Asymptotic distribution of the bias corrected LSEs in measurement error linear regression models under long memory[1]

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Abstract

This paper derives the consistency and asymptotic distribution of the bias corrected least squares estimators (LSEs) of the regression parameters in linear regression models when covariates have measurement error and errors and covariates form mutually independent long memory moving average processes. In the structural measurement error linear regression model, the nature of the asymptotic distribution of suitably standardized bias corrected LSEs depends on the range of the values of $D_{\max} = \max\{d_X + d_\varepsilon, d_X + d_u, d_u + d_\varepsilon, 2d_u\}$, where d_X, d_u, and d_ε are the long memory parameters of the covariate, measurement error and regression error processes, respectively. This limiting distribution is Gaussian when $D_{\max} < 1/2$ and non-Gaussian in the case $D_{\max} > 1/2$. In the former case some consistent estimators of the asymptotic variances of these estimators and a $\log(n)$-consistent estimator of an underlying long memory parameter are also provided. They are useful in the construction of the large sample confidence intervals for regression parameters. The paper also discusses the asymptotic distribution of these estimators in some functional measurement error linear regression models, where the unobservable covariate is non-random. In these models, the limiting distribution of the bias corrected LSEs is always a Gaussian distribution determined by the range of the values of $d_\varepsilon - d_u$.

1 Introduction

The classical regression analysis often assumes that both the response variable and the predicting variables are fully observable and that the errors are independent. But, as is evidenced in the monographs of Fuller (1987), Cheng and Van Ness (1999), Carroll, Ruppert, Stefanski and Craineceanu (2006), and the references therein, there are numerous examples of practical importance where the predicting variables are not observable. Instead one observes surrogates that provide estimates of the true predictors. Such models are known as the regression models with measurement error. On the other hand there are examples from the various scientific disciplines where observed data do not obey the assumption of

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independence. Instead one observes data that are generated by some long memory (LM) processes. In economics the first authors to point out the usefulness of these processes were Granger and Joyeux (1980) and Hosking (1981). The monographs of Giraitis, Koul and Surgailis (2012) and Beran, Feng, Ghosh and Kulik (2013), and the references therein, contain numerous other examples of LM processes and relevant theoretical results.

The focus of this paper is to study the consistency and asymptotic distribution theory of the bias corrected LSEs of the parameters in linear regression models when predicting variables have measurement error (ME) and when the covariate, the regression error and measurement error processes have LM. We discuss both structural and functional models. In the former, the predicting variables are random while in the latter they are non-random.

An example of a simple structural ME linear regression model with LM in regression and measurement error processes is provided by the so called Phillips Curve, Phillips (1958), which uses the unemployment rate to predict the inflation rate. The unemployment rate is known to have long memory and measurement error. See Blanchard and Summers (9187) and Shiskin and Stein (1975). Another example is provided in finance when studying uncovered interest parity, where forward premium, used as a predicting variable, is known to have measurement error and long memory, and where the regression errors are also known to have long memory. See Cornell (1989), Bekaert and Hodrick (1993) and Baillie (1996). The results of the current paper would be applicable to these models.

For the sake of transparency, we first discuss the simple structural measurement error (ME) linear regression model in the next section, where the long memory moving average (LMMA) models along with the needed assumptions are also described. It also contains the proof of the consistency of the bias corrected LSEs in this model. The derivation of the asymptotic distribution of suitably standardized versions of these estimators is facilitated by the derivation of the limiting distributions of some general quadratic forms of LMMA processes given in Section 3. These results in turn are used in Sections 4 and 5 to derive the limiting distributions of the bias corrected LSEs in the simple and multiple structural ME linear regression models, respectively. Section 6 contains similar results for the functional ME simple linear regression model where the true unobservable predicting variable is nonrandom.

These limiting distributions are non Gaussian when $D_{\max} > 1/2$ and Gaussian when $D_{\max} < 1/2$. The results in the latter case are used to construct asymptotic confidence intervals for the underlying regression parameters in Remark 4.2, where we also provide HAC estimators of asymptotic variances of the bias corrected LSEs and a $\log(n)$-consistent estimator of an underlying long memory parameter that are needed for the construction of these intervals. Section 7 contains the proofs of the main results of Sections 3 and 4. The proof of the consistency of a residuals based HAC estimator under the current set up appears in the supplement [Koul and Surgailis (2018)] to this paper.

2 Simple structural ME linear regression model

In this section we shall focus on the simple structural ME linear regression model and establish the consistency of the bias corrected LSEs. In this model the unobserved predicting r.v. X_i, the observable random surrogate Z_i and the response Y_i are related to each other by the following relations. For some real numbers α, β,

$$(2.1) \quad Y_i = \alpha + \beta X_i + \varepsilon_i, \quad Z_i = X_i + u_i, \quad E\varepsilon_i = 0, \quad Eu_i = 0, \quad i \in \mathbb{Z} := \{0, \pm 1, \cdots\}.$$

Moreover, we assume that the process $\{(\varepsilon_i, X_i, u_i); i \in \mathbb{Z}\}$ is strictly stationary and ergodic and each of these processes form a LMMA as in the following assumptions.

Assumption (E) Errors $\{\varepsilon_i\}$ form a moving average process

$$(2.2) \qquad \varepsilon_i = \sum_{k=0}^{\infty} b_k \zeta_{i-k}, \quad i \in \mathbb{Z},$$

where $\{\zeta_s; s \in \mathbb{Z}\}$ are i.i.d., with zero mean and unit variance, with coefficients

$$(2.3) \qquad b_j \sim \kappa_\varepsilon \, j^{-(1-d_\varepsilon)}, \quad \text{as } j \to \infty, \text{ for some } 0 < \kappa_\varepsilon < \infty \text{ and } 0 < d_\varepsilon < 1/2.$$

Assumption (X) Covariates $\{X_i\}$ form a LMMA process

$$(2.4) \quad X_i = \mu_X + \sum_{k=0}^{\infty} a_k \xi_{i-k}, \quad i \in \mathbb{Z}, \quad \text{with MA coefficients} \quad a_j \sim \kappa_X j^{-(1-d_X)}, \ j \to \infty,$$

for some $\mu_X \in \mathbb{R}, \ \kappa_X > 0, 0 < d_X < 1/2$, and standardized i.i.d. innovations $\{\xi_s\}$.

Assumption (U) Measurement errors $\{u_i\}$ form a LMMA process

$$(2.5) \qquad u_i = \sum_{k=0}^{\infty} c_k \eta_{i-k}, \quad i \in \mathbb{Z}, \quad \text{with MA coefficients} \quad c_j \sim \kappa_u j^{-(1-d_u)}, \ j \to \infty$$

for some $\kappa_u > 0, 0 < d_u < 1/2$, and standardized i.i.d. innovations $\{\eta_s\}$. Moreover, $\sigma_{u0}^2 :=$ Var(u_0) is known.

Assumption (I) The innovation sequences $\{\zeta_s; s \in \mathbb{Z}\}$, $\{\xi_s; s \in \mathbb{Z}\}$ and $\{\eta_s; s \in \mathbb{Z}\}$ are mutually independent.

From now on let $\varepsilon, X, u, \zeta, \xi, \eta$ denote copies of $\varepsilon_0, X_0, u_0, \zeta_0, \xi_0, \eta_0$, respectively. For any r.v. U with finite variance, let $\sigma_U^2 :=$ Var(U). We also let $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$, $a > 0, b > 0$.

The above assumptions imply that for each $i \in \mathbb{Z}$, the r.v.'s ε_i, X_i, u_i are mutually independent and

$$0 < \sigma_\varepsilon^2 = E\varepsilon^2 = \sum_{k=0}^{\infty} b_k^2 < \infty, \quad 0 < \sigma_X^2 = EX^2 = \sum_{k=0}^{\infty} a_k^2 < \infty.$$
$$0 < \sigma_u^2 = Eu^2 = \sum_{k=0}^{\infty} c_k^2 < \infty.$$

From (7.2.10) of Giraitis, Koul and Surgailis (2012) (GKS), we obtain that

$$\text{(2.6)}\quad \mathrm{Cov}(\varepsilon_0, \varepsilon_k) \sim \kappa_\varepsilon^2 B(d_\varepsilon, 1-2d_\varepsilon) k^{-(1-2d_\varepsilon)}, \quad \mathrm{Cov}(X_0, X_k) \sim \kappa_X^2 B(d_X, 1-2d_X) k^{-(1-2d_X)},$$
$$\mathrm{Cov}(u_0, u_k) \sim \kappa_u^2 B(d_u, 1-2d_u) k^{-(1-2d_u)}, \quad k \to \infty.$$

The sums of the absolute values of each of these covariances diverge, which implies that each of the processes $\{\varepsilon_i\}, \{X_i\}$ and $\{u_i\}$ has long memory.

To proceed further, for any two sets of variables $U_i, V_i, 1 \le i \le n$, let

$$\bar{U} := n^{-1} \sum_{i=1}^n U_i, \quad S_{UV} := n^{-1} \sum_{i=1}^n (U_i - \bar{U})(V_i - \bar{V}).$$

In the sequel, all limits are taken as $n \to \infty$, unless mentioned otherwise.

The naive LSEs of α, β, where one simply replaces X_i's in the classical LSE by Z_i's, are $\tilde{\beta} := S_{ZY}/S_{ZZ}$, $\tilde{\alpha} := \bar{Y} - \tilde{\beta}\bar{Z}$. As argued say in Fuller (1987), under the classical i.i.d. and finite variance set up, $\tilde{\beta} - \beta \to -\beta\, \sigma_u^2/(\sigma_X^2 + \sigma_u^2)$, a.s. Hence these estimators are inconsistent. The bias correct estimators suitable here are

$$\text{(2.7)}\qquad \widehat{\beta} := S_{ZY}/(S_{ZZ} - \sigma_u^2), \qquad \widehat{\alpha} := \bar{Y} - \widehat{\beta}\bar{Z}.$$

We shall first establish the consistency of these estimators under the assumed stationarity, ergodicity and long memory set up. Rewrite $Y_i = \alpha + \beta Z_i + \varepsilon_i - \beta u_i$, $Z_i = X_i + u_i$. Let

$$T_n := n^{-1} \sum_{i=1}^n (Z_i - \bar{Z})(\varepsilon_i - \beta u_i).$$

Use the relation $Z_i = X_i + u_i$, to obtain the decomposition

$$\text{(2.8)}\qquad \begin{aligned} T_n &= n^{-1} \sum_{i=1}^n (X_i - \bar{X})(\varepsilon_i - \beta u_i) + n^{-1} \sum_{i=1}^n (u_i - \bar{u})\varepsilon_i - \beta n^{-1} \sum_{i=1}^n (u_i - \bar{u})^2 \\ &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta S_{uu}. \end{aligned}$$

By the mutual independence of ε_i, X_i, u_i and the assumption that $E\varepsilon_i \equiv 0$, $Eu_i \equiv 0$,

$$E(T_n) = -\beta E(S_{uu}) = -\beta \left[\sigma_u^2 - \mathrm{Var}(\bar{u})\right].$$

By (2.6), $\mathrm{Var}(\bar{u}) = O(n^{2d_u - 1}) \to 0$ and by the Ergodic Theorem and the assumed stationarity,

$$T_n \to -\beta\sigma_u^2, \qquad S_{ZZ} - \sigma_u^2 \to \sigma_X^2 > 0, \quad \text{a.s.}$$

These facts now clearly imply that

$$\text{(2.9)}\qquad \widehat{\beta} - \beta = \frac{S_{ZY}}{S_{ZZ} - \sigma_u^2} - \beta = \frac{T_n + \beta\sigma_u^2}{S_{ZZ} - \sigma_u^2} \to 0, \quad \text{a.s.},$$

thereby proving the strong consistency of $\widehat{\beta}$ for β. This fact and the Ergodic Theorem in turn imply that $\widehat{\alpha} \to \alpha$, a.s.

The derivation of the asymptotic distributions of suitably standardized versions of these estimators and their analogs in multiple linear regression model is facilitated by the more general asymptotic distributional results about certain quadratic forms established in the next section.

3 Limit theorem for quadratic forms

Let $\gamma_{t,i} = \sum_{k=0}^{\infty} b_{k,i}\xi_{t-k,i}, t \in \mathbb{Z}, i = 1, \cdots, m$ be m mutually independent LMMA processes with MA coefficients $b_{k,i} \sim \kappa_i k^{d_i - 1}$, $d_i \in (0, 1/2), \kappa_i > 0$ with i.i.d. mutually independent innovations $\{\xi_{s,i}\} \sim \text{IID}(0,1), i = 1, \cdots, m$. Let $\Pi_m \subset \{(i,j); 1 \le i \le j \le m\}$ be a non-empty subset of the set of all ordered pairs $(i,j), 1 \le i \le j \le m$ and $\gamma_i := \{\gamma_{t,i};\ t \in \mathbb{Z}\}$. Define the sample cross-covariance between γ_i and γ_j to be

$$S_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^{n}(\gamma_{t,i} - \bar{\gamma}_i)(\gamma_{t,j} - \bar{\gamma}_j), \quad (i,j) \in \Pi_m.$$

We also need to define the normalizing sequence as follows.

$$\delta_{\max} := \max\{d_i + d_j; (i,j) \in \Pi_m\}, \tag{3.1}$$

$$A(n) := \begin{cases} n^{1-\delta_{\max}}, & \delta_{\max} > 1/2, \\ n^{1/2}, & \delta_{\max} < 1/2, \\ (n/\log n)^{1/2}, & \delta_{\max} = 1/2. \end{cases}$$

We are interested in deriving the asymptotic joint distribution of normalized quadratic forms

$$\mathcal{S}_n := \big\{A(n)\big(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}\big);\ (i,j) \in \Pi_m\big\}. \tag{3.2}$$

As shown below, the limit distribution of $\mathcal{S}_n$ is Gaussian or non-Gaussian depending on whether $\delta_{\max} \le 1/2$ or $\delta_{\max} > 1/2$. Before describing this distribution, we need to recall some preliminaries. From GKS, pp.410-411, we recall the definition of the stochastic integrals

$$I_i(f) = \int_{\mathbb{R}} f(s)W_i(ds), \qquad I_{ij}(g) = \int_{\mathbb{R}^2} g(s_1,s_2)W_i(ds_1)W_j(ds_2) \tag{3.3}$$

w.r.t. independent Brownian motions $W_i, i = 1, \cdots, m$ (for $i = j$ the second integral in (3.3) coincides with the usual double Wiener-Itô integral w.r.t. W_i). The integrals $I_i(f), I_{ij}(g)$, $(i,j) \in \Pi_m$ are jointly defined for any non-random integrands $f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2)$. Moreover, $EI(f) = EI_{ij}(g) = 0$ and

$$\begin{aligned} EI_i(f)I_{i'}(f') &= \begin{cases} 0, & i \ne i', \\ \langle f, f'\rangle, & i = i', \end{cases} \quad f, f' \in L^2(\mathbb{R}), \\ EI_i(f)I_{i'j'}(g) &= 0, \qquad \forall i, i', j', \quad f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2), \end{aligned} \tag{3.4}$$

$$EI_{ij}(g)I_{i'j'}(g') = \begin{cases} 0, & (i,j) \neq (i',j'), \\ \langle g, g' \rangle, & (i,j) = (i',j'),\ i \neq j, \qquad g, g' \in L^2(\mathbb{R}^2), \\ 2\langle g, \mathrm{sym} g' \rangle, & i = i' = j = j', \end{cases}$$

where $\langle f, f' \rangle = \int_{\mathbb{R}} f(s) f'(s) ds$ ($\|f\| := \sqrt{\langle f', f \rangle}$), $\langle g, g' \rangle = \int_{\mathbb{R}^2} g(s_1,s_2) g'(s_1,s_2) ds_1 ds_2$ ($\|g\| := \sqrt{\langle g, g \rangle}$) denote scalar products (norms) in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, and sym denotes the symmetrization, see GKS, sections 11.5 and 14.3.

