(f(x), |f'(x)|, |f''(x)|, |f'''(x)|) \le C(1 + |x|)^{-2}, \quad \forall\, x \in \mathbb{R},$$

where f'', f''' are the second and third derivatives of f, respectively. This fact in turn clearly implies f and these derivatives are square integrable.

Theorem 2.1 *Suppose (1.1), (1.2), (2.10), (2.11) hold. Let $\varphi(x), x \in [0,1]^\nu$ be a piece-wise continuously differentiable function satisfying $\bar{\varphi} = 0$. Then, with $v(\varphi)$ as in (1.8), under $\mathcal{H}_0$,*

$$n^{\nu/2-d} \sup_{x \in \mathbb{R}} \big|\widetilde{F}_{n\varphi}(x) - F_0(x) - \bar{W}_{n\varphi} f_0(x)\big| = o_p(1), \tag{2.12}$$

$$n^{\nu/2-d} \widetilde{D}_{n\varphi} = \|f_0\|_\infty n^{\nu/2-d} |\bar{W}_{n\varphi}| + o_p(1) \;\to_D\; \sqrt{v(\varphi)} \|f_0\|_\infty |Z|. \tag{2.13}$$

Remark 2.1 For stationary Gaussian random fields with zero mean, assumptions (1.1), (1.2), (2.10), (2.11) can be relaxed. Namely, for such random fields $\{X_t, t \in \mathbb{Z}^\nu\}$, Theorem 2.1 and the subsequent Corollary 2.1 remain valid under the single condition (1.3). This is due to the fact that in the Gaussian case, the proof of the Uniform Reduction Principle (see, e.g., Giraitis et al. (2012), sec.10.2.1) carries over from $\nu = 1$ to $\nu > 1$ with minor changes.

Proof of Theorem 2.1. Recall in (1.6), $\mathrm{Var}(X_0) = 1$. We first estimate the convergence rate of $\widehat{\sigma}_n$. Note that

$$\mathrm{Cov}(X_0^2, X_t^2) = O((\mathrm{Cov}(X_0, X_t))^2) \;=\; O(|t|^{-2(\nu-2d)}) \quad \text{as} \quad |t| \to \infty. \tag{2.14}$$

For $\nu = 1$, this result is well known, see e.g. Giraitis et al. (2012), Lemma 4.5.3. It easily extends to $\nu > 1$. Next, write $\widehat{\sigma}_n^2 - \sigma^2 = V_1 - V_2$, where $V_1 := \sigma^2 n^{-\nu} \sum_{t\in A_n}(X_t^2 - EX_t^2)$, $V_2 := \sigma^2(\bar{X}_n)^2$. Using (1.4) and (2.14), similarly as in Giraitis et al. (2012), p.509, we obtain $V_2 = O_p(n^{-(\nu-2d)})$, and

$$EV_1^2 \;\le\; \begin{cases} n^{-\nu}, & 0 < d < \nu/4, \\ n^{4d-2\nu}, & \nu/4 < d < \nu/2, \\ (\log(n)/n)^{\nu}, & d = \nu/4. \end{cases}$$

Hence

$$\epsilon_n = (\widehat{\sigma}_n - \sigma)/\sigma \;=\; \begin{cases} O_p(n^{-\nu/2}), & 0 < d < \nu/4, \\ O_p(n^{2d-\nu}), & \nu/4 < d < \nu/2, \\ O_p((\log(n)/n)^{\nu/2}), & d = \nu/4. \end{cases} \tag{2.15}$$

We are now ready to prove (2.12). Let

$$\begin{aligned} U_{n1}(x) &:= \widehat{F}_n(x + x\epsilon_n + \widetilde{\delta}_{n\varphi}) - F_0(x + x\epsilon_n + \tilde{\delta}_{n\varphi}) + f_0(x + x\epsilon_n + \widetilde{\delta}_{n\varphi})\bar{X}_n, \\ U_{n2}(x) &:= \int_x^{x+x\epsilon_n+\widetilde{\delta}_{n\varphi}} (f_0(u) - f_0(x + x\epsilon_n + \widetilde{\delta}_{n\varphi}))du, \\ U_{n3}(x) &:= f_0(x + x\epsilon_n + \widetilde{\delta}_{n\varphi})x\epsilon_n, \\ U_{n4}(x) &:= (f_0(x + x\epsilon_n + \widetilde{\delta}_{n\varphi}) - f_0(x))\bar{W}_{n\varphi}. \end{aligned}$$

Using (2.8) and the fact that $\widetilde{\delta}_{n\varphi} = \bar{X}_n + \bar{W}_{n\varphi}$, rewrite

$$\widetilde{F}_{n\varphi}(x) - F_0(x) - \bar{W}_{n\varphi}f_0(x) = \sum_{i=1}^{4} U_{ni}(x).$$

By (2.15), $\epsilon_n = o_p(n^{d-\nu/2})$ for any $0 < d < \nu/2$. According to (DLS, Cor.1.2), $\|U_{n1}\|_\infty = o_p(n^{d-\nu/2})$. Next, similarly to Giraitis et al. (2012), p. 510,

$$\begin{aligned} |U_{n2}(x)| &\le C(1 + x^2)^{-1}(|x\epsilon_n|^2 + |\widetilde{\delta}_{n\varphi}|^2) = o_p(n^{d-\nu/2}), \\ |U_{n3}(x)| &\le C(1 + x^2)^{-1}|x\epsilon_n| = o_p(n^{d-\nu/2}), \\ |U_{n4}(x)| &\le C(1 + x^2)^{-1}(|x\epsilon_n| + |\widetilde{\delta}_{n\varphi}|)|\bar{W}_{n\varphi}| = o_p(n^{d-\nu/2}) \end{aligned}$$

uniformly in $x \in \mathbb{R}$. This proves (2.12). Relation (2.13) follows from (2.12) and (2.3). Theorem 2.1 is proved.

Next, we discuss the asymptotic null distribution of $\Delta_n(\varphi_1, \varphi_2)$ of (1.10).

Corollary 2.1 *Suppose (1.1), (1.2), (2.10), (2.11) hold as in the previous theorem. Let $\varphi_i(x), x \in [0,1]^\nu, i = 1, 2$ be two piece-wise continuously differentiable functions satisfying $\bar{\varphi}_i = 0$, $v(\varphi_i) > 0, i = 1, 2$. Then, under $\mathcal{H}_0$,*

$$\Delta_n(\varphi_1, \varphi_2) \ \to_D \ \|f_0\|_\infty \sqrt{\frac{v(\varphi_1)}{v(\varphi_2)}} \Big|\frac{Z_1}{Z_2}\Big|, \tag{2.16}$$

with (Z_1, Z_2) as in Lemma 2.1.

In particular, if $v(\varphi_1) = v(\varphi_2)$ then, under $\mathcal{H}_0$,

$$\Delta_n(\varphi_1, \varphi_2) \ \to_D \ \|f_0\|_\infty \Big|\frac{Z_1}{Z_2}\Big|. \tag{2.17}$$

In general, when $v(\varphi_1) \neq v(\varphi_2)$, if $\widetilde{v}_n(\varphi_i), i = 1, 2$, are two statistics such that under $\mathcal{H}_0$, $\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1) \to_p v(\varphi_2)/v(\varphi_1)$, then, under $\mathcal{H}_0$,

$$\widetilde{\Delta}_n \to_p \Big|\frac{Z_1}{Z_2}\Big|. \tag{2.18}$$

Proof. By (2.13),

$$n^{\nu/2-d}\widetilde{D}_{n\varphi_1} \ = \ n^{\nu/2-d}|\bar{W}_{n\varphi_1}|\|f_0\|_\infty + o_p(1)$$

and, in view of (2.1), and because $\bar{\varphi}_2 = 0$,

$$n^{\nu/2-d}(\widetilde{Y}_{n,\varphi_2} - \bar{Y}_n) \ = \ n^{\nu/2-d}\big[\mu\big(1 + \bar{\varphi}_2 - 1\big) + \sigma\bar{W}_{n,\varphi_2}\big] \ = \ n^{\nu/2-d}\sigma\bar{W}_{n\varphi_2}. \tag{2.19}$$

Hence

$$\Delta_n(\varphi_1, \varphi_2) \ = \ \frac{\widehat{\sigma}_n\, n^{\nu/2-d}|\bar{W}_{n\varphi_1}|\|f_0\|_\infty + o_p(1)}{\sigma\, n^{\nu/2-d}|\bar{W}_n(\varphi_2)|} \ \to_D \ \|f_0\|_\infty \sqrt{\frac{v(\varphi_1)}{v(\varphi_2)}} \Big|\frac{Z_1}{Z_2}\Big|, \tag{2.20}$$

by the convergence in (2.3) and $\widehat{\sigma}_n \to_p \sigma$. This proves (2.16) and hence (2.17), too.

Choice of $\varphi_i, i = 1, 2$ in (2.17) and (1.14). Recall the conditions on φ_i in (2.17):

$$\bar{\varphi}_1 = \bar{\varphi}_2 = 0 \qquad \text{and} \qquad v(\varphi_1) = v(\varphi_2). \tag{2.21}$$

Note the choice $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = \pm\varphi(x)$ for a given φ with $\bar{\varphi} = 0$ satisfying (2.21) is not good since this leads to a degenerate distribution in (2.17). A better choice of φ_i seems

$$\varphi_1(x) := \varphi(x), \qquad \varphi_2(x) := \varphi(\mathbf{1} - x) \tag{2.22}$$

where $\mathbf{1} = (1, 1, \cdots, 1) \in [0,1]^\nu$ and $\varphi(x), x \in [0,1]^\nu$ is a given piecewise continuously differentiable function with $\bar{\varphi} = 0$. Clearly (2.22) satisfy the first condition in (2.21). The second one follows from the definition (1.8) by change of variables $\mathbf{1} - u \to u, \mathbf{1} - v \to v$:

$$\begin{aligned} v(\varphi_2) &= \int_{[0,1]^\nu} \int_{[0,1]^\nu} \varphi(\mathbf{1} - u)\varphi(\mathbf{1} - v) R_0\Big(\frac{u - v}{|u - v|}\Big) \frac{du\, dv}{|u - v|^{\nu - 2d}} \\ &= \int_{[0,1]^\nu} \int_{[0,1]^\nu} \varphi(u)\varphi(v) R_0\Big(\frac{v - u}{|u - v|}\Big) \frac{du\, dv}{|u - v|^{\nu - 2d}} \;=\; v(\varphi_1) \end{aligned}$$

since $R_0(-x) = R_0(x), x \in \mathbb{R}^\nu$ is an even function.