Let $\Pi_m^+ := \{(i,j) \in \Pi_m; d_i + d_j > 1/2\}$. Introduce

$$
\begin{aligned}
(3.5) \qquad f_{d_i}(s) &:= \kappa_i \int_0^1 (t-s)_+^{d_i - 1} dt, \qquad 1 \le i \le m, \\
\widetilde{g}_{d_i,d_j}(s_1,s_2) &:= \kappa_i \kappa_j \int_0^1 (t-s_1)_+^{d_i-1} (t-s_2)_+^{d_j-1} dt, \\
g_{d_i,d_j}(s_1,s_2) &:= \widetilde{g}_{d_i,d_j}(s_1,s_2) - f_{d_i}(s_1) f_{d_j}(s_2), \qquad (i,j) \in \Pi_m^+.
\end{aligned}
$$

Then $f_{d_i} \in L^2(\mathbb{R})$, $\widetilde{g}_{d_i,d_j} \in L^2(\mathbb{R}^2)$, $g_{d_i,d_j} \in L^2(\mathbb{R}^2)$, see GKS, Prop.11.5.6. Observe that

$$
\begin{aligned}
(3.6) \quad & \langle \widetilde{g}_{d_i,d_j}, f_{d_i} \otimes f_{d_j} \rangle / \kappa_i^2 \kappa_j^2 \\
&= \int_{\mathbb{R}^2} ds_1 ds_2 \int_0^1 (t-s_1)_+^{d_i-1} (t-s_2)_+^{d_j-1} dt \int_0^1 (t_1 - s_1)_+^{d_i-1} dt_1 \int_0^1 (t_2 - s_2)_+^{d_j-1} dt_2 \\
&= \int_{(0,1]^3} dt dt_1 dt_2 \int_{\mathbb{R}} (t-s_1)_+^{d_i-1} (t_1-s_1)_+^{d_i-1} ds_1 \int_{\mathbb{R}} (t-s_2)_+^{d_j-1} (t_2-s_2)_+^{d_j-1} ds_2 \\
&= \frac{B(d_i, 1-2d_i) B(d_j, 1-2d_j)}{2 d_i d_j} \Big(\frac{1}{1 + 2(d_i + d_j)} + B(2d_i + 1, 2d_j + 1) \Big).
\end{aligned}
$$

In a similar way,

$$
\begin{aligned}
(3.7) \qquad \|f_{d_i}\|^2 &= \kappa_i^2 \int_{(0,1]^2} dt_1 dt_2 \int_{\mathbb{R}} (t_1 - s)_+^{d_i-1} (t_2 - s)_+^{d_i-1} ds \\
&= \kappa_i^2 B(d_i, 1-2d_i) \int_{(0,1]^2} |t_1 - t_2|^{2d_i - 1} dt_1 dt_2 = \frac{\kappa_i^2 B(d_i, 1-2d_i)}{d_i(1+2d_i)},
\end{aligned}
$$

and

$$
\begin{aligned}
(3.8) \quad & \|\widetilde{g}_{d_i,d_j}\|^2 \\
&= \kappa_i^2 \kappa_j^2 \int_{(0,1]^2} dt_1 dt_2 \int_{\mathbb{R}} (t_1 - s_1)_+^{d_i-1} (t_2 - s_1)_+^{d_i-1} ds_1 \int_{\mathbb{R}} (t_1 - s_2)_+^{d_j-1} (t_2 - s_2)_+^{d_j-1} ds_2 \\
&= \frac{\kappa_i^2 \kappa_j^2 B(d_i, 1-2d_i) B(d_j, 1-2d_j)}{(d_i + d_j)(2(d_i + d_j) - 1)}.
\end{aligned}
$$

From (3.6), (3.7), and (3.8) we obtain

$$
\begin{aligned}
(3.9)\qquad & \|g_{d_i,d_j}\|^2 \\
&= \|\widetilde{g}_{d_i,d_j}\|^2 - 2\langle \widetilde{g}_{d_i,d_j}, f_{d_i}\otimes f_{d_j}\rangle + \|f_{d_i}\|^2\|f_{d_j}\|^2 \\
&= \kappa_i^2\kappa_j^2 B(d_i,1-2d_i)B(d_j,1-2d_j)\Big\{\frac{1}{(d_i+d_j)(2(d_i+d_j)-1)} \\
&\quad + \frac{1}{d_id_j(1+2d_i)(1+2d_j)} - \frac{1}{d_id_j(1+2(d_i+d_j))} - \frac{B(2d_i+1,2d_j+1)}{d_id_j}\Big\}.
\end{aligned}
$$

Consequently, the r.v.'s $I_{ij}(g_{d_i,d_j}),\ d_i+d_j>1/2$ in (3.13) below are jointly well-defined and their second order characteristics can be obtained from (3.4) and (3.9).

We are now ready to state the main result of this section. Its proof appears in Section 7.

Theorem 3.1 *Let $\gamma_i = \{\gamma_{t,i};\, t\in\mathbb{Z}\}$, $i=1,\cdots,m$, be m stationary LMMA processes as above and $\mathcal{S}_n$ be as in (3.2). In addition, assume the following two conditions hold.*

$$(3.10)\qquad E|\xi_{0,i}|^{2+\epsilon} < \infty, \quad (\exists\, \epsilon>0) \quad \textit{for all } 1\le i\le m,$$

$$(3.11)\qquad E\xi_{0,i}^4 < \infty, \qquad \textit{for any } 1\le i\le m \textit{ such that } (i,i)\in\Pi_m.$$

Then

$$(3.12)\qquad \mathcal{S}_n \to_D \mathcal{R}_m = \big\{R_{ij};\, (i,j)\in\Pi_m\big\},$$

where, for any $(i,j)\in\Pi_m$,

$$(3.13)\qquad R_{ij} := \begin{cases} I_{ij}(g_{d_i,d_j})\mathbf{1}(d_i+d_j=\delta_{\max}), & \delta_{\max}>1/2,\\ \sigma_{ij}Z_{ij}\mathbf{1}(d_i+d_j=1/2), & \delta_{\max}=1/2,\\ \sigma_{ij}Z_{ij}, & \delta_{\max}<1/2,\end{cases}$$

with $g_{d_i,d_j}\in L^2(\mathbb{R}^2)$, $f_i\in L^2(\mathbb{R})$ as in (3.5), $\sigma_{ij}\ge 0$ as in (7.5) below, and Z_{ij} as independent $N(0,1)$ r.v.'s, $EZ_{ij}Z_{i'j'}=0$, for $(i,j)\ne(i',j'),(i,j),(i',j')\in\Pi_m$.

Remark 3.1 The literature on limit theorems for quadratic forms in dependent r.v.'s is large. See e.g., Bhansali et al. (2007), GKS, Ch.6, and the references therein. Theorem 3.1 deals with rather special 'diagonal' quadratic forms in LMMA processes. It extends various central and noncentral limit results in GKS (Thms 4.8.1, 4.8.2, Prop.11.5.5) to joint convergence of array (3.2) involving LMMA processes with different memory parameters. We also note the proof of Theorem 3.1 largely relies on variance or L^2 considerations and does not require using higher moments or other advanced mathematical tools.

Let $\Pi_{0m} \subset \{1, \cdots, m\}$ be a non-empty set, $d_{\max} := \max\{d_k; k \in \Pi_{0m}\}$ and $\mathcal{S}_{0n} := \{n^{(1/2)-d_{\max}}\bar{\gamma}_k; k \in \Pi_{0m}\}$ be a collection of normalized sample means. Then from Remark 4.3.1 in GKS, we obtain

$$
\begin{aligned}
\mathcal{S}_{0n} \to_D \mathcal{R}_{0m} = \{R_{0k}, k \in \Pi_{0m}\} \quad &:= \quad \{I_k(f_{d_k})\mathbf{1}(d_k = d_{\max}); k \in \Pi_{0m}\} \\
&=_D \quad \{\sigma_k Z_k \mathbf{1}(d_k = d_{\max}); k \in \Pi_{0m}\},
\end{aligned} \tag{3.14}
$$

where $Z_k, k \in \Pi_{0m}$ are independent $N(0,1)$ r.v.'s and $\sigma_k^2 = \|f_{d_k}\|^2$ as in (3.9). The following corollary extends Theorem 3.1 to joint convergence of normalized sample means $\mathcal{S}_{0n}$ and sample cross-covariances $\mathcal{S}_n$.

Corollary 3.1 *Under the assumptions of Theorem 3.1,*

$$
(\mathcal{S}_{0n}, \mathcal{S}_n) \to_D (\mathcal{R}_{0m}, \mathcal{R}_m). \tag{3.15}
$$

The joint distribution of $(\mathcal{R}_{0m}, \mathcal{R}_m)$ is Gaussian if $\delta_{\max} \le 1/2$. Moreover, for any $k \in \Pi_{0m}$, $(i,j) \in \Pi_m$,

$$
E(R_{0k}R_{ij}) = \begin{cases} (\kappa_k/d_k(1+d_k))E(\xi_{0,k}\xi_{0,i}\xi_{0,j}) \sum_{s=0}^{\infty} b_{s,i}b_{s,j}\mathbf{1}(d_k = d_{\max}), & \delta_{\max} < 1/2, \\ 0, & \delta_{\max} \ge 1/2. \end{cases} \tag{3.16}
$$

Remark 3.2 Note that under the assumption of independence of $\gamma_i, i = 1, \cdots, m$ the covariance in (3.16) when $\delta_{\max} < 1/2$ vanishes unless $k = i = j$ and $E\xi_{0,k}^3 \ne 0$ and the Z_k, Z_{ij} in (3.13), (3.14) are independent $N(0,1)$ r.v.'s.

Remark 3.3 Theorem 3.1 and Remark 3.1 can be extended to *mutually dependent* LMMA processes $\gamma_{t,i} = \sum_{k=0}^{\infty} b_{k,i}\xi_{t-k,i}, i = 1, \cdots, m$ with MA coefficients $b_{k,i} \sim \kappa_i k^{d_i - 1}, d_i \in (0, 1/2), \kappa_i > 0$ with innovations forming a $\mathbb{R}^m$-valued i.i.d. sequence $\{(\xi_{s,1}, \cdots, \xi_{s,m}); s \in \mathbb{Z}\}$ with zero mean, whose components are mutually dependent, viz., $E\xi_{0,i}\xi_{0,j} =: \sigma_{\xi,ij}, i,j = 1, \cdots, p$ where $\Sigma_\xi = E\xi_0\xi_0'$ is a general positive definite matrix. In such a case if (3.11) is strengthened to $E\xi_{0,i}^2\xi_{0,j}^2 < \infty$, $(i,j) \in \Pi_m$ the convergences in (3.12) and (3.15) hold under the same normalizations except that the limit r.v.'s there are generally correlated and have a representation w.r.t. *mutually correlated* Brownian motions $W_i, W_j, EW_i(t)W_j(t) = t\,\sigma_{\xi,ij}$. The double stochastic integral

$$
I_{ij}(g) = \int_{\mathbb{R}^2} g(s_1, s_2) W_i(ds_1) W_j(ds_2) \tag{3.17}
$$

w.r.t. such Brownian motions is well-defined for any $g \in L^2(\mathbb{R}^2)$ and has zero mean and a finite variance $EI_{ij}^2(g) = \sigma_{\xi,ii}\sigma_{\xi,jj}\|g\|^2 + \sigma_{\xi,ij}^2\langle g, g^*\rangle$ where $g^*(s_1, s_2) := g(s_2, s_1)$. In particular, the variance of the double Wiener-Itô integral $I_{ij}(g_{d_i,d_j}) = \int_{\mathbb{R}^2} g_{d_i,d_j}(s_1, s_2)W_i(ds_1)W_j(ds_2)$ in (3.9) equals

$$EI_{ij}^2(g_{d_i,d_j}) = \sigma_{\xi,ii}\sigma_{\xi,jj}\|g_{d_i,d_j}\|^2 + \sigma_{\xi,ij}^2\langle g_{d_i,d_j}, g_{d_j,d_i}\rangle, \tag{3.18}$$

where (with $B_{ij} := B(d_i, 1-d_i-d_j), B_{ji} := B(d_j, 1-d_i-d_j)$)

$$\begin{aligned}\frac{\langle g_{d_i,d_j}, g_{d_j,d_i}\rangle}{\kappa_i^2\kappa_j^2} &= \frac{B_{ij}B_{ji}}{(d_1+d_j)(2(d_i+d_j)-1)} + \Big(\frac{B_{ij}+B_{ji}}{(d_i+d_j)(d_i+d_j+1)}\Big)^2 \\ &\quad - \frac{2}{(d_i+d_j)^2}\Big(\frac{2B_{ij}B_{ji}}{2(d_i+d_j)+1} + (B_{ij}^2+B_{ji}^2)B(d_i+d_j+1, d_i+d_j+1)\Big).\end{aligned}$$

Note that for $i = j$ the last expression agrees with $\|g_{d_i,d_j}\|^2/\kappa_i^2\kappa_j^2$ in (3.9).

Remark 3.4 The 4th moment condition (3.11) is required only for those LMMA processes γ_i which enter sample variances S_{γ_i,γ_i} in the collection $\mathcal{S}_n$ (3.2). For instance for Π_3 in (4.2) the 4th moment condition applies to the innovations of the measurement errors $\{u_t\}$ alone whereas $\{X_t\}$ and $\{\varepsilon_t\}$ may have infinite 4th moment. Condition (3.11) is crucial for the validity of (3.12). Indeed if $E\xi_{0,i}^4 = \infty$ for some $i = 1, \cdots, m$ then $ES_{\gamma_i,\gamma_i}^2 = \infty$ and the limit distribution of S_{γ_i,γ_i} may be α-stable with $\alpha < 2$, see Surgailis (2004), and Horvath and Kokoszka (2008).

4 Asymptotic distribution of $\widehat{\alpha}$, $\widehat{\beta}$

In this section we shall use the results of the previous section to derive the limiting distribution of suitably standardized $\widehat{\alpha}$, $\widehat{\beta}$. To begin with note that from (2.8) we obtain

$$\begin{aligned} T_n + \beta\sigma_u^2 &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta(S_{uu} - \sigma_u^2) \\ &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta(S_{uu} - ES_{uu}) + \beta E\bar{u}^2. \end{aligned} \tag{4.1}$$

According to (2.9), (4.1), the asymptotic distribution of $\widehat{\beta} - \beta$ coincides with that of the quadratic form $\widetilde{T}_n := (T_n + \beta\sigma_u^2)/\sigma_X^2$. Under Assumptions (E), (X), and (U), $\widetilde{T}_n$ is a particular case of the quadratic forms studied in Theorem 3.1. More specifically, $\widetilde{T}_n$ corresponds to the case $m = 3$, $\gamma_{t,1} \equiv \varepsilon_t$, $\gamma_{t,2} \equiv X_t$, $\gamma_{t,3} \equiv u_t$ and the set

$$\Pi_3 = \{(X,\varepsilon),\ (X,u),\ (u,\varepsilon),\ (u,u)\}. \tag{4.2}$$

Accordingly, the limit distribution of $\widetilde{T}_n$ and $\widehat{\beta}-\beta$ is essentially determined by the maximum

$$\delta_{\max} = \max\{d_X + d_\varepsilon, d_X + d_u, d_u + d_\varepsilon, 2d_u\}, \tag{4.3}$$

with the convergence rate $\widehat{\beta} - \beta = O_p\big(n^{-(1-\min\{1/2, 1-\delta_{\max}\})}(1 + \mathbf{1}(\delta_{\max} = 1/2)\log n)\big)$. From (2.7) we obtain

$$\widehat{\alpha} - \alpha \;=\; \bar{\varepsilon} - \beta\bar{u} - (\widehat{\beta} - \beta)\bar{Z}. \tag{4.4}$$

Note that in (4.4), the linear term $\bar\varepsilon - \beta\bar u = O_p(n^{\max\{d_\varepsilon, d_u\}-1/2})$, where

$$(1/2) - \max\{d_\varepsilon, d_u\} < \min\{1/2, 1-\delta_{\max}\}.$$

Since $\bar Z = \bar X + \bar u = O_p(1)$ $(\mu_X := EX \neq 0)$, $= o_p(1)$ $(\mu_X = 0)$, the above facts imply that *the term* $(\widehat\beta - \beta)\bar Z$ *in* (4.4) *is asymptotically negligible independent of the value of* μ_X, and the limit distribution of $\widehat\alpha - \alpha$ is determined by that of $\bar\varepsilon - \beta\bar u$.

Under suitable assumptions on the innovations, see (4.6) below, Theorem 3.1 and Remark 3.1 completely describes the limit distribution of $(\bar\varepsilon - \beta\bar u, \widetilde T_n)$, or that of $(\widehat\alpha - \alpha, \widehat\beta - \beta)$. The description of this limiting distribution is relatively simpler and more transparent if we assume that the LM parameters d_X, d_ε *and* d_u *are all different*, i.e.,

$$d_u \neq d_\varepsilon \neq d_X. \tag{4.5}$$

This assumption guarantees that the maximum in (4.3) is achieved by a single pair in Π_3 of (4.2), i.e., either by (X, ε), or by (X, u), or by (u, ε), or by (u, u).

In order to apply Theorem 3.1, in addition to Assumptions (E), (X), (U), we need the following conditions on the innovations:

$$E|\zeta|^{2+\epsilon} + E|\xi|^{2+\epsilon} < \infty \quad (\exists\, \epsilon > 0), \qquad E|\eta|^4 < \infty. \tag{4.6}$$

Let

$$\begin{aligned} D_{\max} &= \max\{d_X + d_\varepsilon, d_X + d_u, d_u + d_\varepsilon, 2d_u\}, \\ d_{\max} &:= \max\{d_\varepsilon, d_X, d_u\}, \qquad d_{\min} := \min\{d_\varepsilon, d_X, d_u\}. \end{aligned}$$

We are now ready to state the following corollary.

Corollary 4.1 *Suppose assumptions (E), (X), (U) and (I) hold. In addition, assume* (4.5) *and* (4.6) *hold. Then the following hold.*

(i) Case $D_{\max} = 2d_u > 1/2$ *(this implies* $d_{\max} = d_u$*). Then*

$$\big(n^{1/2-d_u}(\widehat\alpha - \alpha), n^{1-2d_u}(\widehat\beta - \beta)\big) \to_D \Big(- \beta I_u(f_u),\ \frac{\beta}{\sigma_X^2}\big(\frac{\kappa_u^2 B(d_u, 1-d_u)}{d_u(1+2d_u)} - I_{uu}(g_{d_u,d_u})\big)\Big),$$

where I_{uu} *(*I_u*) are the double (single) Wiener-Itô integrals in* (3.3) *w.r.t. the same standard Brownian motion* $W_i = W_j \equiv W_u$ *and the integrand* $g_{d_u,d_u} = g_{d_i,d_j}$ *(*$f_{d_u} = f_{d_i}$*) in* (3.5)*, where* $d_i = d_j = d_u, \kappa_i = \kappa_j = \kappa_u$.

(ii) Case $D_{\max} = d_X + d_u > 1/2$ *(this implies* $d_{\max} = d_X > d_u > d_\varepsilon$*). Then*

$$\big(n^{1/2-d_u}(\widehat\alpha - \alpha), n^{1-d_X-d_u}(\widehat\beta - \beta)\big) \to_D \Big(- \beta I_u(f_u), -\frac{\beta}{\sigma_X^2} I_{Xu}(g_{d_X,d_u})\Big), \tag{4.7}$$

where I_{Xu} (I_u) is the double (single) Wiener-Itô integral in (3.3) *w.r.t. independent standard Brownian motions $W_i \equiv W_X$, $W_j \equiv W_u$ and the integrand $g_{d_X,d_u} = g_{d_i,d_j}$ ($f_{d_u} = f_{d_j}$) in* (3.5), *where $d_i = d_X$, $\kappa_i = \kappa_X$, $d_j = d_u$, $\kappa_j = \kappa_u$.*

(iii) Case $D_{\max} = d_u + d_\varepsilon > 1/2$ (this implies $d_{\max} = d_\varepsilon > d_u > d_X$). Then

$$\big(n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha), n^{1-d_u-d_\varepsilon}(\widehat{\beta}-\beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \frac{1}{\sigma_X^2} I_{u\varepsilon}(g_{d_u,d_\varepsilon})\Big), \tag{4.8}$$

where $I_{u\varepsilon}$ (I_ε) is the double (single) Wiener-Itô integral in (3.3) *w.r.t. independent standard Brownian motions $W_i \equiv W_u$, $W_j \equiv W_\varepsilon$ and the integrand $g_{d_u,d_\varepsilon} = g_{d_i,d_j}$ ($f_{d_\varepsilon} = f_{d_j}$) in* (3.5) *where $d_i = d_u$, $\kappa_i = \kappa_u$, $d_j = d_\varepsilon$, $\kappa_j = \kappa_\varepsilon$.*

(iv) Case $D_{\max} = d_X + d_\varepsilon > 1/2$ (this implies $d_{\min} = d_u < d_\varepsilon$). Then

$$\big(n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha),\ n^{1-d_X-d_\varepsilon}(\widehat{\beta}-\beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \frac{1}{\sigma_X^2} I_{X\varepsilon}(g_{d_X,d_\varepsilon})\Big), \tag{4.9}$$

where $I_{X\varepsilon}$ (I_ε) is the double (single) Wiener-Itô integral in (3.3) *w.r.t. independent standard Brownian motions $W_i \equiv W_X$, $W_j \equiv W_\varepsilon$ and the integrand $g_{d_X,d_\varepsilon} = g_{d_i,d_j}$ ($f_{d_\varepsilon} = f_{d_j}$) in* (3.5), *where $d_i = d_X$, $\kappa_i = \kappa_X$, $d_j = d_\varepsilon$, $\kappa_j = \kappa_\varepsilon$.*

(v) Case $D_{\max} < 1/2$. In addition, assume that the innovations of the ME process u_t satisfy $E\eta^3 = 0$, when $d_u > d_\varepsilon$. Then

$$\big(n^{1/2-(d_u \vee d_\varepsilon)}(\widehat{\alpha}-\alpha),\ n^{1/2}(\widehat{\beta}-\beta)\big) \to_D \big(\sigma_\alpha Z_\alpha,\ \sigma_\beta Z_\beta\big), \tag{4.10}$$

where Z_α, Z_β are independent $N(0,1)$ r.v.'s,

$$\sigma_\alpha^2 := \begin{cases} \beta^2 \|f_{d_u}\|^2, & d_u > d_\varepsilon, \\ \|f_{d_\varepsilon}\|^2, & d_\varepsilon > d_u, \end{cases}, \qquad \sigma_\beta^2 := \sigma_R^2/\sigma_X^4,$$

where $\sigma_R^2 := \sum_{t\in\mathbb{Z}} \mathrm{Cov}(R_0, R_t)$ and $R_t := (\varepsilon_t - \beta u_t)(X_t - EX_t + u_t) = (\varepsilon_t - \beta u_t)(Z_t - EZ_t)$, $t \in \mathbb{Z}$ is a stationary process with $ER_t = -\beta\sigma_u^2$ and $\sum_{t\in\mathbb{Z}} |\mathrm{Cov}(R_0, R_t)| < \infty$.

Remark 4.1 It is of some interest to compare the above asymptotic distributional results with those available in the case of i.i.d. set up. For that reason we shall first recall the results available in the i.i.d. case. Accordingly, suppose $\{\varepsilon, \varepsilon_i\}, \{X, X_i\}, \{u, u_i\}$ are mutually independent sequences of i.i.d.r.v.'s with positive and finite variances σ_ε^2, σ_X^2, σ_u^2, respectively. Suppose further that $E\varepsilon = Eu = 0$ and $\mu_4 = Eu^4 < \infty$. Let $\mu_X = EX$, $\mu_3 = Eu^3$. Let

$$\varphi := \frac{1}{\sigma_X^4}\Big[\sigma_X^2(\sigma_\varepsilon^2 + \beta^2\sigma_u^2) + \sigma_u^2\sigma_\varepsilon^2 + \beta^2(\mu_4 - \sigma_u^4)\Big].$$

$$\Gamma := \begin{pmatrix} (\sigma_\varepsilon^2 + \beta^2\sigma_u^2) + 2\frac{1}{\sigma_X^2}\beta^2\mu_3\mu_X + \varphi\mu_X^2 & -\frac{1}{\sigma_X^2}\beta^2\mu_3 - \varphi\,\mu_X \\ -\frac{1}{\sigma_X^2}\beta^2\mu_3 - \varphi\,\mu_X & \varphi \end{pmatrix}.$$

Using the classical CLT, we obtain

$$n^{1/2}(\widehat{\alpha}-\alpha, \widehat{\beta}-\beta) \to_D N(0,\Gamma). \tag{4.11}$$

For the sake of completeness a sketch of the proof of (4.11) is included in Section 7.

In the case of no measurement errors, $\sigma_u^2 = 0$, $\mu_4 = \mu_3 = 0$, $\varphi = \sigma_\varepsilon^2/\sigma_X^2$ and

$$\Gamma = \begin{pmatrix} \sigma_\varepsilon^2 + \mu_X^2(\sigma_\varepsilon^2/\sigma_X^2) & -\mu_X(\sigma_\varepsilon^2/\sigma_X^2) \\ -\mu_X(\sigma_\varepsilon^2/\sigma_X^2) & (\sigma_\varepsilon^2/\sigma_X^2) \end{pmatrix} = \frac{\sigma_\varepsilon^2}{\sigma_X^2}\begin{pmatrix} \sigma_X^2+\mu_X^2 & -\mu_X \\ -\mu_X & 1 \end{pmatrix}.$$

Now suppose $E\eta^3 = 0$. Then $\mu_3 = 0$ and in the i.i.d. set up the above LSEs are asymptotically correlated and normally distributed, regardless of whether there is measurement error in the covariate or not. But, surprisingly, under the above assumed long memory set up with $D_{\max} < 1/2$, by (4.10), these estimators are asymptotically independent and normally distributed even when there is no measurement error. If $E\eta^3 \neq 0$ and $d_u \geq d_\varepsilon$, then the limiting r.v.'s in (4.10) are correlated. The correlation can be obtained from (3.16).

For $D_{\max} > 1/2$, Corollary 3.1 and (3.16) yield that these r.v.'s are still asymptotically uncorrelated but have non-Gaussian distribution.

Remark 4.2 *Confidence intervals for α and β in the simple structural ME linear regression model with moderate long memory.* The limit distribution of $\widehat{\alpha}, \widehat{\beta}$ in Corollary 4.1 is very different depending on whether $D_{\max} < 1/2$ or $D_{\max} > 1/2$. In the latter case (which may be termed *very strong long memory* in the current set up) this limit distribution appears intractable. On the other hand, in the case $D_{\max} < 1/2$ (termed *moderate long memory* here) the result of Corollary 4.1(v) can be used to determine the asymptotic confidence intervals (CIs) for α, β. Obviously, these CIs require the estimation of the parameters of the limiting Gaussian distribution, which is discussed below.

Asymptotic CI for β. Recall from Corollary 4.1(v) that under $\delta_{\max} < 1/2$ and under some additional conditions on the innovations, $n^{1/2}(\hat{\beta}-\beta) \to_D N(0, \sigma_\beta^2)$ where $\sigma_\beta^2 = \sigma_R^2/\sigma_X^4$ and $\sigma_X^2 = \sigma_Z^2 - \sigma_u^2$. Because σ_u^2 is assumed to be known, $\hat{\sigma}_X^2 := n^{-1}\sum_{i=1}^n (Z_i - \bar{Z})^2 - \sigma_u^2$ is a consistent estimator of σ_X^2 and $\hat{\sigma}_X^4$ is consistent for σ_X^4.

Next, consider $\sigma_R^2 := \sum_{t\in\mathbb{R}} \mathrm{Cov}(R_0, R_t)$, where $R_t := e_t(Z_t - EZ_t) = (\varepsilon_t - \beta u_t)(X_t - EX_t + u_t)$. The parameter σ_R^2 is called the long-run variance of the stationary process $\{R_t\}$. Let $\hat{R}_t := \hat{e}_t(Z_t - \bar{Z})$, $\hat{e}_t := Y_t - \widehat{\alpha} - \widehat{\beta} Z_t$, $1 \leq t \leq n$, $\bar{\hat{R}} = n^{-1}\sum_{t=1}^n \hat{R}_t$. Then, following Abadir et al (2009), GKS, sec.9.4, an estimate of σ_R^2 is given by the HAC estimator based on $\hat{R}_t, 1 \leq t \leq n$, viz,

$$\hat{\sigma}_{\hat{R},q}^2 := n^{-1}\sum_{|k|\leq q}\Big(1 - \frac{|k|}{q}\Big)\sum_{1\leq t,s\leq n: t-s=k}(\hat{R}_t - \bar{\hat{R}})(\hat{R}_s - \bar{\hat{R}}). \tag{4.12}$$

In (4.12), $q = q(n) = 0, 1, \cdots, n$ is the bandwidth parameter. The consistency of $\hat{\sigma}_{\hat{R},q}^2$ is derived in the following proposition, where limits are taken as $n, q, n/q \to \infty$.

Proposition 4.1 *Suppose assumptions (E), (X), (U), and (I) hold,* $E(\xi^4+\zeta^4)<\infty$*,* $E\eta^3=0$*,* $E\eta^8<\infty$ *and* $D_{\max}=\max\{d_X+d_\varepsilon, d_X+d_u, d_u+d_\varepsilon, 2d_u\}<1/2$*. Then* $\hat\sigma^2_{\hat R,q}\to_p \sigma^2_R$*.*

The proof of this proposition is given in the Appendix section of this paper. It uses cumulants and is rather lengthy due to the fact that process R_t is a quadratic form in i.i.d.r.v.s.

An estimator of σ^2_β is given by $\widehat\sigma^2_\beta := \hat\sigma^2_{\hat R,q}/\hat\sigma^4_X$. Under the conditions of Proposition 4.1, $\widehat\sigma^2_\beta\to_p\sigma^2_\beta$ and the asymptotic confidence level of the CI

$$\Big\{\beta; \widehat\beta - \frac{z_{\gamma/2}}{n^{1/2}\widehat\sigma_\beta}\le\beta\le\widehat\beta+\frac{z_{\gamma/2}}{n^{1/2}\widehat\sigma_\beta}\Big\}$$

for β is $1-\epsilon$, where z_ϵ is the $(1-\epsilon)$100th percentile of the $N(0,1)$ distribution, $0<\epsilon<1$.