A GOF test when $v(\varphi_1) \neq v(\varphi_2)$. The test (1.14) assumes $v(\varphi_1) = v(\varphi_2)$ which restricts the choice of φ_i's. In order to avoid this restriction, we consider the modified test (1.16) based on $\widetilde{\Delta}_n$, where $\widetilde{v}_n(\varphi_i), i = 1, 2$, are defined in (3.4) below. By (2.18) the test that rejects $\mathcal{H}_0$, whenever $\widetilde{\Delta}_n > c_\alpha(\widehat{\rho}_n(\varphi_1, \varphi_2))$, is asymptotically of size α, without requiring a $\log(n)$-consistent estimate of d and regardless of whether $v(\varphi_1)$ equals $v(\varphi_2)$ or not. Section 3 contains the proof of the fact that $\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1) \to_p v(\varphi_2)/v(\varphi_1)$.

3 Estimation of $\rho(\varphi_1, \varphi_2)$

In this section we introduce consistent estimators of $v(\varphi)$ and $\rho(\varphi_1, \varphi_2)$. By Lemma 2.1,

$$\rho_n(\varphi_1, \varphi_2) \;:=\; \mathrm{Corr}(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2}) \;\to\; \rho(\varphi_1, \varphi_2), \tag{3.1}$$

$$\begin{aligned} \mathrm{Cov}(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2}) &= \sigma^{-2} n^{-2\nu} \sum_{t,s \in A_n} \varphi_{1,nt}\varphi_{2,ns} \mathrm{Cov}(Y_t, Y_s) \\ &\sim \mathrm{Cov}(\sqrt{v(\varphi_1)} Z_1, \sqrt{v(\varphi_2)} Z_2) n^{2d - \nu}. \end{aligned} \tag{3.2}$$

Now, let $q \to \infty$, $q = 1, 2, \cdots$, $q = o(n)$ be a bandwidth sequence and $\widehat{\gamma}_n(u)$ be the estimator of the covariance $\gamma(u) := \mathrm{Cov}(Y_0, Y_u)$:

$$\widehat{\gamma}_n(u) := \frac{1}{n^\nu} \sum_{t,s \in A_n : t - s = u} (Y_t - \bar{Y}_n)(Y_s - \bar{Y}_n). \tag{3.3}$$

Note that $\widehat{\sigma}_n^2$ of (2.7) equals $\widehat{\gamma}_n(0)$. Define

$$\begin{aligned} &\widetilde{v}_n(\varphi) := \frac{1}{\widehat{\sigma}_n^2} \sum_{u,v \in A_q} \varphi_{qu}\varphi_{qv} \widehat{\gamma}_n(u - v), \quad \widetilde{v}_n(\varphi_1, \varphi_2) := \frac{1}{\widehat{\sigma}_n^2} \sum_{u,v \in A_q} \varphi_{1,qu}\varphi_{2,qv} \widehat{\gamma}_n(u - v). \\ &v_n(\varphi) := q^{-\nu - 2d}\, \widetilde{v}_n(\varphi), \qquad v_n(\varphi_1, \varphi_2) := q^{-\nu - 2d}\, \widetilde{v}_n(\varphi_1, \varphi_2), \\ &\widehat{v}_n(\varphi) := q^{-\nu - 2\widetilde{d}}\, \widetilde{v}_n(\varphi), \end{aligned} \tag{3.4}$$

where $\widetilde{d}$ is a $\log(n)$-consistent estimator of d under $\mathcal{H}_0$. Then from (3.1), (3.2), a natural estimator of $\rho(\varphi_1, \varphi_2)$ is

$$\widehat{\rho}_n(\varphi_1, \varphi_2) \;:=\; \frac{\widetilde{v}_n(\varphi_1, \varphi_2)}{\sqrt{\widetilde{v}_n(\varphi_1)\widetilde{v}_n(\varphi_2)}}. \tag{3.5}$$

Note that

$$
\begin{aligned}
&n^\nu \widehat{\sigma}_n^2 \widetilde{v}_n(\varphi_1, \varphi_2) \\
&= \sum_{u,v \in A_q} \varphi_{1,qu}\varphi_{2,qv} \sum_{t,s \in A_n : t-s=u-v} (Y_t - \bar{Y}_n)(Y_s - \bar{Y}_n) \\
&= \sum_{k \in A_n \ominus A_q} \Big(\sum_{u \in A_q : k+u \in A_n} \varphi_{1,qu}(Y_{k+u} - \bar{Y}_n) \Big) \Big(\sum_{v \in A_q : k+v \in A_n} \varphi_{2,qu}(Y_{k+v} - \bar{Y}_n) \Big),
\end{aligned}
$$

where $A_n \ominus A_q := \{k \in \mathbb{Z}^\nu : k = t - u, \exists\, t \in A_n, \exists\, u \in A_q\}$. Hence, $\widetilde{v}_n(\varphi) = \widetilde{v}_n(\varphi, \varphi) \ge 0$ and $|\widetilde{v}_n(\varphi_1, \varphi_2)| \le \sqrt{\widetilde{v}_n(\varphi_1)\widetilde{v}_n(\varphi_2)}$. In particular, $\widehat{\rho}_n(\varphi_1, \varphi_2)$ in (3.5) is well-defined unless $\widetilde{v}_n(\varphi_1)\widetilde{v}_n(\varphi_2) = 0$ (in the latter case, we set $\widehat{\rho}_n(\varphi_1, \varphi_2) = 0$ by definition). Clearly, the estimator in (3.5) satisfies the property $|\widehat{\rho}_n(\varphi_1, \varphi_2)| \le 1$ of a correlation coefficient. Evidently, $\widehat{\rho}_n(\varphi_1, \varphi_2)$ does not involve the long memory parameter d or its estimate, and hence its computation is relatively simpler.

The following lemma discusses consistency of the estimator $\widehat{\rho}_n(\varphi_1, \varphi_2)$.

Lemma 3.1 *Let $\varphi_i(x), i = 1, 2$ and $\{X_t\}$ satisfy the conditions of Lemma 2.1. In addition, assume that $v(\varphi_i) > 0, i = 1, 2$ and $E\zeta^4 < \infty$. Then, as $n, q, n/q \to \infty$,*

$$
v_n(\varphi_1, \varphi_2) \to_p v(\varphi_1, \varphi_2), \qquad v_n(\varphi_i) \to_p v(\varphi_i),\ i = 1, 2, \tag{3.6}
$$

$$
\widehat{\rho}_n(\varphi_1, \varphi_2) \to_p \rho(\varphi_1, \varphi_2), \tag{3.7}
$$

where $v(\varphi_1, \varphi_2), \rho(\varphi_1, \varphi_2)$ are defined in (2.4), (2.5), respectively.

Consequently, $\sqrt{\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1)} \to_p \sqrt{v(\varphi_2)/v(\varphi_1)}$, and if $\widetilde{d}$ is a $\log(n)$-consistent estimator of d, then $\widehat{v}_n(\varphi) \to_p v(\varphi)$.

Proof. It suffices to prove the first claim of (3.6) only since the second claim follows similarly and (3.7) follows from (3.6) and the fact that $v(\varphi_i) = v(\varphi_i, \varphi_i) > 0, i = 1, 2$. The remaining claims follow from the second part of (3.6) in a routine fashion. Moreover, since $\widehat{\sigma}_n^2 \to_p \sigma^2 > 0$, see (2.15), we can restrict the proof of (3.6) to the case $\mu = 0, \sigma = 1$, or $Y_t = X_t$.

The following proof of (3.6) follows the argument in Lavancier, Philippe, and Surgailis (2010, proof of Prop. 4.1) in the case $\nu = 1$. Write $v_n(\varphi_1, \varphi_2) = \ell_{n1} + \ell_{n2}$, $\ell_{ni} := q^{-\nu-2d} \sum_{t,s \in A_q} \varphi_{1,qt}\varphi_{2,qs}\widehat{\gamma}_{ni}(t - s), i = 1, 2$, where

$$
\widehat{\gamma}_{n1}(t - s) := \frac{1}{n^\nu} \sum_{u,v \in A_n : u-v=t-s} X_u X_v, \tag{3.8}
$$

and $\widehat{\gamma}_{n2}(t - s) := \widehat{\gamma}_n(t - s) - \widehat{\gamma}_{n1}(t - s)$. Then (3.6) follows from

$$
\ell_{n1} \to_p v(\varphi_1, \varphi_2), \qquad \ell_{n2} = o_p(1). \tag{3.9}
$$

To prove the first relation of (3.9), write $\ell_{n1} = \sum_{i=1}^{3} \mathfrak{L}_{ni}$, where $\mathfrak{L}_{n1}$ is obtained by replacing $X_u X_v$ in (3.8) by $EX_u X_v = EX_t X_s = \gamma(t-s)$, viz.,

$$
\begin{aligned}
(3.10) \qquad \mathfrak{L}_{n1} &:= q^{-\nu-2d} \sum_{t,s\in A_q} \varphi_{1,qt}\varphi_{2,qs}\gamma(t-s) \\
&= \frac{1}{q^{2\nu}} \sum_{t,s\in A_q} \varphi_{1,qt}\varphi_{2,qs} R_0\Big(\frac{\frac{t}{q}-\frac{s}{q}}{|\frac{t}{q}-\frac{s}{q}|}\Big) \frac{1}{|\frac{t}{q}-\frac{s}{q}|^{\nu-2d}} \ \to \ v(\varphi_1,\varphi_2)
\end{aligned}
$$

as $q\to\infty$. The terms $\mathfrak{L}_{ni}, i = 2, 3$ correspond to the decomposition

$$
X_u X_v - EX_u X_v = \sum_{w\in\mathbb{Z}^\nu} b_{u+w} b_{v+w}\eta_w + \sum_{w_1,w_2\in\mathbb{Z}^\nu, w_1\neq w_2} b_{u+w_1} b_{v+w_2}\zeta_{w_1}\zeta_{w_2}
$$

of $X_u X_v$ in (3.8) with $\eta_w := \zeta_w^2 - E\zeta_w^2$, yielding

$$
\begin{aligned}
\mathfrak{L}_{n2} &:= q^{-\nu-2d} \sum_{w\in\mathbb{Z}^\nu} \eta_w \sum_{t,s\in A_q} \varphi_{1,qt}\varphi_{2,qs} \frac{1}{n^\nu} \sum_{u,v\in A_n: u-v=t-s} b_{u+w} b_{v+w}, \\
\mathfrak{L}_{n3} &:= q^{-\nu-2d} \sum_{w_1\neq w_2} \zeta_{w_1}\zeta_{w_2} \sum_{t,s\in A_q} \varphi_{1,qt}\varphi_{2,qs} \frac{1}{n^\nu} \sum_{u,v\in A_n: u-v=t-s} b_{u+w_1} b_{v+w_2}.
\end{aligned}
$$

We shall show that $\mathfrak{L}_{ni}\to 0$, in mean square, for $i = 2, 3$. To prove this claim for $\mathfrak{L}_{n2}$, we use the facts that the $\varphi_{i,qt}$'s are bounded, the η_u's are uncorrelated zero mean r.v.'s with finite variance, the form of the moving-average coefficients b_t's (1.2) with a bounded B_0, and the Minkowski inequality. Accordingly, then

$$
\begin{aligned}
(3.11) \ E\mathfrak{L}_{n2}^2 &\le Cq^{-2\nu-4d} n^{-2\nu} \sum_{w\in\mathbb{Z}^\nu} \Big(\sum_{t,s\in A_q} \sum_{u,v\in A_n: u-v=t-s} |b_{u+w} b_{v+w}|\Big)^2 \\
&\le Cq^{-2\nu-4d} n^{-2\nu} \Big(\sum_{t,s\in A_q} \sum_{u,v\in A_n: u-v=t-s} \big(\sum_{w\in\mathbb{Z}^\nu} |b_{u+w} b_{v+w}|^2\big)^{1/2}\Big)^2 \\
&\le Cq^{-2\nu-4d} n^{-2\nu} \Big(\sum_{t,s\in A_q} \sum_{u,v\in A_n: u-v=t-s} \big(\sum_{w\in\mathbb{Z}^\nu} |u+w|_+^{-2(\nu-d)} |v+w|_+^{-2(\nu-d)}\big)^{1/2}\Big)^2 \\
&\le Cq^{-2\nu-4d} n^{-2\nu} \big(\sum_{t,s\in A_q} \sum_{u,v\in A_n: u-v=t-s} \big(|u-v|_+^{\nu-4(\nu-d)}\big)^{1/2}\big)^2 \\
&\le Cq^{-2\nu-4d} \big(\sum_{t,s\in A_q} |t-s|_+^{(\nu/2)-2(\nu-d)}\big)^2 \\
&\le Cq^{-2\nu-4d} \big(q^{2\nu+(\nu/p)-2(\nu-d)}\big)^2 \ = \ Cq^{-2\nu} \ \to \ 0.
\end{aligned}
$$

Finally, using the facts that for $u_1 \neq u_2$, the r.v.'s ζ_{u_1}, ζ_{u_2} have zero mean, finite variance and are mutually uncorrelated, we obtain

$$
(3.12) \qquad E\mathfrak{L}_{n3}^2 \ \le \ Cq^{-2\nu-4d} n^{-2\nu} \sum_{w_1,w_2\in\mathbb{Z}^\nu} \Big(\sum_{t,s\in A_q} \sum_{u\in A_n} b_{u+w_1} b_{u-t+s+w_2}\Big)^2
$$

$$
\begin{aligned}
&\leq \; Cq^{-2\nu-4d}n^{-2\nu} \sum_{t,s,t',s'\in A_q} \sum_{u,u'\in A_n} \sum_{w_1,w_2} |b_{u+w_1}b_{u-t+s+w_2}b_{u'+w_1}b_{u'-t'+s'+w_2}| \\
&\leq \; Cq^{-2\nu-4d}n^{-2\nu} \sum_{t,s,t',s'\in A_q} \sum_{u,u'\in A_n} |u-u'|_+^{2d-\nu}|u-u'+s-s'-t+t'|_+^{2d-\nu} \\
&\leq \; Cq^{\nu-4d}n^{-\nu} \sum_{|u|<n} |u|_+^{2d-\nu} \sum_{|t|<2q} |u+t|_+^{2d-\nu} \;\leq\; C(J_1+J_2).
\end{aligned}
$$

Here,

$$
\begin{aligned}
J_1 \;&:=\; q^{\nu-4d}n^{-\nu} \sum_{|u|<4q} |u|_+^{2d-\nu} \sum_{|t|<6q} |t|_+^{2d-\nu} \\
&\leq\; Cq^{\nu-4d}n^{-\nu}q^{4d} = O((q/n)^{\nu}) = o(1), \\
J_2 \;&:=\; Cq^{\nu-4d}(q/n)^{\nu} \sum_{4q\leq|u|<n} |u|^{4d-2\nu} \;\leq Cq^{\nu-4d}(q/n)^{\nu} \begin{cases} n^{4d-\nu}, & 2\nu-4d<\nu, \\ q^{4d-\nu}, & 2\nu-4d>\nu, \\ \log(n/q), & 2\nu-4d=\nu, \end{cases}
\end{aligned}
$$

and so $J_2 = o(1)$ as $q, n, n/q \to \infty$ in all three cases (in the last case where $2\nu - 4d = \nu$, this follows from the fact that $x \to 0$ entails $x^{\nu}\log(1/x) \to 0$). Clearly, (3.10)-(3.12) prove the first relation in (3.9).

It remains to show the second relation in (3.9). It follows from

$$
q^{-2d} \sum_{|t|\leq q} E|\widehat{\gamma}_{n2}(t)| = o(1). \tag{3.13}
$$

Use the definition $\widehat{\gamma}_{n2}(t) = \widehat{\gamma}_n(t) - \widehat{\gamma}_{n1}(t)$, the Cauchy-Schwarz inequality, and (1.4), to obtain

$$
\begin{aligned}
&\Big(E|\widehat{\gamma}_n(t)-\widehat{\gamma}_{n1}(t)|\Big)^2 \\
&\quad\leq\; E\bar{X}_n^2\, E\Big(n^{-\nu} \sum_{u,v\in A_n: u-v=t} X_v\Big)^2 + E\bar{X}_n^2\, E\Big(n^{-\nu} \sum_{u,v\in A_n: u-v=t} X_u\Big)^2 + \big(E\bar{X}_n^2\big)^2 \\
&\quad\leq\; Cn^{4d-2\nu},
\end{aligned}
$$

with C independent of $|t| < n/2$. Hence, (3.13) reduces to $(q/n)^{\nu-2d} = o(1)$ which is a consequence of $d < \nu/2$ and $q/n \to 0$. This proves (3.9) and completes the proof of Lemma 3.1.

4 Consistency and asymptotic power

We shall now discuss the consistency and asymptotic power of the GOF test (1.14). Let F be another marginal d.f. of the error process $\{X_t\}$, and $F \neq F_0$ so that $\|F - F_0\|_\infty =$

$\sup_{x\in\mathbb{R}} |F(x) - F_0(x)| > 0$, and such that the underlying innovations ζ_j still satisfy the assumptions (2.10) and (2.11). The power of the Δ_n-test at this F is

$$
\begin{aligned}
(4.1) \qquad & P\Big(\frac{\Delta_n(\varphi_1,\varphi_2)}{\|f_0\|_\infty} > c_\alpha(\widehat{\rho}_n(\varphi_1,\varphi_2))\Big) \\
&= P\Big(n^{\nu/2-d}\frac{\widehat{\sigma}_n \widetilde{D}_{n\varphi_1}}{n^{\nu/2-d}|\bar{W}_{n\varphi_2}|} > c_\alpha(\widehat{\rho}_n(\varphi_1,\varphi_2))\|f_0\|_\infty\Big).
\end{aligned}
$$

Now, decompose the empirical process $\widetilde{D}_{n\varphi_1}(x)$ in (2.9) as $\widetilde{D}_{n\varphi_1}(x) = (\widetilde{F}_{n\varphi_1}(x) - F(x)) + (F(x) - F_0(x))$. Then $\widetilde{D}_{n\varphi_1} = \|F - F_0\|_\infty + o_p(1)$ and

$$
\begin{aligned}
&\widehat{\sigma}_n \widetilde{D}_{n\varphi_1}/n^{\nu/2-d}|\bar{W}_{n\varphi_2}| \to_D \sigma\|F - F_0\|_\infty/\sqrt{v(\varphi_2)}|Z_2|, \\
&c_\alpha(\widehat{\rho}_n(\varphi_1,\varphi_2))\|f_0\|_\infty \to_p c_\alpha(\rho(\varphi_1,\varphi_2))\|f_0\|_\infty < \infty.
\end{aligned}
$$

Since $n^{\nu/2-d} \to \infty$, it immediately follows that the l.h.s. of (4.1) tends to 1 for any $0 < \alpha < 1$, implying that the test in (1.14) is consistent against the above fixed alternative F. A similar argument establishes the consistency of the $\widetilde{\Delta}_n$-test without requiring $v(\varphi_1) = v(\varphi_2)$.