Asymptotic CI for α. By Corollary 4.1(v), $n^{1/2-(d_u\vee d_\varepsilon)}(\hat\alpha-\alpha)\to_D N(0,\sigma^2_\alpha)$. By (2.6) $d_e := d_u\vee d_\varepsilon$ is the LM parameter of stationary process $\{e_t=\varepsilon_t-\beta u_t\}$. More precisely, by (3.9), as $t\to\infty$ the covariance $\mathrm{Cov}(e_0,e_t)=\mathrm{Cov}(\varepsilon_0,\varepsilon_t)+\beta^2\mathrm{Cov}(u_0,u_t)$ satisfies

$$\text{(4.13)}\qquad \mathrm{Cov}(e_0,e_t) \sim \begin{cases}\kappa_\varepsilon^2B(d_\varepsilon,1-2d_\varepsilon)t^{-(1-2d_\varepsilon)}, & d_\varepsilon>d_u,\\ \beta^2\kappa_u^2B(d_u,1-2d_u)t^{-(1-2d_u)}, & d_\varepsilon<d_u,\end{cases}$$
$$= \sigma^2_\alpha\begin{cases}d_\varepsilon(1+2d_\varepsilon)t^{-(1-2d_\varepsilon)}, & d_\varepsilon>d_u,\\ d_u(1+2d_u)t^{-(1-2d_u)}, & d_\varepsilon<d_u,\end{cases}.$$

Moreover, $\mathrm{Var}(\sum_{t=1}^n e_t)\sim\sigma^2_\alpha n^{1+2(d_u\vee d_\varepsilon)}, n\to\infty$, so that σ^2_α is the long-run variance of $\{e_t\}$: see GKS, (3.3.5). The consistent and $\log(n)$-consistent estimators of σ^2_α and d_e are the HAC and LW estimators, $\widehat\sigma_\alpha$ and $\widehat d_{\hat e}$, respectively, based on the residuals

$$\text{(4.14)}\qquad \hat e_t = Y_t-\widehat\alpha-\widehat\beta Z_t = e_t+(\alpha-\widehat\alpha)+(\beta-\widehat\beta)Z_t,\qquad 1\le t\le n,$$

see GKS, sec. 8.6, 9.4. Then, the CI

$$\Big\{\alpha; \widehat\alpha-\frac{z_{\gamma/2}}{n^{1/2-\widehat d_{\hat e}}\widehat\sigma_\alpha}\le\alpha\le\widehat\alpha+\frac{z_{\gamma/2}}{n^{1/2-\widehat d_{\hat e}}\widehat\sigma_\alpha}\Big\}$$

for α is of the asymptotic level $1-\gamma$.

We shall now describe the above estimators of d_e and σ^2_α. The LW estimator of d_e is

$$\text{(4.15)}\quad \widehat d_{\hat e} := \mathrm{argmin}_{d\in[-1/2.1/2]}\mathcal{U}_n(d),\quad \mathcal{U}_n(d):=\log\Big(\frac{1}{m}\sum_{j=1}^m j^{2d}I_{\hat e}(\lambda_j)\Big)-\frac{2d}{m}\sum_{j=1}^m\log j,$$
$$\lambda_j = \frac{2\pi j}{n},\qquad I_{\hat e}(\lambda):=\frac{1}{2\pi n}\Big|\sum_{t=1}^n e^{\imath t\lambda}\hat e_t\Big|^2,\quad \lambda\in[-\pi,\pi].$$

see GKS, (8.5.2). Here, $m = 1, 2, \cdots, m = m(n) \to \infty, m = o(n)$ is the bandwidth parameter. The HAC estimator of σ^2_α is

$$(4.16)\quad \widehat{\sigma}^2_{\hat{e},q} \;:=\; q^{-2\hat{d}_{\hat{e}}} n^{-1} \sum_{|k|\le q} (1 - \frac{|k|}{q}) \sum_{1\le t,s\le n: t-s=k} (\hat{e}_t - \bar{\hat{e}})(\hat{e}_s - \bar{\hat{e}}), \quad \bar{\hat{e}} := n^{-1} \sum_{t=1}^{n} \hat{e}_t.$$

We shall use the results of Section 8.5, 8.6 of GKS to prove the consistency of $\widehat{d}_{\hat{e}}$ for d_e. Accordingly, we need to show that (4.14) satisfies the conditions of the 'signal+noise' model in GKS, sec.8.6. Let $d_\varepsilon > d_u$ for concreteness (the case $d_\varepsilon < d_u$ can be discussed analogously). Then $\hat{e}_t = \varepsilon_t + \mathcal{Z}_t$, where $\{\varepsilon_t\}$ is a LMMA 'signal' process, and the 'noise' process $\mathcal{Z}_t$ is given by

$$\mathcal{Z}_t = -\beta u_t + (\alpha - \widehat{\alpha}) + (\beta - \widehat{\beta}) Z_t.$$

According to GKS Thm. 8.5.2 (i), the LMMA process $\{\varepsilon_t\}$ satisfies Assumptions A and B of GKS provided its spectral density has the representation

$$(4.17)\qquad f_\varepsilon(\lambda) = |\lambda|^{-2d_\varepsilon} g_\varepsilon(\lambda), \quad |\lambda| \le a$$

where g_ε is a positive Lipschitz function on $[0, a]$ for some $a > 0$. Conditions (4.17) is satisfied by ARFIMA and some other classes of LMMA processes. We also need to assume the existence of the 4th moment of innovations of $\{\varepsilon_t\}$. Under these conditions the LW estimator $\hat{d}_\varepsilon$ of d_ε satisfies

$$(4.18)\qquad \hat{d}_\varepsilon - d_\varepsilon = O_p(m^{-1/2}) + O_p(m/n),$$

see GKS, Thm. 8.5.2 (iii). Let $d_Z := d_u \vee d_X$ be the LM parameter of $\{Z_t = u_t + X_t\}$ and

$$(4.19)\qquad s_n \;:=\; O_p((m/n)^{d_\varepsilon - d_u}) + O_p(\widehat{\beta} - \beta) \times \begin{cases} O_p((m/n)^{d_\varepsilon - d_Z}), & d_\varepsilon > d_Z, \\ O_p((1/n)^{d_\varepsilon - d_Z}), & d_\varepsilon \le d_Z. \end{cases}$$

Then by GKS Thm. 8.6.2 we obtain

$$(4.20)\qquad \widehat{d}_{\hat{e}} - d_e \;=\; (\hat{d}_\varepsilon - d_\varepsilon)(1 + o_p(1)) + O_p(s_n).$$

Note $n^{d_Z - d_\varepsilon}(\widehat{\beta} - \beta) = o_p(n^{d_X - 1/2})$. These facts and (4.18), (4.19) guarantee the $\log(n)$-consistency of $\widehat{d}_{\hat{e}}$ in (4.15): $\widehat{d}_{\hat{e}} - d_e = o_p(\log(n))$ for any bandwidth choice $m = [n^a], 0 < a < 1$.

The consistency of $\widehat{\sigma}^2_\alpha$ of (4.16) is established following the proofs of GKS, Theorem 9.4.1 and Proposition 4.1. See also Lavancier et al. (2010).

5 Structural ME multiple linear regression model

Here we shall now discuss the asymptotic distributions of the bias adjusted LSEs in the structural multiple linear regression model. Accordingly, now β, X_t, Z_t, u_t are p-dimensional random vectors and the model of interest is

$$Y_t = \alpha + X_t'\beta + \varepsilon_t, \quad Z_t = X_t + u_t, \quad t \in \mathbb{Z}, \tag{5.1}$$

where X and u are vector-valued LMMA processes satisfying the following assumptions and x' denotes the transpose of a vector $x \in \mathbb{R}^p$.

Assumption (X)$_p$ Covariates $X_t = (X_{t,1}, \cdots, X_{t,p})'$ form a LMMA process

$$X_{t,i} \;=\; \mu_{X,i} + \sum_{k=0}^{\infty} a_{k,i}\xi_{t-k,i}, \quad t \in \mathbb{Z}, \quad \text{with} \quad a_{k,i} \sim \kappa_{X,i} k^{-(1-d_{X,i})}, \ k \to \infty, \tag{5.2}$$

where $\mu_{X,i} \in \mathbb{R}, \ \kappa_{X,i} > 0, 0 < d_{X,i} < 1/2$, and i.i.d. innovations $\{\xi_s = (\xi_{s,1}, \cdots, \xi_{s,p})'; s \in \mathbb{Z}\}$ with $E\xi_{0,i} = 0, E\xi_{0,i}\xi_{0,j} = \sigma_{\xi,ij}, \ i, j = 1, \cdots, p$.

Assumption (U)$_p$ Measurement errors $u_t = (u_{t,1}, \cdots, u_{t,p})'$ form a LMMA process

$$u_{t,i} = \sum_{k=0}^{\infty} c_{k,i}\eta_{t-k,i}, \quad t \in \mathbb{Z}, \quad \text{with} \quad c_{k,i} \sim \kappa_{u,i} k^{-(1-d_{u,i})}, \ k \to \infty, \tag{5.3}$$

where $\kappa_{u,i} > 0, 0 < d_{u,i} < 1/2$, the innovations $\{\eta_s = (\eta_{s,1}, \cdots, \eta_{s,p})'; s \in \mathbb{Z}\}$ are i.i.d. with $E\eta_{0,i} = 0, E\eta_{0,i}\eta_{0,j} = \sigma_{\eta,ij}, \ i, j = 1, \cdots, p$, and $\Sigma_u := E(u_0 u_0')$ is known and positive definite.

Assumption (I)$_p$ The innovation sequences $\{\zeta_s; s \in \mathbb{Z}\}$, $\{\xi_s; s \in \mathbb{Z}\}$, and $\{\eta_s; s \in \mathbb{Z}\}$ in Assumptions (E), (X)$_p$ and (U)$_p$ are mutually independent.

We also assume that

$$E|\zeta_{0,i}|^{2+\epsilon} + E|\xi_{0,i}|^{2+\epsilon} < \infty \quad (\exists\, \epsilon > 0), \qquad E\eta_{0,i}^4 < \infty, \quad \forall\, 1 \le i \le p. \tag{5.4}$$

The bias corrected LSEs of α, β here are defined as

$$\begin{aligned} &S_{ZZ} := n^{-1}\sum_{i=1}^{n}(Z_i - \bar{Z})(Z_i - \bar{Z})', \quad S_{ZY} := n^{-1}\sum_{i=1}^{n}(Z_i - \bar{Z})(Y_i - \bar{Y}), \\ &\widehat{\beta} := \big(S_{ZZ} - \Sigma_u\big)^{-1} S_{ZY}, \quad \widehat{\alpha} := \bar{Y} - \bar{Z}'\widehat{\beta}. \end{aligned}$$

Whence as in the case of simple linear regression model we obtain

$$\begin{aligned} \widehat{\beta} - \beta &= (S_{ZZ} - \Sigma_u)^{-1}\big(S_{X\varepsilon} + S_{u\varepsilon} - S_{Xu}\beta - (S_{uu} - \Sigma_u)\beta\big), \\ \widehat{\alpha} - \alpha &= \bar{\varepsilon} - \bar{u}'\beta - \bar{Z}'(\widehat{\beta} - \beta). \end{aligned} \tag{5.5}$$

Since $S_{ZZ} - \Sigma_u \to_p \Sigma_X := EX_0X_0'$, we see from (5.5) that the limit distribution of $\widehat{\beta} - \beta$ coincides with that of $\widetilde{T}_n := \Sigma_X^{-1}(T_n + E(\bar{u}\bar{u}')\beta)$, where

$$T_n := S_{X\varepsilon} + S_{u\varepsilon} - S_{Xu}\beta - (S_{uu} - ES_{uu})\beta$$

is a zero-mean quadratic form in LMMA satisfying Assumptions (E), $(\mathrm{X})_p$ and $(\mathrm{U})_p$. As it follows from Theorem 3.1 and Remark 3.3, under these assumptions the limit distribution of T_n and $\widetilde{T}_n$ is essentially determined by the maximum

$$D_{\max} := \max\{d_{X,i} + d_\varepsilon, d_{u,i} + d_\varepsilon, d_{X,i} + d_{u,j}, d_{u,i} + d_{u,j};\ 1 \le i, j \le p\}. \tag{5.6}$$

Accordingly, the limit distribution of $\widehat{\beta} - \beta$ is non-gaussian or Gaussian depending on whether $D_{\max} > 1/2$ or $D_{\max} < 1/2$. In general, $\widehat{\alpha}$ and $\widehat{\beta}_i, 1 \le i \le p$ may have different convergence rates and a complicated joint limit distribution. We first discuss the case $D_{\max} < 1/2$ where the limit result admits a relatively simple formulation as seen in the following corollary.

Corollary 5.1 *Suppose (E), $(X)_p$, $(U)_p$ and $(I)_p$ hold and $D_{\max} < 1/2$. In addition, assume that $d_{u,i}, 1 \le i \le p$ are all different, $d_{u,\max} := \max\{d_{u,i}, 1 \le i \le p\}$, the 3rd moment of the innovations of $u_{t,i}$ with $d_{u,i} = d_{u,\max}$ is zero when $d_{u,\max} > d_\varepsilon$, and* (5.4) *hold. Then*

$$\big(n^{1/2-(d_\varepsilon \vee d_{u,\max})}(\widehat{\alpha} - \alpha),\, n^{1/2}(\widehat{\beta} - \beta)\big) \to_D \big(\sigma_\alpha Z_\alpha,\, \Sigma_X^{-1} Z_\beta\big). \tag{5.7}$$

Here $Z_\alpha \sim N(0,1)$, Z_β is a normal vector independent of Z_α, with $EZ_\beta = 0$ and covariance matrix $EZ_\beta Z_\beta' := \sum_{t\in\mathbb{Z}} \mathrm{Cov}(R_0, R_t)$, where $R_t := (\varepsilon_t - \beta' u_t)(X_t - EX_t + u_t) = (\varepsilon_t - \beta' u_t)(Z_t - EZ_t)$, $t \in \mathbb{Z}$ is a stationary $\mathbb{R}^p$-valued process with $ER_t = -\Sigma_u\beta$ and $\sum_{t\in\mathbb{Z}} \|\mathrm{Cov}(R_0, R_t)\| < \infty$. Moreover,

$$\sigma_\alpha^2 := \begin{cases} \beta_i^2 \|f_{d_{u,i}}\|^2, & d_{u,\max} = d_{u,i} > d_\varepsilon, i = 1, \cdots, p, \\ \|f_{d_\varepsilon}\|^2, & d_\varepsilon > d_{u,\max}. \end{cases}$$

Next, we discuss the limit distribution of the LSE $(\widehat{\alpha}, \widehat{\beta})$ in (5.5) when $\delta_{\max} > 1/2$. The description of this limit distribution is complicated for the case $p \ge 2$ and when long memory parameters of components of $\{X_t\}$ and $\{u_t\}$ are all different. For this reason we shall describe these distributions only in the case when these long memory parameters are equal, viz., $d_{X,i} \equiv d_X, d_{u,i} \equiv d_u, i = 1, \cdots, p$, and in the case when $p = 2$ but $d_{X,,1} \ne d_{X,2}, d_{u,1} \ne d_{u,2}$. We note that in the latter case, the convergence rates of $\widehat{\beta}_1, \widehat{\beta}_2$ are generally different.