The following proposition describes asymptotic distribution of the sequence of the statistics $\Delta_n(\varphi_1,\varphi_2)$ and $\widetilde{\Delta}_n$ under certain sequences of local alternatives. Analogous to Koul et al. (2013), Thm.2.5, consider a sequence of stationary moving-average fields

$$
(4.2) \qquad X_{tn} = \sum_{s\in\mathbb{Z}^\nu} b_{t-s}\zeta_{sn}, \quad t \in \mathbb{Z}^\nu,
$$

where the b_t's are as in (1.2) and do not depend on n, and $\{\zeta_{sn}, s \in \mathbb{Z}^\nu\}$ are standardized innovations satisfying (2.11) and $E\zeta_{0n}^4 < C$ for each n with C, δ independent of n. We observe $Y_{tn}, t \in \mathbb{Z}^\nu$, obeying the model

$$
(4.3) \qquad Y_{tn} = \mu + \sigma X_{tn}, \quad t \in A_n, \quad \text{for some } \mu \in \mathbb{R},\ \sigma > 0.
$$

Let $F_n(x) := P(X_{tn} \le x)$ be the marginal d.f. of (4.2) and $\widetilde{Y}_{n\varphi}$, $\bar{W}_{n\varphi}$, $\widetilde{\sigma}_n$, $\widetilde{F}_{n\varphi}(x)$, be defined analogously to (2.1), (2.2), (2.7), (2.8), with $\{X_t\}$, $\{Y_t\}$ replaced by $\{X_{tn}\}$, $\{Y_{tn}\}$, respectively. Then under the same assumptions on $\varphi_i, i = 1, 2$ as in Corollary 2.1 by inspecting the proofs of Lemma 3.1 and Doukhan et al. (2002), Thm.1.1, it follows that

$$
\begin{aligned}
(4.4) \qquad & n^{\nu/2-d}\sup_{x\in\mathbb{R}}\big|\widetilde{F}_{n\varphi_1}(x) - F_n(x) - \bar{W}_{n\varphi_1}f_n(x)\big| \ = \ o_p(1), \\
(4.5) \qquad & n^{\nu/2-d}(\bar{W}_{n\varphi_1}, \bar{W}_{n\varphi_2}) \ \to_D \ (\sqrt{v(\varphi_1)}\,Z_1, \sqrt{v(\varphi_2)}\,Z_2), \\
(4.6) \qquad & \widehat{\rho}_n(\varphi_1,\varphi_2) \ \to \ \rho(\varphi_1,\varphi_2), \quad \widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1) \ \to v(\varphi_1)/v(\varphi_2)
\end{aligned}
$$

analogously to (2.12), (2.3), (3.7), (3.6), where $f_n := F_n'$ is the probability density of F_n and where $v(\varphi), (Z_1, Z_2)$ are the same as in Lemma 2.1 and independent of n.

Proposition 4.1 *Let $F_n, n \geq 1$ be a sequence of distribution functions on $\mathbb{R}$. Suppose $\{X_{tn}, t \in \mathbb{Z}^\nu\}$ given by (4.2) with b_t, ζ_{sn} satisfying the stated conditions, and having the marginal distribution F_n. Assume that there exists a bounded and continuous function $G = G(x), x \in \mathbb{R}$ such that*

$$\|n^{\nu/2-d}(F_n - F_0) - G\|_\infty \to 0, \qquad n \to \infty \tag{4.7}$$

and $\|f_n - f_0\|_\infty \to 0$. Let $\varphi_i, Z_i, i = 1, 2$ be the same as in Corollary 2.1. Then

$$\Delta_n(\varphi_1, \varphi_2) \ \to_D \ \sup_{x\in\mathbb{R}} \Big| \frac{\sqrt{v(\varphi_1)} Z_1 f_0(x) + G(x)}{\sqrt{v(\varphi_2)} Z_2} \Big|. \tag{4.8}$$

Consequently, if $v(\varphi_1) = v(\varphi_2)$, then the asymptotic power of the Δ_n-test is

$$P\Big(\sup_{x\in\mathbb{R}} \Big| \frac{Z_1 f_0(x)}{Z_2} + \frac{G(x)}{\sqrt{v(\varphi_1)} Z_2} \Big| > c_\alpha(\rho(\varphi_1, \varphi_2)) \|f_0\|_\infty \Big), \tag{4.9}$$

which is also the asymptotic power of the $\widetilde{\Delta}_n$- test, regardless of whether $v(\varphi_1)$ equals to $v(\varphi_2)$ or not.

Proof. Decompose $\widetilde{F}_{n\varphi_1}(x) - F_0(x) = \bar{W}_{n\varphi_1} f_0(x) + (F_n(x) - F_0(x)) + V_n(x)$, where

$$V_n(x) := (\widetilde{F}_{n\varphi_1}(x) - F_n(x) - \bar{W}_{n\varphi_1} f_n(x)) + \bar{W}_{n\varphi_1}(f_n(x) - f_0(x)).$$

Then $\sup_{x\in\mathbb{R}} |V_n(x)| = o_p(n^{d-\nu/2})$ according to (4.4), (4.5) and $\|f_n - f_0\|_\infty = o(1)$. Using these facts and (4.7) we obtain, under $\mathcal{H}_1$, that

$$\begin{aligned} n^{\nu/2-d} \widetilde{D}_{n\varphi_1} &= n^{\nu/2-d} \sup_{x\in\mathbb{R}} |\bar{W}_{n\varphi_1} f_0(x) + (F_n(x) - F_0(x)) + V_n(x)| \\ &\to_D \sup_{x\in\mathbb{R}} |\sqrt{v(\varphi_1)} Z_1 f_0(x) + G(x)|. \end{aligned}$$

and $n^{\nu/2-d}(\widetilde{Y}_{n,\varphi_2} - \bar{Y}_n) = \sigma n^{\nu/2-d} \bar{W}_{n\varphi_2} \to_D \sigma \sqrt{v(\varphi_2)} Z_2$. This proves (4.8). Relation (4.9) follows from (4.8) and (4.6).

5 Fractionally integrated random fields

In this section we shall present two examples of fractionally integrated random fields in $\mathbb{Z}^2$, i.e., examples of the functions B_0 and R_0, where the results of the previous sections apply.

Example 1. Isotropic fractionally integrated random field. Let L denote the operator $LX_{t,s} = (1/4)\sum_{|u|+|v|=1} X_{t+u,s+v}$ so that $L-1$ is the (discrete) Laplacian on $\mathbb{Z}^2$. Consider the stationary lattice isotropic fractionally integrated random field

$$(1-L)^\tau X_{t,s} = \zeta_{t,s}, \tag{5.1}$$

where $\{\zeta_{t,s}, (t,s) \in \mathbb{Z}^2\}$ are standard i.i.d. r.v.'s, $0 < \tau < 1/2$ is the order of fractional integration, $(1-z)^\tau = \sum_{j=0}^\infty \psi_j(\tau) z^j$, $\psi_j(\tau) := \Gamma(j-\tau)/\Gamma(j+1)\Gamma(-\tau)$. More explicitly,

$$(1-L)^\tau X_{t,s} \;=\; \sum_{j=0}^\infty \psi_j(\tau) L^j X_{t,s} \;=\; \sum_{(u,v)\in\mathbb{Z}^2} a_{u,v} X_{t-u,s-v}, \tag{5.2}$$

where $a_{u,v} := \sum_{j=0}^\infty \psi_j(\tau) p_j(u,v)$ and $p_j(u,v)$ are j-step transition probabilities of the symmetric nearest-neighbor random walk $\{W_k, k=0,1,\dots\}$ on $\mathbb{Z}^2$ with equal 1-step probabilities $P(W_1 = (u,v)|W_0 = (0,0)) = 1/4, |u|+|v| = 1$. Note $\sum_{(u,v)\in\mathbb{Z}^2} |a_{u,v}| = \sum_{j=0}^\infty |\psi_j(\tau)| < \infty$, $\tau > 0$ and therefore the l.h.s. of (5.2) is well-defined for any stationary random field $\{X_{t,s}\}$ with $E|X_{0,0}| < \infty$. A stationary solution of (5.2) with zero-mean and finite variance can be defined as a moving-average random field:

$$X_{t,s} = (1-L)^{-\tau}\zeta_{t,s} = \sum_{(u,v)\in\mathbb{Z}^2} b_{u,v}\zeta_{t-u,s-v}, \tag{5.3}$$

where

$$b_{u,v} := \sum_{j=0}^\infty \psi_j(-\tau) p_j(u,v). \tag{5.4}$$

Note the Fourier transform

$$\begin{aligned}\hat{b}(x,y) \;&=\; \sum_{(u,v)\in\mathbb{Z}^2} e^{\mathbf{i}(ux+vy)} b_{u,v} = \sum_{j=0}^\infty \psi_j(-\tau)\hat{p}_j(x,y) \\ &=\; \sum_{j=0}^\infty \psi_j(-\tau)(\hat{p}_1(x,y))^j = (1-\hat{p}_1(x,y))^{-\tau},\end{aligned} \tag{5.5}$$

where $\hat{p}_1(x,y) = (1/4)\sum_{|u|+|v|=1} e^{\mathbf{i}(ux+vy)} = (\cos x + \cos y)/2$. Since $|1 - (\cos x + \cos y)/2| \ge (x^2+y^2)/4, (x,y) \in [-\pi,\pi]^2$, this implies that $\int_{[-\pi,\pi]^2} |\hat{b}(x,y)|^2 dxdy < \infty$ for $0 < \tau < 1/2$ and hence $\sum_{(u,v)\in\mathbb{Z}^2} b_{u,v}^2 < \infty$ by Parseval's identity. As a consequence, the random field in (5.3) is well-defined for any $0 < \tau < 1/2$ and has spectral density $f(x,y) = (2\pi)^{-2}2^{-2\tau}|(1-\cos x) + (1-\cos y)|^{-2\tau}, (x,y) \in [-\pi,\pi]^2$, which behaves as $const(x^2+y^2)^{-2\tau}$ as $x^2+y^2 \to 0$. The following proposition verifies the isotropic behavior of (1.2) with $d = 2\tau$ of the moving-average coefficients in (5.4).

Proposition 5.1 *Let* $0 < \tau < 1/2$. *Then*

$$b_{t,s} = (B_0 + o(1))(t^2+s^2)^{-(1-\tau)}, \quad t^2+s^2 \to \infty, \tag{5.6}$$

where $B_0 := \pi^{-1}\Gamma(1-\tau)/\Gamma(\tau)$.

Proof. We use the factorization $p_j(u,v) = q(j,u+v)q(j,u-v)$, where $q(j,v) = 2^{-j}\binom{j}{(j+u)/2}$ if $j+u$ is even, $= 0$ otherwise, is the distribution of the sum of j Bernoulli r.v.'s taking values ± 1 with probability $1/2$. See Puplinskaitė and Surgailis (2014), proof of Lemma 6.1. Let $r := (u^2+v^2)^{1/2}$. Then (5.4) can be rewritten

$$
(5.7) \qquad b_{u,v} \;=\; \sum_{j=0}^{\infty} \psi_j(-\tau) q(j,u+v) q(j,u-v) = \sum_{j>r^{5/3}} + \sum_{0\le j\le r^{5/3}} =: b^0_{u,v} + b^1_{u,v}.
$$

The statement of the proposition follows from

$$
(5.8) \qquad \lim_{r\to\infty} r^{2-2\tilde d} b^0_{u,v} = B_0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{2-2\tilde d} b^1_{u,v} = 0.
$$

To show the first relation in (5.8), note $q(j,u) = \mathrm{bin}((j+u)/2, j; 1/2)$, where $\mathrm{bin}(k,j;p) := \binom{j}{k} p^k (1-p)^{j-k}$ are binomial probabilities. We shall use the following version of the Moivre-Laplace theorem (Feller, 1966, ch.7, §2, Thm.1): *There exists a constant C such when $j\to\infty$ and $k\to\infty$ vary in such a way that*

$$
(5.9) \qquad \frac{(k-pj)^3}{j^2} \;\to\; 0,
$$

then

$$
(5.10) \qquad \Bigg| \frac{\mathrm{bin}(k,j;p)}{\frac{1}{\sqrt{2\pi j p(1-p)}} \exp\{-\frac{(k-jp)^2}{2jp(1-p)}\}} - 1 \Bigg| \;<\; \frac{C}{j} + \frac{C|k-pj|^3}{j^2}.
$$

For $q(j,u)$ (5.9)-(5.10) imply that there exists $j_0 > 0$ and $C > 0$ such that

$$
(5.11) \qquad \sup_{u\in\mathbb{Z}} \Bigg| \frac{q(j,u)}{\sqrt{\frac{2}{\pi j}} e^{-u^2/2j}} - 1 \Bigg| I\big(|u| < j^{3/5},\, u = j \bmod 2\big) \;<\; \frac{C}{j^{1/5}}, \qquad \forall\, j > j_0.
$$

Using (5.7), $\psi_j(-\tau) = \Gamma(\tau)^{-1} j^{\tau-1}(1+o(1))$ and (5.11) for $u+v$ even as $r\to\infty$ we obtain

$$
\begin{aligned}
r^{2-2\tau} b^0_{u,v} &= \frac{2r^{2-2\tau}}{\pi\Gamma(\tau)} \sum_{j>r^{5/3},\, j\,\mathrm{even}} j^{\tau-2}(1+o(1)) e^{-r^2/j}(1+o(1)) \\
&= \frac{2}{\pi\Gamma(\tau)} \frac{1}{r^2} \sum_{j/r^2>r^{-1/3},\, j\,\mathrm{even}} (\frac{j}{r^2})^{\tau-2} e^{-r^2/j}\big(1+o(1)\big) \\
&\to \frac{1}{\pi\Gamma(\tau)} \int_0^\infty x^{\tau-2} e^{-1/x} dx = \frac{\Gamma(1-\tau)}{\pi\Gamma(\tau)} = B_0.
\end{aligned}
$$

The above convergence for $u+v$ odd is analogous. This proves the first relation in (5.8). To show the second relation in (5.8), we use Hoeffding's inequality (Hoeffding, 1963), according to which $|b^1_{u,v}| \le C\sum_{0\le j\le r^{5/3}} |\psi_j(-\tau)| e^{-r^2/2j} = O(r^{5/3} e^{-(r^{1/3}/2)}) = o(r^{-(2-2\tau)})$. Proposition 5.1 is proved.

Example 2. Aggregated causal fractionally integrated random field. Let $L_1 X_{t,s} = X_{t-1,s}, L_2 X_{t,s} = X_{t,s-1}$, $(t,s) \in \mathbb{Z}^2$ be backward shift operators on $\mathbb{Z}^2$. Consider a stationary fractionally integrated random field

$$(1 - pL_1 - qL_2)^d X_{t,s} \;=\; \zeta_{t,s}, \tag{5.12}$$

where $\{\zeta_{t,s}, (t,s) \in \mathbb{Z}^2\}$ are standard i.i.d. r.v.'s, $p, q \ge 0, p + q = 1$ are parameters and $0 < d < 1$ is the order of fractional integration. More explicitly,

$$\begin{aligned}(1 - pL_1 - qL_2)^d X_{t,s} &= \sum_{j=0}^{\infty} \psi_j(d) \sum_{k=0}^{j} \binom{j}{k} p^k q^{j-k} L_1^k L_2^{j-k} X_{t,s} \\ &= \sum_{u,v \ge 0} a_{u,v} X_{t-u,s-v}, \qquad a_{u,v} := \psi_{u+v}(d) \mathrm{bin}(u, u+v; p).\end{aligned}$$

Note $\sum_{u,v\ge0} |a_{u,v}| = \sum_{j=0}^{\infty} |\psi_j(d)| < \infty$, $d > 0$ and therefore the l.h.s. of (5.12) is well-defined for any stationary random field $\{X_{t,s}\}$ with $E|X_{0,0}| < \infty$. A stationary solution of (5.12) with zero-mean and finite variance can be defined as a moving-average random field:

$$X_{t,s} = (1 - pL_1 - qL_2)^{-d} \zeta_{t,s} \;=\; \sum_{u,v\ge0} b_{u,v} \zeta_{t-u,s-v}, \tag{5.13}$$

where $b_{u,v} := \psi_{u+v}(-d)\, \mathrm{bin}(u, u+v; p)$. The random field in (5.13) is well-defined for any $0 < d < 3/4$ since

$$\begin{aligned}\sum_{u,v\ge0} b_{u,v}^2 \;&:=\; \sum_{j=0}^{\infty} \psi_j^2(-d) \sum_{k=0}^{j} (\mathrm{bin}(k,j;p))^2 \le \sum_{j=0}^{\infty} \psi_j^2(-d) \max_{0\le k\le j} \mathrm{bin}(k,j;p) \\ &\le\; C \sum_{j=0}^{\infty} (j \vee 1)^{2(d-1)} j_+^{-1/2} \;<\; \infty, \qquad 0 < d < 3/4.\end{aligned} \tag{5.14}$$

It follows from (5.10) that the result in (5.14) cannot be improved, in the sense that for any $d \ge 3/4$,

$$\sum_{u,v\ge0} b_{u,v}^2 \;\ge\; \sum_{j=0}^{\infty} \psi_j^2(-d) \sum_{0\le k\le j: |k-pj| \le c/\sqrt{j}} (\mathrm{bin}(k,j;p))^2 \;\ge\; c \sum_{j\ge j_0} j^{2(d-1)} j^{-1/2} \;=\; \infty,$$

where $c > 0, j_0 > 0$ are some constants. The moving average coefficients $b_{u,v}$ in (5.13) do not satisfy the assumption (1.2) since they are very much "concentrated" along the line $uq - vp = 0$ and exponentially decay if $u, v \to \infty$ so that $|uq - vp| > c > 0$. The random field in (5.13) exhibits strongly anisotropic long memory behavior different from the random fields in (1.1)-(1.2). See Puplinskaitė and Surgailis (2014). Obviously, the results in the previous sections do not apply to (5.13).

Assume now that $p \in [0,1]$ is random and has a bounded probability density $\ell(p)$ on $[0,1]$. Consider a moving-average random field

$$\widetilde{X}_{t,s} \; = \; \sum_{u,v\ge 0} \widetilde{b}_{u,v}\zeta_{t-u,s-v}, \qquad (t,s) \in \mathbb{Z}^2, \tag{5.15}$$

where

$$\widetilde{b}_{u,v} := Eb_{u,v} = \psi_{u+v}(-d)\binom{u+v}{u}\int_0^1 p^u(1-p)^v\ell(p)dp. \tag{5.16}$$

It readily follows that

$$\widetilde{b}_{u,v} \le C\psi_{u+v}(-d)\binom{u+v}{u}\int_0^1 p^u(1-p)^v dp = C\psi_{u+v}(-d)(u+v)^{-1},$$

and therefore $\sum_{u,v\ge 0}\widetilde{b}^2_{u,v} \le C\sum_{j=0}^\infty \psi_j^2(-d)(j+1)^{-1} < \infty$, for any $0<d<1$.