Consider first the former case, $p \ge 1$ arbitrary. Let $\Sigma_\eta = E\eta_0\eta_0', \Sigma_\xi = E\xi_0\xi_0'$ denote the respective covariance matrices of innovations in Assumption $(\mathrm{U})_p$ and $(\mathrm{X})_p$. Introduce a scalar-valued standard Brownian motion $W_\varepsilon = W_\varepsilon(t), t \in \mathbb{R}$, and vector-valued Brownian motions $W_X(t) = (W_{X,1}(t), \cdots, W_{X,p}(t))'$, $W_u(t) = (W_{u,1}(t), \cdots, W_{u,p}(t))', t \in \mathbb{R}$ with respective covariance matrices $EW_X(t)W_X(t)' = |t|\Sigma_\xi, EW_u(t)W_u(t)' = |t|\Sigma_\eta$, W_ε, W_X, W_u

mutually independent. Recall from (3.3), (3.17) the definition of the stochastic integrals with respect to these Brownian motions: $I_u(f) = \big(\int_{\mathbb{R}} f_i(s) W_{u,i}(ds)\big)_{1\le i\le p}$, $f = (f_1, \cdots, f_p)'$, $I_{uu}(g) = \big(\int_{\mathbb{R}^2} g_{ij}(s_1,s_2) W_{u,i}(ds_1) W_{u,j}(ds_2)\big)_{1\le i,j\le p}$, $g = (g_{ij})_{1\le i,j\le p}$ defined for vector- and matrix-valued valued integrands from $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, the stochastic integrals $I_X(f), I_{X\varepsilon}(g)$, $I_{u\varepsilon}(g)$ defined in a similar fashion. Note I_{uu}, I_{Xu} are matrix-valued and $I_X, I_u, I_{X\varepsilon}, I_{X\varepsilon}$ are vector-valued r.v.'s.

Corollary 5.2 *Let Assumptions (E), $(X)_p$, $(U)_p$ and $(I)_p$ be satisfied. In addition, assume that $d_{u,i} = d_u,\ d_{X,i} = d_X,\ 1 \le i \le p$ and (5.4) hold.*

(i) Case $D_{\max} = 2d_u > 1/2$. Then

$$(5.8)\qquad \big(n^{1/2-d_u}(\widehat{\alpha}-\alpha),\, n^{1-2d_u}(\widehat{\beta}-\beta)\big) \to_D \Big(-\beta' I_u(f_u),\, \Sigma_X^{-1}\big(\langle f_u, f_u'\rangle - I_{uu}(g_{d_u,d_u})\big)\beta\Big),$$

where $f_u = (f_{d_1}, \cdots, f_{d_p})'$ and $g_{d_u,d_u} = (g_{d_i,d_j})_{1\le i,j\le p}$ are defined as in (3.5) where $d_i = d_j := d_u,\ \kappa_i := \kappa_{u,i},\ \kappa_j := \kappa_{u,j}$.

(ii) Case $D_{\max} = d_X + d_u > 1/2$. Then

$$(5.9)\qquad \big(n^{1/2-d_u}(\widehat{\alpha}-\alpha),\ n^{1-d_X-d_u}(\widehat{\beta}-\beta)\big) \to_D \Big(-\beta' I_u(f_u), -\Sigma_X^{-1} I_{Xu}(g_{d_X,d_u})\beta\Big),$$

where f_u is the same as in (5.8) and $g_{d_X,d_u} = (g_{d_i,d_j})_{1\le i,j\le p}$ as in (3.5) where $d_i := d_X,\ d_j := d_u, \kappa_i := \kappa_{X,i},\ \kappa_j := \kappa_{u,j}$.

(iii) Case $D_{\max} = d_u + d_\varepsilon > 1/2$. Then

$$(5.10)\qquad \big(n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha),\ n^{1-d_u-d_\varepsilon}(\widehat{\beta}-\beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}),\ \Sigma_X^{-1} I_{u\varepsilon}(g_{d_u,d_\varepsilon})\Big),$$

where $g_{d_u,d_\varepsilon} = (g_{d_i,d_j})_{1\le i\le p}$ and $f_{d_\varepsilon} = f_{d_j}$ as in (3.5) where $d_i := d_{u,i} = d_u,\ \kappa_i := \kappa_{u,i},\ d_j := d_\varepsilon,\ \kappa_j := \kappa_\varepsilon$.

(iv) Case $D_{\max} = d_X + d_\varepsilon > 1/2$. Then

$$(5.11)\qquad \big(n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha), n^{1-d_X-d_\varepsilon}(\widehat{\beta}-\beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \Sigma_X^{-1} I_{X\varepsilon}(g_{d_X,d_\varepsilon})\Big),$$

where f_{d_ε} is the same as in (5.10) and $g_{d_X,d_\varepsilon} = (g_{d_i,d_j})_{1\le i\le p}$ as in (3.5) where $d_i := d_{X,i} = d_X,\ \kappa_i := \kappa_{X,i},\ d_j := d_\varepsilon,\ \kappa_j := \kappa_\varepsilon$.

Next, consider the case $p = 2, \widehat{\beta} = (\widehat{\beta}_1, \widehat{\beta}_2)'$, $d_{X,1} \ne d_{X,2}$, $d_{u,1} \ne d_{u,2}$ and $D_{\max} > 1/2$, where $D_{\max}$ is defined in (5.6). Let $\Sigma_X^{-1} = (\rho_{X,ij})_{1\le i,j\le 2}$. As noted above, the limit distribution of $\widehat{\beta}-\beta$ coincides with that of $\widetilde{T}_n := \Sigma_X^{-1}(T_n + E(\bar{u}\bar{u}')\beta) = (\widetilde{T}_{n1}, \widetilde{T}_{n2})'$ where

$$(5.12)\qquad \begin{aligned}\widetilde{T}_{n1} \;=\; &\rho_{X,11}\big(S_{X_1,\varepsilon} + S_{u_1,\varepsilon} - S_{X_1u_1}\beta_1 - S_{X_1u_2}\beta_2 - (S_{u_1u_1} - ES_{u_1u_1})\beta_1 \\ &\qquad - (S_{u_1u_2} - ES_{u_1u_2})\beta_2 + (\bar{u}_1)^2\beta_1 + \bar{u}_1\bar{u}_2\beta_2\big) \\ &+\rho_{X,12}\big(S_{X_2,\varepsilon} + S_{u_2,\varepsilon} - S_{X_2u_1}\beta_1 - S_{X_2u_2}\beta_2 - (S_{u_2u_1} - ES_{u_2u_1})\beta_1 \\ &\qquad - (S_{u_2u_2} - ES_{u_2u_2})\beta_2 + \bar{u}_1\bar{u}_2\beta_1 + (\bar{u}_2)^2\beta_2\big).\end{aligned}$$

We omit a similar expression for $\widetilde{T}_{n2}$, where the only difference is that $\rho_{X,11}, \rho_{X,12}$ in (5.12) are replaced by $\rho_{X,21}, \rho_{X,22}$, respectively. We have the two cases: (a) $\rho_{X,12} = \rho_{X,21} \neq 0$ (or Σ_X is not a diagonal matrix), and (b) $\rho_{X,12} = \rho_{X,21} = 0$ (Σ_X is diagonal). From these formulas it is easy to see that in case (a) that the convergence rate of $\widetilde{T}_{ni}, i = 1,2$ hence also of $\widehat{\beta}_i, i = 1,2$ is the same and is equal to $n^{1-D_{\max}}$. In case (b), $\widehat{\beta}_i, i = 1,2$ may have different convergence rates and their limit distribution is more complex. As an illustration, the following corollary details this limit distribution when $D_{\max} = 2d_{u,1}$. In the cases $D_{\max}$ is achieved at other pairs of LM indices in (5.6), this limit distribution can be derived in a similar fashion.

Corollary 5.3 *Let $p = 2$ and Assumptions (E), $(X)_2$, $(U)_2$, $(I)_2$ and (5.4) be satisfied. In addition, assume that $d_{u,1} > \max\{d_{u,2},\ d_{X,1},\ d_{X,2},\ d_\varepsilon\}$ and $D_{\max} = 2d_{u,1} > 1/2$.*
(a) Let $\sigma_{X,12} = \mathrm{Cov}(X_{0,1}, X_{0,2}) \neq 0$. Then

$$\begin{aligned}&\big(n^{1/2-d_{u,1}}(\widehat{\alpha} - \alpha),\ n^{1-2d_{u,1}}(\widehat{\beta}_i - \beta_i), i = 1,2\big)\\ &\quad \to_D \quad \beta_1\Big(- I_{u_1}(f_{d_{u,1}}),\ \rho_{X,1i}\big(\|f_{d_{u,1}}\| - I_{u_1u_1}(g_{d_{u_1},d_{u_1}})\big), i = 1,2\Big).\end{aligned}$$

(b) Let $\sigma_{X,12} = \mathrm{Cov}(X_{0,1}, X_{0,2}) = 0$ and $d_{X,2} \neq d_{u,2}$. Then

$$\begin{aligned}&\big(n^{1/2-d_{u,1}}(\widehat{\alpha} - \alpha),\ n^{1-2d_{u,1}}(\widehat{\beta}_1 - \beta_1), n^{1-d_{u,1}-d_{X,2}\vee d_{u,2}}(\widehat{\beta}_2 - \beta_2)\big)\\ &\quad \to_D \quad \beta_1\Big(- I_{u_1}(f_{d_{u,1}}),\ \rho_{X,11}\big(\|f_{d_{u,1}}\| - I_{u_1u_1}(g_{d_{u_1},d_{u_1}})\big), \rho_{X,22}\mathcal{W}\Big),\end{aligned}$$

where

$$\mathcal{W} := \begin{cases} -I_{u_1,X_2}(g_{d_{u,1},d_{X,2}}), & d_{X,2} > d_{u,2}, \\ \langle f_{d_{u,1}}, f_{d_{u,2}}\rangle - I_{u_1u_2}(g_{d_{u_1},d_{u_2}}), & d_{X,2} < d_{u,2}. \end{cases}$$

6 Functional ME model: nonrandom design

In this section we describe the analogs of the previous results in the functional linear regression model with LMMA regression and measurement errors, and nonrandom design satisfying the following assumption. For clarity of exposition, the subsequent discussion is confined to the case $p = 1$, or the simple linear regression model in (2.1).

Assumption $(\mathbf{X})_{det}$ There exists a (nonrandom) piece-wise continuous function $V : [0,1] \to \mathbb{R}$ such that $X_t = V(t/n),\ t = 1, \cdots, n$.

The above form of regressors also assumes that V is not a constant so that $\sigma_V^2 := \int_0^1 (V(t) - \bar{V})^2 dt > 0$, where $\bar{V} := \int_0^1 V(t)dt$. As shown below, the limit behavior of LSE $(\widehat{\alpha}, \widehat{\beta})$ in the nonrandom design case is Gaussian and generally simpler than in the random design case. The dominating role in the limit distribution now is being played by terms $S_{X\varepsilon}, S_{Xu}, \bar{\varepsilon}, \bar{u}$ in (4.1) and (4.4).

Note first that Assumption (X)$_{det}$ implies $\bar{X} \to \bar{V}$ and $S_{XX} \to \sigma_V^2$ as $n \to \infty$. Moreover, $S_{Xu} = O_p(n^{d_u - 1/2}) = o_p(1)$, $S_{X\varepsilon} = O_p(n^{d_\varepsilon - 1/2}) = o_p(1)$, see (6.2) below, while $S_{uu} \to \sigma_u^2$. Therefore the normalization entity $S_{ZZ} - \sigma_u^2$ in (2.9) tends to σ_V^2, viz.,

$$S_{ZZ} - \sigma_u^2 \to_p \sigma_V^2. \tag{6.1}$$

Let $V_c(t) := V(t) - \bar{V},\ t \in [0,1]$. Assumptions (X)$_{det}$, (E), and (U) imply

$$n^{1/2-d_\varepsilon} S_{X\varepsilon} \to_D I_\varepsilon(f_{V_c,\varepsilon}), \qquad n^{1/2-d_u} S_{Xu} \to_D I_u(f_{V_c,u}), \tag{6.2}$$

where I_ε, I_u are the same (Gaussian) stochastic integrals as in Corollary 4.1 with respective integrands

$$f_{V_c,\varepsilon}(s) \ := \ \kappa_\varepsilon \int_0^1 V_c(t)(t-s)_+^{d_\varepsilon - 1} dt, \qquad f_{V_c,u}(s) := \kappa_u \int_0^1 V_c(t)(t-s)_+^{d_u - 1} dt. \tag{6.3}$$

Note $I_\varepsilon(f_{V_c,\varepsilon}), I_u(f_{V_c,u})$ in (6.2) are independent and have a Gaussian distribution with zero mean and respective variances

$$\begin{aligned} EI_\varepsilon^2(f_{V_c,\varepsilon}) &= \|f_{V_c,\varepsilon}\|^2 = \kappa_\varepsilon^2 B(d_\varepsilon, 1-2d_\varepsilon)\langle V_c, V_c\rangle_{d_\varepsilon}, \\ EI_u^2(f_{V_c,u}) &= \|f_{V_c,u}\|^2 = \kappa_u^2 B(d_u, 1-2d_u)\langle V_c, V_c\rangle_{d_u}, \end{aligned} \tag{6.4}$$

where for any two bounded functions $f,\ g$, $\langle f, g\rangle_d = \int_{[0,1]^2} f(t)g(s)|t-s|^{2d-1}dtds$ is a strictly positive definite quadratic form, for all $0 < d < 1/2$. The convergences in (6.2) can be proved by using the criterion in GKS, Cor.4.7.1, for linear forms in i.i.d.r.v.'s. Moreover, $\bar{Z} \to_p \bar{V}$ and $S_{uu} - \sigma_u^2 = O_p(n^{-(1-2d_u-1)\vee(1/2)}(1 + \mathbf{1}(d_u = 1/2)\log^{1/2} n) = o_p(n^{d_u - 1/2})$ and $S_{u\varepsilon} = o_p(n^{d_\varepsilon - 1/2})$ follow from Theorem 3.1. These facts together with (4.1), (4.4), (6.1), (6.2) result in the following corollary.

Corollary 6.1 *Let Assumptions (E), (X)$_{det}$, (U) and (I) be satisfied. In addition, assume that* (5.4) *hold.*

(i) Suppose $d_\varepsilon > d_u$. Then

$$n^{1/2-d_\varepsilon}(\widehat{\alpha} - \alpha,\ \widehat{\beta} - \beta) \to_D (W_{1,\varepsilon}, W_{2,\varepsilon}), \tag{6.5}$$

where $(W_{1,\varepsilon}, W_{2,\varepsilon})$ have a bivariate Gaussian distribution with zero mean and (co)variances

$$\begin{aligned} EW_{1,\varepsilon}^2 &= \|f_{d_\varepsilon}\|^2 + \bar{V}^2\sigma_V^{-4}\|f_{V_c,\varepsilon}\|^2 - 2\bar{V}\sigma_V^{-2}\langle f_{d_\varepsilon}, f_{V_c,\varepsilon}\rangle, \\ EW_{2,\varepsilon}^2 &= \sigma_V^{-4}\|f_{V_c,\varepsilon}\|^2, \qquad EW_{1,\varepsilon}W_{2,\varepsilon} = \sigma_V^{-2}\big(\langle f_{d_\varepsilon}, f_{V_c,\varepsilon}\rangle - \bar{V}\|f_{V_c,\varepsilon}\|^2\big), \end{aligned} \tag{6.6}$$

(ii) Suppose $d_\varepsilon < d_u$. Then

$$n^{1/2-d_u}(\widehat{\alpha} - \alpha,\ \widehat{\beta} - \beta) \to_D -(W_{1,u}, W_{2,u})\beta,$$

where $(W_{1,u}, W_{2,u})$ have a similar bivariate Gaussian distribution as in (6.5)-(6.6) *with the only difference that $f_{d_\varepsilon}, f_{V_c,\varepsilon}$ in* (6.6) *are replaced by $f_{d_u}, f_{V_c,u}$, respectively.*

7 Proofs of Theorem 3.1 and Corollary 3.1

Proof of Theorem 3.1. Let $\widetilde{S}_{\gamma_i,\gamma_j} := n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ so that $S_{\gamma_i,\gamma_j} = \widetilde{S}_{\gamma_i,\gamma_j} - \bar{\gamma}_i\bar{\gamma}_j$. Note for any $t, t' \in \mathbb{Z}$.

$$(7.1)\quad \mathrm{Cov}(\gamma_{t,i}\gamma_{t,j}, \gamma_{t',i'}\gamma_{t',j'}) \;=\; \begin{cases} \mathrm{Cov}(\gamma_{t,i}, \gamma_{t',i})\mathrm{Cov}(\gamma_{t,j}, \gamma_{t',j}), & (i,j) = (i',j'), i \ne j, \\ \mathrm{Cov}(\gamma_{t,i}^2, \gamma_{t',i}^2), & i = j = i' = j', \\ 0, & (i,j) \ne (i',j') \end{cases}$$

From (7.1) we obtain

$$(7.2)\qquad \mathrm{Cov}(S_{\gamma_i,\gamma_j}, S_{\gamma_{i'},\gamma_{j'}}) = \mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \widetilde{S}_{\gamma_{i'},\gamma_{j'}}) = 0, \qquad (i,j) \ne (i',j').$$

From GKS, Prop.3.2.1(ii), it follows that

$$(7.3)\qquad \mathrm{Cov}(\gamma_{t,i}, \gamma_{0,i}) \;=\; \sum_{s\le 0} b_{t-s,i}b_{-s,i} \;\sim\; \chi_i t^{2d_i-1}, \qquad t \to \infty,$$

where $\chi_i := \kappa_i^2\int_0^\infty (1+s)^{d_i-1}s^{d_i-1}ds = \kappa_i^2 B(d_i, 1-2d_i)$. We shall prove that

$$(7.4)\qquad \mathrm{Var}(S_{\gamma_i,\gamma_j}) \sim \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \sim \sigma_{ij}^2/n, \quad d_i + d_j < 1/2,$$

$$\mathrm{Var}(S_{\gamma_i,\gamma_j}) \sim \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \sim \sigma_{ij}^2(\log n)/n, \quad d_i + d_j = 1/2,$$

$$\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \;\sim\; \widetilde{\sigma}_{ij}^2 n^{2(d_i+d_j-1)},\ \mathrm{Var}(S_{\gamma_i,\gamma_j}) \;\sim\; \sigma_{ij}^2 n^{2(d_i+d_j-1)}, \quad d_i + d_j > 1/2,$$

where $\widetilde{\sigma}_{ij}^2 := (1+\delta_{ij})\|\widetilde{g}_{d_i,d_j}\|^2$,

$$(7.5)\qquad \sigma_{ij}^2 \;:=\; \begin{cases} \sum_{t\in\mathbb{Z}} \mathrm{Cov}(\gamma_{t,i}\gamma_{t,j}, \gamma_{0,i}\gamma_{0,j}), & d_i + d_j < 1/2, \\ 2(1+\delta_{ij})\chi_i\chi_j, & d_i + d_j = 1/2, \\ (1+\delta_{ij})\|g_{d_i,d_j}\|^2, & d_i + d_j > 1/2; \end{cases}$$

and $\|g_{d_i,d_j}\|^2$ is defined in (3.9), $\delta_{ij} := \mathbf{1}(i = j)$.

Consider (7.4) for $d_i + d_j > 1/2$. Here, the asymptotics of $\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j})$ is immediate from (7.1), (7.3) and GKS, Prop.3.3.1(i). To check the asymptotics of $\mathrm{Var}(S_{\gamma_i,\gamma_j})$, consider first the case of $i \ne j$. Write $\mathrm{Var}(S_{\gamma_i,\gamma_j}) = \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) - 2\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \bar{\gamma}_i\bar{\gamma}_j) + \mathrm{Var}(\bar{\gamma}_i\bar{\gamma}_j)$, where the variance $\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j})$ satisfies (7.4) and $\mathrm{Var}(\bar{\gamma}_i\bar{\gamma}_j) = \mathrm{Var}(\bar{\gamma}_i)\mathrm{Var}(\bar{\gamma}_j) \sim \|f_{d_i}\|^2\|f_{d_j}\|^2 n^{2(d_i+d_j-1)}$, see (3.9). The asymptotics of the covariance

$$(7.6)\qquad \mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \bar{\gamma}_i\bar{\gamma}_j) \;=\; n^{-3}\sum_{t,t_1,t_2=1}^n E\gamma_{t,i}\gamma_{t_1,i}E\gamma_{t,j}\gamma_{t_2,j}$$

$$\sim\; \chi_i\chi_j n^{-3}\sum_{t,t_1,t_2=1}^n |t-t_1|^{2d_i-1}|t-t_2|^{2d_j-1}$$

$$\sim\; n^{2(d_i+d_j-1)}\langle \widetilde{g}_{d_i,d_j}, f_{d_i}\otimes f_{d_j}\rangle$$

follows by integral approximation and a calculation as in (3.6). This proves (7.4) for $d_i+d_j > 1/2$ and $i \neq j$.

Next, we shall prove (7.4) for $i = j, d_i > 1/4$. We have $\mathrm{Cov}(\gamma_{t,i}^2, \gamma_{t',i}^2) = 2(\mathrm{Cov}(\gamma_{t,i}, \gamma_{t',i}))^2 + h_{t-t',i}$, where $h_{t-t',i} := \nu_{4,i} \sum_{u \le t \wedge t'} b_{t-u,i}^2 b_{t'-u,i}^2$ and $\nu_{4,i} = E(\xi_{0,i}^2 - 1)^2 - 2 = E\xi_{0,i}^4 - 3$ is the 4th cumulant of $\xi_{0,i}$, see GKS, (6.2.25). The sumability $\sum_{k \in \mathbb{Z}} h_{k,i} < \infty$ implies $n^{-2} \sum_{t,t'=1}^n h_{t-t',i} = O(n^{-1}) = o(n^{2(2d_i-1)})$. Then $\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \sim 2n^{-2} \sum_{t,t'=1}^n (\mathrm{Cov}(\gamma_{t,i}, \gamma_{t',i}))^2$ $\sim 2\|\widetilde{g}_{d_i,d_i}\|^2 n^{2(2d_i-1)} = \widetilde{\sigma}_{d_i,d_i}^2 n^{2(2d_i-1)}$ follows as in the case $i \neq j$ above. By writing $\mathrm{Var}(S_{\gamma_i,\gamma_i})$ $= \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_i}) - 2\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_i}, (\bar{\gamma}_i)^2) + \mathrm{Var}((\bar{\gamma}_i)^2)$, (7.4) follows from

$$\text{(7.7)} \quad \mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_i}, (\bar{\gamma}_i)^2) \sim 2n^{2(2d_i-1)} \langle \widetilde{g}_{d_i,d_i}, f_{d_i} \otimes f_{d_i} \rangle \quad \text{and} \quad \mathrm{Var}((\bar{\gamma}_i)^2) \sim 2n^{2(2d_i-1)} \|f_{d_i}\|^4,$$

c.f. the case $i \neq j$ above. To prove the second relation in (7.7) use $\mathrm{Var}((\bar{\gamma}_i)^2) = \mathrm{Cum}_4(\bar{\gamma}_i) + 2(\mathrm{Var}(\bar{\gamma}_i))^2$ where $\mathrm{Cum}_4(\bar{\gamma}_i) = \nu_{4,i} n^{-4} \sum_{s \le n} \big(\sum_{t=1 \vee s}^n b_{t-s,i} \big)^4 \le Cn^{-4} \big\{ \sum_{|s| \le n} (\sum_{t=1}^{2n} t^{d_i-1} \big)^4 + \sum_{s \ge n} (\sum_{t=1}^n (t+s)^{d_i-1})^4 \big\} \le Cn^{4d_i-3} = o(n^{2(2d_i-1)})$ and $(\mathrm{Var}(\bar{\gamma}_i))^2 \sim \|f_{d_i}\|^4 n^{2(2d_i-1)}$; see above. Next, $\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_i}, (\bar{\gamma}_i)^2) = n^{-3} \sum_{t,t_1,t_2=1}^n E(\gamma_{t,i}^2 - E\gamma_{t,i}^2) \gamma_{t_1,i} \gamma_{t_2,i} = n^{-3} \sum_{t,t_1,t_2=1}^n \sum_{s_1,s_2 \le t} \sum_{u_1 \le t_1, u_2 \le t} b_{t-s_1,i} b_{t-s_2,i} b_{t_1-u_1,i} b_{t_2-u_2,i} \psi(s_1,s_2,u_1,u_2)$, where $\psi(s_1,s_2,u_1,u_2) := E(\xi_{s_1,i}\xi_{s_2,i} - E\xi_{s_1,i}\xi_{s_2,i})\xi_{u_1,i}\xi_{u_2,i} = 0$ except for $(s_1,s_2) = (u_1,u_2)$ and $(s_1,s_2) = (u_2,u_1)$. Particularly, for $s := s_1 = s_2 = u_1 = u_2$ we have $\psi(s,s,s,s) = \nu_{4,i} + 2$ while $s_1 \neq s_2$ yield $\psi(s_1,s_2,s_1,s_2) = \psi(s_1,s_2,s_2,s_1) = 1$. Hence we obtain $\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_i}, (\bar{\gamma}_i)^2) = J_1 + 2J_2$ where $|J_1| = \nu_{4,i} n^{-3} \big| \sum_{t,t_1,t_2=1}^n \sum_{s \le t_1 \wedge t_2 \wedge t_2} b_{t-s,i}^2 b_{t_1-s,i} b_{t_2-s,i} \big| \le Cn^{-3} \sum_{t_1,t_2=1}^n \sum_{s \le t_1 \wedge t_2} |b_{t_1-s,i} b_{t_2-s,i}|$ $= O(n^{2(d_i-1)}) = o(n^{2(2d_i-1)})$ and J_2 coincides with the r.h.s. of (7.6) with $i = j$, thus proving the first relation (7.7) and completing the proof of (7.4) for $d_i + d_j > 1/2$.

Consider (7.4) for $d_i + d_j = 1/2$. Let $i \neq j$. Then by (7.3)

$$\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \ \sim \ \chi_i \chi_j n^{-2} \sum_{t,s=1}^n |t-s|^{-1} \ \sim \ \sigma_{ij}^2 n^{-1} \log n,$$

with $\sigma_{ij}^2 = 2\chi_i\chi_j$. The case $i = j$ follows similarly. Finally, (7.4) for the case $d_i + d_j < 1/2$ follows from (7.3), (7.1) and the fact that the r.h.s. of (7.1) is summable.

Next, we prove the convergence in (3.12). Because of the differences in the normalization and the limit distribution, the cases $\delta_{\max} > 1/2$, $\delta_{\max} = 1/2$, and $\delta_{\max} < 1/2$, where $D_{\max}$ is as in (3.1), will be discussed separately. Let $\Pi_{\max} := \{(i,j) \in \Pi_m; d_i + d_j = \delta_{\max}\}$.

Proof of (3.12)**: Case** $\delta_{\max} > 1/2$**.** Since (7.4) imply $A(n)(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}) \to_D 0$ for $(i,j) \notin \Pi_{\max}$, relation (3.12) follows from

$$\text{(7.8)} \qquad \{n^{1-\delta_{\max}}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}); (i,j) \in \Pi_{\max}\} \to_D \{I_{ij}(g_{d_i,d_j}); (i,j) \in \Pi_{\max}\},$$

where I_{ij} are the double Wiener-Itô integrals in (3.3). Assume first that that $\Pi_{\max}$ consists of a single element $(i,j), i \neq j$. Then, because $\delta_{\max} = d_i + d_j$ and $ES_{\gamma_i,\gamma_j} = 0$ for

$i \neq j$, $n^{1-\delta_{\max}}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}) = n^{1-d_i-d_j} S_{\gamma_i,\gamma_j}$ can be written as a quadratic form in i.i.d. innovations $\{\xi_{s,i}, \xi_{s,j}, s \in \mathbb{Z}\}$, viz.,

$$\begin{aligned} Q(g_n) &:= \sum_{s_1,s_2\in\mathbb{Z}} g_n(s_1,s_2)\xi_{s_1,i}\xi_{s_2,j}, \quad \text{with coefficients} \\ g_n(s_1,s_2) &:= n^{-d_1-d_j}\sum_{t=1}^n b_{t-s_1,i}b_{t-s_2,j} - n^{-1-d_i-d_j}\sum_{t_1,t_2=1}^n b_{t_1-s_1,i}b_{t_2-s_2,j}. \end{aligned}$$

Let

$$\widetilde{g}_n(x_1,x_2) := ng_n([nx_1],[nx_2]) = \frac{n}{n^{d_i+d_j}}\sum_{t=1}^n b_{t-[nx_1],i}b_{t-[nx_2],j} - \frac{1}{n^{d_i+d_j}}\sum_{t_1,t_2=1}^n b_{t_1-[nx_1],i}b_{t_2-[nx_2],j}.$$

We use GKS, Prop.11.5.5. Accordingly, the result $n^{1-d_i-d_j}S_{\gamma_i,\gamma_j} \to_D I_{ij}(g_{d_i,d_j})$ follows from the following convergence in $L^2(\mathbb{R}^2)$:

$$\|\widetilde{g}_n - g_{d_i,d_j}\| \to 0. \tag{7.9}$$

Since $b_{k,i} \sim \kappa_i k^{d_i-1}, k \to \infty$ the point-wise convergence

$$\begin{aligned} \widetilde{g}_n(x_1,x_2) \to g_{d_i,d_j}(x_1,x_2) &= \kappa_i\kappa_j\Big\{\int_0^1 (t-s_1)_+^{d_i-1}(t-s_2)_+^{d_j-1}dt \\ &\qquad - \int_0^1 (t_1-x_1)_+^{d_i-1}dt_1\int_0^1 (t_2-x_2)_+^{d_j-1}dt_2\Big\}, \end{aligned}$$

see (3.5), for any fixed $(x_1,x_2) \in \mathbb{R}^2, x_i \neq 0,1, i = 1,2$ follows by integral approximation. Then, (7.9) follows by the DCT similarly as GKS, Prop.11.5.6. The general case in (7.8) follows similarly and we omit the details.