The random field in (5.15) is of interest since it arises by aggregating N copies

$$(1-a_i(p_iL_1+q_iL_2))Y^{(i)}_{t,s} \; = \; \zeta_{t,s}, \quad (t,s)\in\mathbb{Z}^2, \quad i=1,\cdots,N$$

of random-coefficient autoregressive random field with common innovations $\{\zeta_{t,s}\}$ following Granger's (1980) contemporaneous aggregation scheme. Here, $(a_i, p_i, q_i = 1-p_i), i = 1,\cdots,N$ are independent copies of $(a,p,q=1-p)$, where $a\in[0,1), p\in[0,1], q=1-p$ are random coefficients, a is independent of p and having a beta distribution with density

$$\phi(x) := B(d,1-d)^{-1}x^{d-1}(1-x)^{-d}, \qquad 0<x<1, \quad 0<d<1. \tag{5.17}$$

By the law of large numbers, the limit aggregated random field $\mathcal{Y}_{t,s} := \lim_{N\to\infty} N^{-1}\sum_{i=1}^N Y^{(i)}_{t,s}$ satisfies

$$\mathcal{Y}_{t,s} \; = \; \sum_{j=0}^\infty Ea^jE(pL_1+qL_2)^j\zeta_{t,s} \; = \; \sum_{u,v\ge 0} Ea^{u+v}E\mathrm{bin}(u,u+v;p)\zeta_{t-u,s-v} \tag{5.18}$$

In the case of beta density in (5.17), $Ea^j = \Gamma(j+d)/\Gamma(j+1)\Gamma(d) = \psi_j(-d)$ and therefore the moving average coefficients $Ea^{u+v}E\mathrm{bin}(u,u+v;p)$ in (5.18) coincide with $\widetilde{b}_{u,v}$ of (5.16), implying $\{\mathcal{Y}_{t,s}\} = \{\widetilde{X}_{t,s}\}$.

The following proposition shows that under some regularity conditions on the density ℓ of $p\in[0,1]$, the random field in (5.15) belongs to the class of random fields (1.1) discussed in this paper. Let $S_2 := \{(u,v)\in\mathbb{R}^2 : |(u,v)| = \sqrt{u^2+v^2} = 1\}$, $S_2^+ := \{(u,v)\in S_2 : u \ge 0, v\ge 0\}$.

Proposition 5.2 *Assume that $\ell(x), x \in [0,1]$ is a continuous function with support in $(0,1)$. Then*

$$\widetilde{b}_{u,v} \sim \frac{1}{\Gamma(d)}\ell(\frac{u}{u+v})\frac{1}{(u+v)^{2-d}}, \qquad u+v \to \infty. \tag{5.19}$$

In particular, $\widetilde{b}_{u,v}$ in (5.16) satisfy (1.2) with

$$B_0(u,v) := \begin{cases} \frac{1}{\Gamma(d)}\ell(\frac{u}{u+v})\frac{1}{(u+v)^{2-d}}, & (u,v) \in S_2^+, \\ 0, & (u,v) \in S_2 \setminus S_2^+. \end{cases}$$

Proof. Let $j = u+v$ and $u/(u+v) = k/j$. Then

$$\widetilde{b}_{u,v} = \psi_j(-d)\int_0^1 \mathrm{bin}(k,j;p)\ell(p)dp.$$

Note that the constant C in (5.10) does not depend on $p \in (\epsilon, 1-\epsilon), \epsilon > 0$ separated from 0 and 1. Next, for a small $\delta > 0$, split $E\mathrm{bin}(k,j;p) = E\mathrm{bin}(k,j;p)I(|k-pj|^3/j^2 \le \delta) + E\mathrm{bin}(k,j;p)I(|k-pj|^3/j^2 > \delta) =: \beta_1(k,j) + \beta_2(k,j)$. Using (5.10), we can write $\beta_1(k,j) = \gamma_1(k,j) + \gamma_2(k,j)$, where

$$\gamma_1(k,j) := \int_{\{|p-k/j|^3 \le \delta/j\}} \frac{1}{\sqrt{2\pi jp(1-p)}} \exp\{-\frac{(k-jp)^2}{2jp(1-p)}\}\ell(p)dp$$

and

$$\begin{aligned} &|\gamma_2(k,j)| \\ \le\ & C(\delta + j^{-1})\int_{\{|p-k/j|^3 \le \delta/j\}} \frac{1}{\sqrt{2\pi jp(1-p)}} \exp\{-\frac{(k-jp)^2}{2jp(1-p)}\}\ell(p)dp \\ \le\ & C(\delta + j^{-1})\frac{1}{\sqrt{2\pi j\epsilon}}\int_{\mathbb{R}} \exp\{-(j/2)(p-k/j)^2\}dp \\ \le\ & C(\delta + j^{-1})j^{-1} = o(1/j), \end{aligned} \tag{5.20}$$

where we used the facts that $1 \ge p(1-p) > \epsilon$ on $\{p \in [0,1] : \ell(p) > 0\}$ and that $\delta > 0$ can be chosen arbitrarily small. Next, using the continuity of $\ell(p)$ and $1/p(1-p)$ we see that $\gamma_1(k,j) = \tilde{\gamma}_1(k,j)(1+o(1))$, $j \to \infty$, where

$$\begin{aligned} &\tilde{\gamma}_1(k,j) \\ :=\ & \frac{\ell(k/j)}{\sqrt{2\pi j(k/j)(1-(k/j))}}\int_{\{|p-k/j|^3 \le \delta/j\}} \exp\{-\frac{(k-jp)^2}{2(k/j)(1-(k/j))}\}dp \\ =\ & \frac{\ell(k/j)}{j}\big(1+o(1)\big). \end{aligned} \tag{5.21}$$

To estimate $\beta_2(k,j)$, we use Hoeffding's inequality (Hoeffding, 1963), according to which

$$\begin{aligned}
(5.22) \qquad \beta_2(k,j) &\le \sup_{\epsilon<p<1-\epsilon} b(k,j;p) I(|k-pj|^3/j^2 > \delta) \\
&\le 2e^{-2\delta^{1/3} j^{1/6}} = o(1/j). \\
(5.23) \qquad \sum_{|t-kp|>\tau\sqrt{k}} b(t;k,p) &\le 2e^{-2\tau^2}.
\end{aligned}$$

Relations (5.20), (5.21), and (5.22) entail (5.19), hence the proposition.

Remark 5.1 Boissy et al. (2005), Guo et al. (2009) discuss fractionally integrated random fields satisfying

$$(1-L_1)^{d_1}(1-L_2)^{d_2} Y_{t,s} = \zeta_{t,s}, \qquad (t,s) \in \mathbb{Z}^2 \tag{5.24}$$

with possibly different parameters $|d_i| < 1/2, i = 1, 2$. We note that (5.24) form a distinct class from (1.1) - (1.2) and also from (5.12). Extension of Theorem 2.1 to fractionally integrated spatial models in (5.12) and (5.24) remains open.

6 A simulation study

This section describes the results of a simulation study investigating performance of the test in (1.14) based on the statistic $\Delta_n(\varphi_1, \varphi_2)$ (1.10), for the two cases $\nu = 1$ and $\nu = 2$, and the test based on $\widetilde{\Delta}_n$ of (1.16) for $\nu = 1$. Throughout the study, we test normality of the marginal d.f. F of the random field $\{Y_t\}$ in (1.6), i.e., the hypothesis

$$\mathcal{H}_0 : F(x) = \Phi(\frac{x-\mu}{\sigma}), \quad \forall x \in \mathbb{R}, \text{ for some } \mu \in \mathbb{R}, \ \sigma > 0,$$

vs. $\mathcal{H}_1 : \mathcal{H}_0$ is not true, where Φ denotes the standard normal d.f. Particularly, for $\nu = 1$, we compare the performance of the Δ_n-test with the test based on the statistic

$$\kappa_n = \frac{n^{\nu/2-\tilde{d}} \widetilde{\mathcal{D}}_{n\varphi}}{\sqrt{\widehat{v}_n(\varphi)} \|f_0\|_\infty}, \tag{6.1}$$

where $\widehat{v}_n(\varphi)$ is as in (3.4) and $\tilde{d}$ is described later. Contrary to the Δ_n-test, the κ_n-test uses estimated nuisance parameters d and $v(\varphi)$ and was discussed in Koul et. al. (2013) in the case when σ is known.

Case $\nu = 1$. We considered two null scenarios and one alternative. In each scenario, the data was generated according to the model $Y_t = 3 + 2X_t$ (or $\mu = 3, \sigma = 2$), with the error process $X_t = H_t/\sqrt{EH_t^2}$ and H_t given, respectively, by

$Null_1$: $H_t = U_t$, where $\{U_t\}$ is ARFIMA(0,d,0) with standard Gaussian innovations ,

$Null_2$: $H_t = U_t + Z_t$, where $\{U_t\}$ is ARFIMA(0,d,0) as in $Null_1$, and $\{Z_t\}$ is i.i.d. $N(0,4)$-distributed and independent of $\{U_t\}$,

Alt_1 : H_t is ARFIMA(0,d,0) with centered-exponential innovations.