Proof of (3.12)**: Case** $\delta_{\max} = 1/2$**.** Let $\widetilde{\Pi}_{1/2} := \{(i,j) \in \Pi_m : d_i + d_j = 1/2\}$. Then by (7.4) relation (3.12) reduces to

$$\big\{(n/\log n)^{1/2}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}); (i,j) \in \widetilde{\Pi}_{1/2}\big\} \to_D \big\{\sigma_{ij}Z_{ij}; (i,j) \in \widetilde{\Pi}_{1/2}\big\}, \tag{7.10}$$

where $Z_{ij}, (i,j) \in \widetilde{\Pi}_{1/2}$ are independent $N(0,1)$ r.v.'s and $\sigma^2_{ij} = 2(1+\delta_{ij})\chi_i\chi_j$, see (7.5). Also, since $\bar{\zeta}_i = O_p(n^{d_i-1/2})$, $i = 1,\cdots,m$ so $\bar{\gamma}_i\bar{\gamma}_j = O_p(n^{d_i+d_j-1}) = O_p(n^{-1/2})$, $(i,j) \in \Pi_{1/2}$ and hence $(n/\log n)^{1/2}\bar{\gamma}_i\bar{\gamma}_j = O_p((\log n)^{-1/2}) = o_p(1)$, $(i,j) \in \Pi_{1/2}$. Thus, (7.10) follows from

$$\big\{(n/\log n)^{1/2}(\widetilde{S}_{\gamma_i,\gamma_j} - E\widetilde{S}_{\gamma_i,\gamma_j});\ (i,j) \in \Pi_{1/2}\big\} \to_D \big\{\sigma_{ij}Z_{ij};\ (i,j) \in \Pi_{1/2}\big\}, \tag{7.11}$$

where $\widetilde{S}_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ as above. We shall prove (7.11) for a single pair $(i,j) \in \widetilde{\Pi}_{1/2}$. Let $i \neq j$. Then $E\widetilde{S}_{\gamma_i,\gamma_j} = 0$. Moreover, $\widetilde{S}_{\gamma_i,\gamma_j} = \widetilde{S}'_{\gamma_i,\gamma_j} + \widetilde{S}''_{\gamma_i,\gamma_j}$ where

$$\widetilde{S}'_{\gamma_i,\gamma_j} := n^{-1}\sum_{t=1}^n \sum_{s_i\le t, i=1,2, s_1\neq s_2} b_{t-s_1,i}b_{t-s_2,j}\xi_{s_1,i}\xi_{s_2,j}, \quad \widetilde{S}''_{\gamma_i,\gamma_j} := n^{-1}\sum_{t=1}^n\sum_{s\le t} b_{t-s,i}b_{t-s,j}\xi_{s,i}\xi_{s,j}$$

are off-diagonal and diagonal terms, respectively. Moreover, $\sum_{t=1}^\infty |b_{t,i}b_{t,j}| \le C\sum_{t=1}^\infty t^{-3/2} < \infty$ implies $\widetilde{S}''_{\gamma_i,\gamma_j} = O_p(n^{-1/2}) = o_p(1)$. Hence it suffices to prove

$$(n/\log n)^{1/2}\widetilde{S}'_{\gamma_i,\gamma_j} \to_D N(0,\sigma^2_{ij}). \tag{7.12}$$

To prove (7.12), as in in Bhansali et al. (2007), we use martingale CLT in Hall and Heyde (1980). Towards this aim rewrite the l.h.s. of (7.12) as the martingale transform

$$(n\log n)^{-1/2}\sum_{s<n} v_n(s), \quad \text{where} \quad v_n(s) := u_{n,i}(s)\xi_{s,j} + u_{n,j}(s)\xi_{s,i}, \tag{7.13}$$

$$u_{n,i}(s) := \sum_{s'<s} c_n(s',s)\xi_{s',i}, \quad u_{n,j}(s) := \sum_{s'<s} c_n(s,s')\xi_{s',j}, \quad c_n(s',s) := \sum_{t=1}^n b_{t-s',i}\, b_{t-s,j}.$$

Let $\mathcal{F}_t := \sigma\{\xi_{s,i},\xi_{s,j}, s\le t\}$ be the σ-field generated by innovations. Then $E[v_n(s)|\mathcal{F}_{s-1}] = 0$, $E[v^2_n(s)|\mathcal{F}_{s-1}] = u^2_{n,i}(s) + u^2_{n,j}(s)$. By the classical martingale CLT, (7.12) follows from

$$B_{ij}(n) := \mathrm{Var}(\sum_{s<n} v_n(s)) = n^2\mathrm{Var}(\widetilde{S}'_{\gamma_i,\gamma_j}) \sim \sigma^2_{ij} n\log n, \tag{7.14}$$

$$B^{-1}_{ij}(n)\sum_{s<n} E[v^2_n(s)|\mathcal{F}_{s-1}] \to_D 1, \tag{7.15}$$

$$B^{-1}_{ij}(n)\sum_{s<n} E[v^2_n(s) I(|v_n(s)| > \delta B^{1/2}_{ij}(n))] \to_D 0, \quad \forall\delta>0. \tag{7.16}$$

The proof of (7.14) follows easily from (7.4). Consider (7.15). Using $B_{ij}(n) = \sum_{s<n} Ev^2_n(s)$, the relation (7.15) follows from (7.14) and

$$\sum_{s<n}(E[v^2_n(s)|\mathcal{F}_{s-1}] - Ev^2_n(s)) = o_p(n\log n), \qquad \text{or}$$

$$\sum_{s<n}(u^2_{n,k}(s) - Eu^2_{n,k}(s)) = o_p(n\log n), \qquad k = i,j. \tag{7.17}$$

Consider (7.17) for $k = i$; the proof for $k = j$ is analogous. By writing the l.h.s. of (7.17) as a centered quadratic form $Q_n = \sum_{s',s''<n}\xi_{s',i}\xi_{s'',i}\sum_{s'\vee s''<s<n} c_n(s',s)c_n(s'',s)$ in i.i.d. r.v.'s $\xi_{s',i}$'s, (7.17) and (7.15) follow from $\mathrm{Var}(Q_n) \le 8E\xi^4_{0,i}R_n$, and

$$R_n \;:=\; \sum_{s''\le s'<n}\Big(\sum_{s'<s<n} c_n(s',s)c_n(s'',s)\Big)^2 = O(n^2) = o(n^2\log^2 n), \tag{7.18}$$

see also GKS, (4.5.4). Using the definition of $c_n(s',s)$ in (7.13) it follows that

$$R_n \;\le\; C\int_{-\infty<s''<s'<n} ds'ds''\Big(\int_{s'<s<n}\widetilde{c}_n(s',s)\widetilde{c}_n(s'',s)ds\Big)^2 =: C\widetilde{R}_n,$$

where $\widetilde{c}_n(s',s) := \int_0^n (t-s')_+^{d_i-1}(t-s)_+^{d_j-1}dt$. By change of variables, $\widetilde{R}_n = n^2\widetilde{R}_1$ and hence (7.18) follows from

$$\widetilde{R}_1 < \infty. \tag{7.19}$$

To check (7.19) use the following bound: for any $-\infty < s' < s < 1$

$$\begin{aligned}\widetilde{c}_1(s',s) &\le \mathbf{1}(s' \in (-1,1)) \int_{\mathbb{R}} (t-s')_+^{d_i-1}(t-s)_+^{d_j-1} dt + \mathbf{1}(s'<-1)|s'|^{d_i-1}\int_0^1 (t-s)_+^{d_j-1}dt \\ &\le C\mathbf{1}(s' \in (-1,1))|s-s'|^{d_i+d_j-1} + C\mathbf{1}(s'<-1)|s'|^{d_i-1}(1+|s|)^{d_j-1} \\ &= C\mathbf{1}(s' \in (-1,1))|s-s'|^{-1/2} + C\mathbf{1}(s'<-1)|s'|^{d_i-1}(1+|s|)^{d_j-1} \qquad (7.20)\end{aligned}$$

since $d_i + d_j = 1/2$. Then

$$\begin{aligned}\widetilde{R}_1 \;\le\; C\int_{(-\infty,1)^2} ds'ds''\Big\{\int_{s'\vee s''}^1 \Big(\frac{\mathbf{1}(|s'|<1)}{|s-s'|^{1/2}} + \frac{\mathbf{1}(s'<-1)}{|s'|^{1-d_i}(1+|s|)^{1-d_j}}\Big) \\ \times\Big(\frac{\mathbf{1}(|s''|<1)}{|s-s''|^{1/2}} + \frac{\mathbf{1}(s''<-1)}{|s''|^{1-d_i}(1+|s|)^{1-d_j}}\Big)ds\Big\}^2 \le C\sum_{k=1}^4 J_k,\end{aligned}$$

where

$$\begin{aligned}J_1 &:= \int_{(-1,1)^2} ds'ds''\Big\{\int_{-1}^1 \frac{ds}{|s-s'|^{1/2}|s-s''|^{1/2}}\Big\}^2, \\ J_2 &:= \int_{(-\infty,-1)\times(-1,1)} |s'|^{-2(1-d_i)}ds'ds''\Big\{\int_{-1}^1 \frac{ds}{(1+|s|)^{1-d_j}|s-s''|^{1/2}}\Big\}^2, \\ J_3 &:= \int_{(-1,1)\times(-\infty,-1)} |s''|^{-2(1-d_i)}ds'ds''\Big\{\int_{-1}^1 \frac{ds}{(1+|s|)^{1-d_j}|s-s'|^{1/2}}\Big\}^2, \\ J_4 &:= \int_{(-\infty,-1)^2} |s's''|^{-2(1-d_i)}ds'ds''\Big\{\int_{s'\vee s''}^1 \frac{ds}{(1+|s|)^{2(1-d_j)}}\Big\}^2.\end{aligned}$$

The fact that $J_k < \infty, k = 1,2,3,4$ is elementary by $0 < d_i, d_j < 1/2$. This proves (7.19) and (7.18), (7.15).

To prove (7.16) we use condition (3.10). By the Markov inequality, $E[v_n^2(s)I(|v_n(s)| > \delta B_{ij}^{1/2}(n))] \le E|v_n(s)|^{2+\epsilon}(\delta B_{ij}^{1/2}(n))^{-\epsilon}$ and (7.16) follows from

$$\sum_{s<n} E|v_n(s)|^{2+\epsilon} = o(B_{i,j}^{(2+\epsilon)/2}(n)) = O((n\log n)^{(2+\epsilon)/2}). \qquad (7.21)$$

We have $E|v_n(s)|^{2+\epsilon} \le C(E|u_{n,i}(s)|^{2+\epsilon} + E|u_{n,j}(s)|^{2+\epsilon}) \le C(L_i(s) + L_j(s))$, where $L_i(s) := E\big|\sum_{s'<s} c_n(s',s)\xi_{s,i}\big|^{2+\epsilon}$, $L_j(s) := E\big|\sum_{s'<s} c_n(s,s')\xi_{s,j}\big|^{2+\epsilon}$. By Rosenthal's inequality, see GKS, Lemma 2.5.2,

$$L_i(s) \le C\Big(\sum_{s'<s} c_n^2(s',s)\Big)^{(2+\epsilon)/2}. \qquad (7.22)$$

We use the following bound similar to (7.20).

$$|c_n(s',s)| \le C\begin{cases}\frac{n|s'|^{d_i-1}}{n^{1-d_j}+|s|^{1-d_j}}, & s' < -n, \\ |s'-s|_+^{-1/2}, & |s'| \le n.\end{cases} \qquad (7.23)$$

From (7.22), (7.23) we obtain

$$\sum_{s<n} L_i(s) \;\le\; C\Big\{\sum_{s\le -n}+\sum_{|s|<n}\Big\}\Big(\sum_{s'<s} c_n^2(s',s)\Big)^{(2+\epsilon)/2} =: C\{J_1+J_2\},$$

where

$$\begin{aligned} J_1 &\le C\int_{-\infty}^{-n} ds\Big(\int_{-\infty}^{-n}(n|s'|^{d_i-1}|s|^{d_j-1})^2 ds'\Big)^{(2+\epsilon)/2} \\ &= Cn\int_{-\infty}^{-1}|s|^{2(d_j-1)(2+\epsilon)/2}ds\Big(\int_{-\infty}^{-1}|s'|^{2(d_i-1)}ds'\Big)^{(2+\epsilon)/2} = Cn \end{aligned}$$

since the last integral converges. On the other hand, since $d_i+d_j=1/2$,

$$\begin{aligned} J_2 &\le C\sum_{|s|\le n}\Big(\sum_{s'\le -n} n^{2d_j}|s'|^{2(d_i-1)}\Big)^{(2+\epsilon)/2} + C\sum_{|s|\le n}\Big(\sum_{|s'|\le n}|s-s_+|_+^{-1}\Big)^{(2+\epsilon)/2} \\ &\le Cn + Cn(\log n)^{(2+\epsilon)/2}, \end{aligned}$$

implying $\sum_{s<n} L_i(s) = O(n(\log n)^{(2+\epsilon)/2})$. Since $\sum_{s<n} L_j(s) = O(n(\log n)^{(2+\epsilon)/2})$ follows exactly similarly, we obtain $\sum_{s<n} E|v_n(s)|^{2+\epsilon} = O(n(\log n)^{(2+\epsilon)/2}) = o((n\log n)^{(2+\epsilon)/2})$ for $\epsilon>0$, proving (7.21), (7.16) and completing the proof of (7.12).