Throughout the simulation, d was chosen to be $d = .1, .2, .3, .4$. Two sample lengths $n = 1000$ and $n = 5000$ were used for each scenario, and the number of replications was 1000 in each numerical experiment. Our choice of weight functions for the Δ_n-test are $\varphi_1 = \varphi_{(1)}$ and $\varphi_2 = \varphi_{(2)}$, where

$$\varphi_{(1)}(x) := 2I(x > 2/3) - I(x \le 2/3), \quad \varphi_{(2)}(x) := 2I(x < 1/3) - I(x \ge 1/3), \quad x \in [0,1].$$

This choice satisfies (2.21) and (2.22). For the κ_n-test, we chose $\varphi = \varphi_{(0)}$, where

$$\varphi_{(0)}(x) \;:=\; I(x > 1/2) - I(x \le 1/2), \qquad x \in [0,1].$$

We used the bandwidth $q \approx n^{.6}$ for computation of estimated long-run variances and covariances $\widetilde{v}_n(\varphi)$, $\widehat{\rho}_n(\varphi_1, \varphi_2)$ given at the (3.4) and (3.5). Throughout the simulation, $\tilde{d}$ was taken to be the local Whittle estimator with the bandwidth $m = n^{.6}$. In all tables, α denotes the asymptotic size.

$\mathcal{H}_0$		$\alpha = .1$				$\alpha = .05$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
Δ_n	1000	0.175	0.129	0.139	0.119	0.080	0.068	0.077	0.052
	5000	0.177	0.128	0.125	0.119	0.096	0.066	0.067	0.060
κ_n	1000	0.399	0.239	0.167	0.122	0.239	0.133	0.083	0.056
	5000	0.307	0.159	0.118	0.100	0.179	0.075	0.056	0.054

Table 1: Empirical size of the Δ_n- and κ_n-tests with $\alpha = .1$ and .05 under $Null_1$ as d and n vary.

Table 1 shows reasonable agreement of the the empirical sizes of the Δ_n-test with its asymptotic size except for the case $d = .1$. A similar thing can be said about the κ_n-test when $n = 5000$ only and the general impression is that the empirical sizes for the latter test tend to be much larger than those for the former test.

From Table 2 we see that while the size of the Δ_n-test does not change much under $Null_2$, as compared to the case of $Null_1$ in Table 1, the empirical size for κ_n has significantly increased for all values of d. This very large size of the κ_n-test seems to be the result of the difficulty of estimating d and $v(\varphi)$ in the presence of additive i.i.d. noise in the ARFIMA$(0,d,0)$ process.

$\mathcal{H}_0$		$\alpha = .1$				$\alpha = .05$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
Δ_n	1000	0.190	0.173	0.157	0.122	0.102	0.087	0.081	0.057
	5000	0.166	0.163	0.115	0.128	0.084	0.084	0.060	0.066
κ_n	1000	0.589	0.502	0.383	0.296	0.396	0.325	0.249	0.177
	5000	0.523	0.391	0.306	0.230	0.365	0.240	0.198	0.145

Table 2: Empirical size of the Δ_n- and κ_n-tests with $\alpha = .1$ and .05 under $Null_2$ as d and n vary.

$\mathcal{H}_1$		$\alpha = .1$				$\alpha = .05$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
Δ_n	1000	0.653	0.427	0.258	0.180	0.351	0.228	0.129	0.085
	5000	0.900	0.541	0.315	0.154	0.594	0.298	0.171	0.072
κ_n	1000	1.000	0.994	0.737	0.284	1.000	0.984	0.576	0.154
	5000	1.000	1.000	0.931	0.244	1.000	1.000	0.784	0.149

Table 3: Empirical power of the Δ_n- and κ_n-tests with $\alpha = .1$ and .05 under Alt_1 scenario as d and n vary.

Table 3 reports the empirical powers of the two tests under the Alt_1 scenario. The κ_n-test shows a higher power. However, due to its very high empirical size, as seen in Table 2, this fact cannot be readily taken as a sign of good performance of this test.

To see how 'fine tuning' of the bandwidth m of the local Whittle estimator affects the empirical size of κ_n-test, we used $m = n^a$, for $a = .4, .5, .6, .7, .8$, to compute the empirical size of this test under $Null_1$ scenario. From Koul et al. (2013) we recall that the bandwidth m needs to satisfy are $m \to \infty$ and $m/n \to 0$. Table 4 lists root mean squared error (RMSE) of local Whittle estimator $\hat{d}$ of d, and the resulting empirical size for these values of m. From this table, it is apparent that the empirical size is not much affected by the choice of m, and the improvement in RMSE of $\hat{d}$ does not result in the improved empirical size of κ_n-test. It seems that the inefficiency of the estimator of the long-run variance is causing the large empirical sizes for this test for relatively small values of d.

Robustness of the Δ_n-test against the choice of q and $\varphi_i(x), i = 1, 2$. Table 5 shows that the Δ_n-test is quite robust against the chosen choices of q. On the other hand, this test is sensitive to the choice of weight functions $\varphi_i(x), i = 1, 2$ as is seen in Table 6. In this table, the empirical size and the power of the Δ_n-test is computed under the $Null_1$

κ_n		RMSE($\hat{d}$)				$\alpha = .1$			
	m	d=.1	.2	.3	.4	d=.1	.2	.3	.4
	$n^{.4}$	0.093	0.111	0.113	0.093	0.286	0.128	0.093	0.084
	$n^{.5}$	0.068	0.072	0.075	0.062	0.269	0.118	0.070	0.070
	$n^{.6}$	0.042	0.043	0.042	0.039	0.307	0.159	0.118	0.100
	$n^{.7}$	0.028	0.026	0.028	0.026	0.333	0.166	0.126	0.113
	$n^{.8}$	0.017	0.017	0.018	0.017	0.351	0.181	0.144	0.120
	$n^{.9}$	0.013	0.018	0.023	0.030	0.417	0.209	0.171	0.137

Table 4: Effect of bandwidth m on RMSE($\hat{d}$) and empirical size of κ_n-test under $Null_1$, $n = 5000$, $\alpha = .1$.

Δ_n		$\alpha = .1$				$\alpha = .05$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
$q = n^{.4}$	1000	0.170	0.129	0.137	0.116	0.078	0.068	0.072	0.049
	5000	0.176	0.125	0.123	0.113	0.095	0.066	0.064	0.059
$q = n^{.7}$	1000	0.173	0.127	0.138	0.118	0.080	0.067	0.073	0.054
	5000	0.175	0.128	0.121	0.114	0.098	0.064	0.071	0.055

Table 5: Effect of bandwidth q on empirical size of Δ_n-test under $Null_1$, $\alpha = .1$ and .05. To be compared with the Δ_n entries in Table 1.

scenario, using smooth weight functions:

$$\varphi_{(3)}(x) = 3x^2 - 1, \quad \varphi_{(4)} = \varphi_{(3)}(1-x), \quad \varphi_{(5)}(x) = \sin(2\pi x - 1), \quad \varphi_{(6)} = \varphi_{(5)}(1-x).$$

Table 6 shows that the choice of weight functions $\varphi_1 = \varphi_{(3)}$, $\varphi_2 = \varphi_{(4)}$ and $\varphi_1 = \varphi_{(5)}$, $\varphi_2 = \varphi_{(6)}$ worsens the empirical size of the Δ_n-test as compared to the choice $\varphi_1 = \varphi_{(1)}$, $\varphi_2 = \varphi_{(2)}$ in Table 1.

Table 7 lists empirical 95% confidence interval for $\rho(\varphi_{(1)}, \varphi_{(2)})$, along with its theoretical values. These confidence intervals are obtained by taking 50th and 950th ordered realizations as lower and upper limits, respectively, from the 1000 iteration of the estimate $\widehat{\rho}_n(\varphi_{(1)}, \varphi_{(2)})$ that arise during the simulation. The table also lists the theoretical values of $\rho(\varphi_{(3)}, \varphi_{(4)})$ and $\rho(\varphi_{(5)}, \varphi_{(6)})$ for various values of d. The power of Δ_n-test for all the three choices of (φ_1, φ_2) decreases as d increases. Since value of ρ in the critical value $C(\rho)$ depends on d as well as on φ_1, φ_2, it is natural to suspect a direct relationship between the power and the values of ρ. However, the relationship seem not to be a straightforward one, because as in Table 7, values of $\rho(\varphi_{(1)}, \varphi_{(2)})$ and $\rho(\varphi_{(3)}, \varphi_{(4)})$ are quite different, but the powers of respective tests listed in Table 3 and Table 6 are quite similar. In the choice of (φ_1, φ_2), we

$\Delta_n(\varphi_{(3)}, \varphi_{(4)})$		$\alpha = .1$				$\alpha = .05$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
$\mathcal{H}_0$	1000	0.304	0.216	0.170	0.143	0.153	0.102	0.078	0.069
	5000	0.295	0.190	0.162	0.126	0.140	0.095	0.094	0.073
$\mathcal{H}_1$	1000	0.614	0.414	0.325	0.216	0.350	0.236	0.152	0.123
	5000	0.854	0.605	0.397	0.233	0.561	0.350	0.205	0.108
$\Delta_n(\varphi_{(5)}, \varphi_{(6)})$									
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
$\mathcal{H}_0$	1000	0.247	0.192	0.172	0.138	0.137	0.093	0.091	0.071
	5000	0.239	0.180	0.148	0.110	0.130	0.094	0.079	0.058
$\mathcal{H}_1$	1000	0.448	0.319	0.239	0.146	0.233	0.170	0.111	0.073
	5000	0.676	0.460	0.294	0.176	0.397	0.261	0.150	0.084

Table 6: Effect of φ's on empirical size of the Δ_n-test with $(\varphi_1, \varphi_2) = (\varphi_{(3)}, \varphi_{(4)})$ and $(\varphi_{(5)}, \varphi_{(6)})$, under $Null_1$ and Alt_1, as d and n vary. To be compared with Table 1.

recommend using $(\varphi_{(1)}, \varphi_{(2)})$, because of the sample sizes listed in Table 1 and 2. As seen in Table 6, smoother choice of (φ_1, φ_2) considered has resulted in worse size, and no gain in the power.

Δ_n	n	d=.1	.2	.3	.4
	1000	(-0.665, -0.426)	(-0.702, -0.491)	(-0.737, -0.527)	(-0.772, -0.570)
	5000	(-0.627, -0.461)	(-0.670, -0.505)	(-0.709, -0.553)	(-0.734, -0.595)
	$\rho(\varphi_{(1)}, \varphi_{(2)})$	-0.544	-0.585	-0.627	-0.667
	$\rho(\varphi_{(3)}, \varphi_{(4)})$	-0.888	-0.898	-0.909	-0.919
	$\rho(\varphi_{(5)}, \varphi_{(5)})$	0.384	0.344	0.305	0.265

Table 7: Empirical 95% confidence intervals for $\rho(\varphi_1, \varphi_2)$, for $(\varphi_1, \varphi_2) = (\varphi_{(1)}, \varphi_{(2)})$, as d varies, and theoretical values of $\rho(\varphi_1, \varphi_2)$, for $(\varphi_1, \varphi_2) = (\varphi_{(1)}, \varphi_{(2)})$, $(\varphi_{(3)}, \varphi_{(4)})$ and $(\varphi_{(5)}, \varphi_{(6)})$.

Case $\nu = 2$. Here we generated random field $Y_{t_1,t_2} = 3 + 2X_{t_1,t_2}$, $(t_1, t_2) \in A_n = [1, n]^2$ for $n = 150$ and $n = 300$, where $X_{t_1,t_2} = H_{t_1,t_2}/\sqrt{EH^2_{t_1,t_2}}$ and H_{t_1,t_2} are given as follows in the two different null hypotheses settings.

$$Null_3: \qquad H_{t_1,t_2} = U_{t_1,t_2}, \text{where } \{U_{t_1,t_2}\} \text{ is a truncated moving-average random field}$$

$$U_{t_1,t_2} = \sum_{s_1=t_1-1000}^{t_1+1000} \sum_{s_2=t_2-1000}^{t_2+1000} b_{t_1-s_1,t_2-s_2}\zeta_{s_1,s_2}$$

with i.i.d. standardized normal r.v.'s $\{\zeta_{s_1,s_2}\}$,

$Null_4$: $H_{t_1,t_2} = U_{t_1,t_2} + Z_{t_1,t_2}$, where $\{U_{t_1,t_2}\}$ is as in $Null_3$, and $\{Z_{t_1,t_2}\}$ is i.i.d. random field with $N(0,4)$ innovations and independent of $\{U_{t_1,t_2}\}$,

Alt_2 : H_t is truncated moving-average field with $\{\zeta_{s_1,s_2}\}$ being i.i.d. centered-exponential innovations.

Moving-average coefficients in U_{t_1,t_2} were set to $b_{0,0} := 1, b_{t_1,t_2} = 0.25(|t_1|^2 + |t_2|^2)^{(d-2)/2}$, $(t_1,t_2) \neq (0,0)$ satisfying (1.2) with $B(t/|t|) = .25$, for $d = .2, .4, .6$, and $.8$. For Δ_n-test, we used the statistic as in (1.10), with

$$\varphi_1(x) = 2I(x_1 > 2/3) - I(x_1 \le 2/3), \qquad \varphi_2(x) = 2I(x_1 < 1/3) - I(x_1 \ge 1/3),$$

for $x = (x_1,x_2) \in [0,1]^2$, and the bandwidth parameter $q = n^{.6}$. For κ_n-test, the statistic in (6.1) was used with $\varphi(x) = I(x_1 > 1/2) - I(x_1 \le 1/2)$. The long memory parameter d was estimated using GPH estimator as described in Wang (2009), with bandwidth set as $n^{.6}$. For each scenario, 500 replications were used.

The simulation results for the three cases $Null_3$, $Null_4$, and Alt_2 are presented in Tables 8, 9, and 10, respectively. From Tables 8 and 9, we see that the empirical sizes of Δ_n-test are much closer to the asymptotic sizes than those of the κ_n-test at $n = 300$ for both $Null_3$, $Null_4$ hypotheses. From Table 10, one sees that in the case of $\nu = 2$, unlike in the case of $\nu = 1$ (see Table 2) the κ_n-test does not have large empirical size in the presence of additive noise in $Null_4$, yet its empirical power at Alt_2 is considerably larger than that of the Δ_n-test for $d = .2, .4, .6$. At $d = .8$, the Δ_n-test appears to be dominating somewhat.

$\mathcal{H}_0$		$\alpha = .1$				$\alpha = .05$			
	n	d=.2	.4	.6	.8	d=.2	.4	.6	.8
Δ_n	150 × 150	0.170	0.160	0.130	0.166	0.082	0.078	0.052	0.096
	300 × 300	0.114	0.120	0.120	0.112	0.072	0.056	0.058	0.048
κ_n	150 × 150	0.064	0.066	0.068	0.074	0.032	0.034	0.040	0.034
	300 × 300	0.052	0.082	0.080	0.066	0.018	0.042	0.028	0.036

Table 8: Empirical size of the Δ_n- and κ_n-tests with $\alpha = .1$ and $.05$ under $Null_3$ as d and n vary.

When $v(\varphi_1) \neq v(\varphi_2)$. Lastly, we investigated the performance of the test when $v(\varphi_1) \neq v(\varphi_2)$ for $\nu = 1$ case. In this case the test statistic is $\widetilde{\Delta}_n$ of (1.16). We ran the simulation with the weight functions $(\varphi_1,\varphi_2) = (\varphi_{(1)},\varphi_{(4)})$ and $(\varphi_{(1)},\varphi_{(6)})$, under $Null_1$ scenario. Table

$\mathcal{H}_0$		$\alpha = .1$				$\alpha = .05$			
	n	d=.2	.4	.6	.8	d=.2	.4	.6	.8
Δ_n	150 × 150	0.136	0.132	0.100	0.136	0.062	0.064	0.044	0.064
	300 × 300	0.146	0.128	0.104	0.112	0.094	0.060	0.052	0.052
κ_n	150 × 150	0.138	0.138	0.142	0.156	0.078	0.064	0.086	0.106
	300 × 300	0.114	0.124	0.128	0.132	0.058	0.060	0.074	0.072

Table 9: Empirical size of the Δ_n- and κ_n-tests with $\alpha = .1$ and .05 under $Null_4$ as d and n vary.

$\mathcal{H}_A$		$\alpha = .1$				$\alpha = .05$			
	n	d=.2	.4	.6	.8	d=.2	.4	.6	.8
Δ_n	150 × 150	0.508	0.330	0.268	0.190	0.276	0.146	0.146	0.094
	300 × 300	0.716	0.528	0.352	0.220	0.376	0.262	0.182	0.110
κ_n	150 × 150	0.976	0.814	0.320	0.156	0.952	0.634	0.174	0.072
	300 × 300	1.000	0.992	0.632	0.208	1.000	0.954	0.438	0.128

Table 10: Empirical power of the Δ_n- and κ_n-tests with $\alpha = .1$ and .05 under Alt_2 as d and n vary.

11 shows the results, which are mixed. The behavior of the empirical sizes of $\widetilde{\Delta}_n$ test is similar to those of the Δ_n test corresponding to $(\varphi_1, \varphi_2) = (\varphi_{(1)}, \varphi_{(2)})$ as seen in Table 1.

		$\widetilde{\Delta}_n$ $(\varphi_{(1)}, \varphi_{(4)})$				$\widetilde{\Delta}_n$ $(\varphi_{(1)}, \varphi_{(6)})$			
	n	d=.1	.2	.3	.4	d=.1	.2	.3	.4
$\alpha = .1$									
$\mathcal{H}_0$	1000	0.185	0.121	0.114	0.109	0.179	0.133	0.140	0.120
	5000	0.168	0.126	0.127	0.120	0.165	0.135	0.134	0.095
$\alpha = .05$									
$\mathcal{H}_0$	1000	0.090	0.067	0.054	0.047	0.104	0.063	0.073	0.060
	5000	0.085	0.067	0.070	0.058	0.083	0.077	0.056	0.047

Table 11: Empirical size of the $\widetilde{\Delta}_n$ test with $\alpha = .1$ (top) and .05 for ARFIMA(0,d,0) as d and n vary, and q is changed to $n^{.4}$ (left) and $n^{.7}$.

Table 12 shows 95% empirical confidence intervals for $\theta := v(\varphi_2)/v(\varphi_1)$ based on the estimate $\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1)$, obtained from simulation, along with the theoretical values of θ under $Null_1$, obtained by using (1.8) with $R_0 = B(d, 1-2d)/\Gamma(d)$. These confidence intervals are constructed by collecting the 1000 independent realization of the estimate $\widetilde{v}_n(\varphi_2)/\widetilde{v}_n(\varphi_1)$

that arise during the simulation, and taking 50th and 950th ordered realizations as lower and upper limits, respectively.

Δ_n	n	d=.1	.2	.3	.4
$(\varphi_{(1)}, \varphi_{(4)})$	1000	(0.379, 0.441)	(0.386, 0.449)	(0.402, 0.462)	(0.411, 0.471)
	5000	(0.384, 0.434)	(0.397, 0.445)	(0.409, 0.456)	(0.423, 0.467)
	θ	0.418	0.423	0.434	0.445
$(\varphi_{(1)}, \varphi_{(6)})$	1000	(0.205, 0.288)	(0.201, 0.287)	(0.191, 0.270)	(0.182, 0.264)
	5000	(0.216, 0.280)	(0.210, 0.272)	(0.201, 0.263)	(0.190, 0.251)
	θ	0.251	0.239	0.230	0.219

Table 12: Empirical 95% confidence intervals for $\theta := v(\varphi_2)/v(\varphi_1)$, $(\varphi_1, \varphi_2) = (\varphi_{(1)}, \varphi_{(4)})$ and $(\varphi_{(1)}, \varphi_{(6)})$ with the theoretical values of θ.

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