Proof of (3.12)**: Case $\delta_{\max}<1/2$.** Then by (7.4) relation (3.12) is equivalent to

(7.24) $$\big\{n^{1/2}(S_{\gamma_i,\gamma_j}-ES_{\gamma_i,\gamma_j});\,(i,j)\in\Pi_m\big\} \to_D \big\{\sigma_{ij}Z_{ij};(i,j)\in\Pi_m\big\},$$

where Z_{ij}, $(i,j)\in\Pi_m$ are independent $N(0,1)$ r.v.'s and σ_{ij}^2 are defined in (7.5). Moreover since $\bar X_i\bar X_j = O_p(n^{d_i+d_j-1}) = o_p(n^{-1/2})$ for $d_i+d_j<1/2$, so S_{γ_i,γ_j} in (7.24) can be replaced by $\widetilde S_{ij} = n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ and (7.24) follows from

(7.25) $$\big\{n^{1/2}(\widetilde S_{\gamma_i,\gamma_j}-E\widetilde S_{\gamma_i,\gamma_j});(i,j)\in\Pi_m\big\} \to_D \big\{\sigma_{ij}Z_{ij};(i,j)\in\Pi_m\big\}.$$

We shall prove (7.25) for a single pair $(i,j)\in\Pi_m$. Let $i\ne j$. Then $E\widetilde S_{\gamma_i,\gamma_j}=0$. Hence it suffices to prove

(7.26) $$n^{1/2}\widetilde S_{\gamma_i,\gamma_j} \to_D N(0,\sigma_{ij}^2)$$

We use the argument as in GKS, Thm.4.8.1. For $\ell\ge 1$ introduce 'truncated' processes:

$$\gamma_{t,i}^{(\ell)} := \sum_{s\le t} b_{t-s,i}\mathbf{1}(t-s\le \ell)\xi_{s,i}, \qquad i=1,\cdots,m,$$

and the corresponding $\widetilde S_{\gamma_i^{(\ell)},\gamma_j^{(\ell)}} := n^{-1}\sum_{t=1}^n \gamma_{t,i}^{(\ell)}\gamma_{t,j}^{(\ell)}$. Thus, for each $1\le \ell<\infty$ fixed, $Y_{ij}^{(\ell)}(t) := \gamma_{t,i}^{(\ell)}\gamma_{t,j}^{(\ell)}$, $t\in\mathbb{Z}$ is a ℓ-dependent stationary process with autocovariance $\rho_{ij}^{(\ell)}(t) :=$

$\mathrm{Cov}(Y^{(\ell)}_{ij}(t), Y^{(\ell)}_{ij}(0))$ such that

$$\begin{aligned}\rho^{(\ell)}_{ij}(t) &= \big(\sum_{s=0}^\infty b_{s,i}b_{t+s,i}\mathbf{1}(t+s\le \ell)\big)\big(\sum_{s=0}^\infty b_{s,j}b_{t+s,j}\mathbf{1}(t+s\le \ell)\big)\\ &\le C\big(\sum_{s=0}^\infty |b_{s,i}b_{t+s,i}|\big)\big(\sum_{s=0}^\infty |b_{s,j}b_{t+s,j}|\big) \le Ct^{2(d_i+d_j-1)}, \quad t\ge 1,\end{aligned}$$

and $\rho^{(\ell)}_{ij}(t) \to \rho_{ij}(t) := \mathrm{Cov}(Y_{ij}(t), Y_{ij}(0)$, as $\ell\to\infty$, where $Y_{ij}(t) := \gamma_{t,i}\gamma_{t,j}$. These facts and the CLT for ℓ-dependent stationary processes, see e.g. GKS, Prop.4.3.2, imply that

$$\begin{aligned} n^{1/2}\widetilde S_{\gamma^{(\ell)}_i,\gamma^{(\ell)}_j} &\to_D N(0, (\sigma^{(\ell)}_{ij})^2), \qquad n\to\infty,\\ (\sigma^{(\ell)}_{ij})^2 &:= \sum_{t\in\mathbb{Z}}\rho^{(\ell)}_{ij}(t) \to \sigma^2_{ij}, \quad \ell\to\infty.\end{aligned}$$

Hence, (7.26) follows provided we show that uniformly in $n\ge 1$

$$n\mathrm{Var}(\widetilde S_{\gamma_i,\gamma_j} - \widetilde S_{\gamma^{(\ell)}_i,\gamma^{(\ell)}_j}) = \sum_{|t|<n}\big(1-\frac{|t|}{n}\big)\mathrm{Cov}\big(Y_{ij}(t)-Y^{(\ell)}_{ij}(t), Y_{ij}(0)-Y^{(\ell)}_{ij}(0)\big) \to \infty, \tag{7.27}$$

as $\ell\to\infty$. The proof of (7.27) mimics that of (GKS, (4.8.7)). We omit the details. This proves (7.26) and the extension to (7.25) seems straightforward. Theorem 3.1 is proved. □

Proof of Corollary 3.1. Assume for concreteness that the sets $\Pi_{0m} = \{k\}, \Pi_m = \{(i,j)\}$ each consist of a single element, $d_{\max} = d_k, \delta_{\max} = d_i+d_j$. Let $\delta_{\max} > 1/2$. Following the proof of Theorem 3.1 in this case, write $n^{1/2-d_k}\bar\gamma_k = \sum_{s\in\mathbb{Z}} f_n(s)\xi_{s,k}$ as a linear form in innovations with coefficients $f_n(s) = n^{-1/2-d_k}\sum_{t=1}^n b_{t-s,k}, s\in\mathbb{Z}$. Let $\tilde f_n(x) := n^{1/2}f_n([sx]), x\in\mathbb{R}$ and $\|\cdot\|_1$ denote the norm in $L^2(\mathbb{R})$. According to (GKS, Propositions 11.5.5, 14.3.3), the joint convergence in (3.15), or $(n^{1/2-d_k}\bar\gamma_k, n^{1-d_i-d_j}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j})) \to_D (I_k(f_{d_k}), I_{ij}(g_{d_i,d_j}))$ follows from (7.9) and $\|\tilde f_n - f_{d_k}\|_1 \to 0$, where the last relation can be verified similarly to (7.9). This proves (3.15) for $\delta_{\max} > 1/2$. For $\delta_{\max} = d_i+d_j \le 1/2$ the joint convergence in (3.15) can be proved similarly as in the proof of Theorem 3.1 and we omit the details.

Consider (3.16). For $\delta_{\max} > 1/2$ (3.16) follows the orthogonality of single and double Wiener-Itô integrals, see (3.4). Suppose $\delta_{\max}\le 1/2$. As in the proof of Theorem 3.1, let $\widetilde S_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^n\gamma_{t,i}\gamma_{t,j}$. It suffices to prove that

$$\lim_{n\to\infty} n^{1-d_k}E\big(\bar\gamma_k(\widetilde S_{\gamma_i,\gamma_j} - E\widetilde S_{\gamma_i,\gamma_j})\big) = \frac{\kappa_k}{d_k(1+d_k)}E(\xi_{0,k}\xi_{0,i}\xi_{0,j})\sum_{s=0}^\infty b_{s,i}b_{s,j}. \tag{7.28}$$

To show (7.28), split $\widetilde S_{\gamma_i,\gamma_j} - E\widetilde S_{\gamma_i,\gamma_j} = S'_n + S''_n$, where $nS'_n := \sum_{s\le n}\sum_{t=1\vee s}^n b_{t-s,i}b_{t-s,j}(\xi_{s,i}\xi_{s,j} - E\xi_{s,i}\xi_{s,j})$, $S''_n := n^{-1}\sum_{s_1,s_2\le n, s_1\ne s_2}\sum_{t=1\vee s_1\vee s_2}^n b_{t-s_1,i}b_{t-s_2,j}\xi_{s_1,i}\xi_{s_2,j}$. Since $E\bar\gamma_k S''_n = 0$, it suffices to prove (7.28) with $\widetilde S_{\gamma_i,\gamma_j} - E\widetilde S_{\gamma_i,\gamma_j}$ replaced by S'_n. We have

$$n^{1-d_k}E(\bar\gamma_k S'_n) = E(\xi_{0,k}\xi_{0,i}\xi_{0,j})n^{-1-d_k}\sum_{s\le n}\sum_{t=1\vee s}^n b_{t-s,k}L_{s,ij}(n), \tag{7.29}$$

where $L_{s,ij}(n) := \sum_{t=1\vee s}^{n} b_{t-s,i} b_{t-s,j} \to L_{ij} := \sum_{t=0}^{\infty} b_{t,i} b_{t,j} < \infty$ for any $1 \le s \le n$ and $|L_{s,ij}(n)| \le C \sum_{t=|s|}^{\infty} t^{d_i+d_j-2} \le C(1+|s|)^{d_i+d_j-1}$, $s \le 0$. Thus, by (10.2.53) of GKS,

$$n^{-1-d_k} \sum_{s=1}^{n} \sum_{t=s}^{n} b_{t-s,k} L_{s,ij}(n) \;\sim\; L_{ij}\kappa_k n^{-1-d_k} \sum_{s=1}^{n} \sum_{t=s}^{n} (t-s)_+^{d_k-1} \to (\kappa_k/d_k(1+d_k))L_{ij},$$

$$\begin{aligned}\Big| \sum_{s\le 0} \sum_{t=s}^{n} b_{t-s,k} L_{s,ij}(n) \Big| &\le C \sum_{t=1}^{n} \sum_{s=0}^{\infty} (t+s)^{d_k-1}(1+s)^{d_i+d_j-1} \\ &\le C \sum_{t=1}^{n} t^{d_k+d_i+d_j-1} \le C n^{d_k+d_i+d_j} = o(n^{1+d_k}).\end{aligned}$$

This completes the proof of (7.28). The last relation also implies the statement (3.16) of the corollary when $\delta_{\max} < 1/2$ and also when $\delta_{\max} = 1/2$ due to the presence of the logarithmic factor in the normalization $A(n)$ (3.1). □

Proof of (4.11). For any two sets of variables $\{U_i\}, \{V_i\}$, let $\widetilde{S}_{UV} := n^{-1}\sum_{i=1}^{n}(U_iV_i - EU_iV_i)$, $U_i^c := U_i - EU_i$. Then (4.1) can be rewritten as $T_n + \beta\sigma_u^2 = T_n' - T_n''$, where

$$T_n' \;:=\; \widetilde{S}_{X^c\varepsilon} - \beta\widetilde{S}_{X^cu} + \widetilde{S}_{u\varepsilon} - \beta\widetilde{S}_{uu}, \qquad T_n'' := \overline{X^c}\bar{\varepsilon} - \beta\overline{X^c}\bar{u} + \bar{u}\bar{\varepsilon} - \beta(\bar{u})^2.$$

Note all summands in T_n'' are uncorrelated, implying

$$\mathrm{Var}(T_n'') = \mathrm{Var}(\overline{X^c})\mathrm{Var}(\bar{\varepsilon}) + \beta^2\mathrm{Var}(\overline{X^c})\mathrm{Var}(\bar{u}) + \mathrm{Var}(\bar{u})\mathrm{Var}(\bar{\varepsilon}) + \beta^2\mathrm{Var}((\bar{u})^2) = O(n^{-2}).$$

Hence and from (2.9) and (4.1),

$$n^{1/2}(\widehat{\beta} - \beta) \;=\; n^{1/2}T_n'/\sigma_X^2 + o_p(1). \tag{7.30}$$

Similarly from (4.4) and (7.30) we obtain

$$\begin{aligned} n^{1/2}(\widehat{\alpha} - \alpha) &= n^{1/2}(\bar{\varepsilon} - \beta\bar{u}) - n^{1/2}(\widehat{\beta} - \beta)(\mu_X + o_p(1)) \\ &= n^{1/2}(\bar{\varepsilon} - \beta\bar{u}) - (\mu_X/\sigma_X^2)n^{1/2}T_n' + o_p(1). \end{aligned} \tag{7.31}$$

Note $n^{1/2}T_n'$ and $n^{1/2}(\bar{\varepsilon} - \beta\bar{u}) + (\mu_X/\sigma_X^2)T_n'$ are sums of i.i.d. r.v.'s with zero mean and finite variance. Moreover, since all terms in T_n' are mutually uncorrelated,

$$\begin{aligned}\mathrm{Var}(T_n') &= \mathrm{Var}(\widetilde{S}_{X^c\varepsilon}) + \beta^2\mathrm{Var}(\widetilde{S}_{X^cu}) + \mathrm{Var}(\widetilde{S}_{u\varepsilon}) + \beta^2\mathrm{Var}(\widetilde{S}_{uu}) \\ &= n^{-1}\big(\sigma_X^2\sigma_\varepsilon^2 + \beta^2\sigma_X^2\sigma_u^2 + \sigma_u^2\sigma_\varepsilon^2 + \beta^2(\mu_4 - \sigma_u^4)\big).\end{aligned}$$

Hence, $\mathrm{Var}(n^{1/2}T_n'/\sigma_X^2) = \varphi$, see (4.11). We also find that the covariance matrix of $(n^{1/2}(\bar{\varepsilon} - \beta\bar{u}) + (\mu_X/\sigma_X^2)T_n', n^{1/2}T_n'/\sigma_X^2)$ (the main terms in (7.30), (7.31)) coincides with Γ in (4.11). Then (4.11) follows from (7.30), (7.31) and the classical CLT for sums of i.i.d. r.v.'s. □

8 Appendix

This appendix contains the proof of Proposition 4.1. In the process of proving the consistency of $\hat\sigma^2_{\hat R,q}$ we also establish some results of independent interest.

Accordingly, let $\{(V_t, W_t), t \in \mathbb{Z}\}$ be a covariance stationary process with summable cross-covariances $\sum_{t\in\mathbb{Z}} |\mathrm{Cov}(V_0, W_t)| < \infty$. The limit

$$\lim_{n\to\infty} n^{-1}\mathrm{Cov}\Big(\sum_{t=1}^n V_t, \sum_{s=1}^n W_s\Big) = \sigma_{V,W} := \sum_{t\in\mathbb{Z}} \mathrm{Cov}\big(V_0, W_t\big) = \sum_{t\in\mathbb{Z}} \mathrm{Cov}\big(W_0, V_t\big)$$

is called the long-run cross covariance of V, W (the long-run variance of V when $V_t \equiv W_t$). Let $q \equiv q(n) = 1, 2, \cdots, n$ be a sequence integers and let $\sigma^2_V = \sigma_{V,V}$, $\sigma^2_W = \sigma_{W,W}$. Similar to the HAC estimator of σ^2_V, see Abadir et al (2009), GKS (2012), the HAC estimator of $\sigma_{V,W}$ is defined to be

$$\hat\sigma_{V,W,q} \;:=\; n^{-1} \sum_{|k|\le q} (1 - \frac{|k|}{q}) \sum_{1\le t,s\le n, t-s=k} (V_t - \bar V)(W_s - \bar W). \tag{8.1}$$

See Lavancier et al. (2009), (2.1). Write $\hat\sigma^2_{V,q} = \hat\sigma_{V,V,q}$, $\hat\sigma^2_{W,q} = \sigma_{W,W,q}$. The representation in (8.1) implies $\hat\sigma^2_{V,q} \ge 0$, $\hat\sigma^2_{W,q} \ge 0$, for all q, and by the Cauchy-Schwarz inequality,

$$\hat\sigma^2_{V+W,q} \le 2(\hat\sigma^2_{V,q} + \hat\sigma^2_{W,q}), \qquad |\hat\sigma_{V,W,q}| \;\le \sqrt{\hat\sigma^2_{V,q}\hat\sigma^2_{W,q}}. \tag{8.2}$$

See also Abadir et al. (2009), (A.2) and Koul and Surgailis (2016), p.176. In the sequel all limits are taken as $n \to \infty, q \to \infty, n/q \to \infty$, unless mentioned otherwise.

Note that $R_t := e_t(Z_t - EZ_t) = (\varepsilon_t - \beta u_t)(X_t - EX_t + u_t)$ is a stationary process. Because of the assumed mutual independence of the processes $\{\varepsilon_t\}, \{X_t\}, \{u_t\}$, we obtain $ER_0 = -\beta\sigma^2_u$, $\mathrm{Var}(R_0) = (\sigma^2_\varepsilon + \beta^2\sigma^2_u)\sigma^2_X + \sigma^2_\varepsilon\sigma^2_u + \beta^2\sigma^2_{u^2}$ and

$$\gamma_R(t) := \mathrm{Cov}(R_0, R_t) \;=\; \gamma_\varepsilon(t)\gamma_X(t) + \beta^2\gamma_u(t)\gamma_X(t) + \gamma_\varepsilon(t)\gamma_u(t) + \beta^2\gamma_{u^2}(t)$$

$$\approx\; C\Big[t^{-2(1-(d_\varepsilon+d_X))} + t^{-2(1-(d_u+d_X))} + t^{-2(1-(d_\varepsilon+d_u))} + t^{-2(1-2d_u)}\Big] = O(t^{-2(1-D_{\max})}),$$

where $D_{\max} = \max\{d_\varepsilon + d_X, d_u + d_X, d_\varepsilon + d_u, 2d_u\}$. Hence $D_{\max} < 1/2$ implies that $\gamma_R(t)$ is summable, i.e., $\{R_t\}$ is a short memory stationary process. Its long-run variance is $\sigma^2_R := \sum_{t\in\mathbb{Z}} \gamma_R(t)$. We are now ready to give the

Proof of Proposition 4.1. The claim that $\hat\sigma^2_{\hat R,q} \to_p \sigma^2_R$ follows from the following two claims.

$$\text{(a)} \quad |\hat\sigma^2_{R,q} - \sigma^2_R| \to_p 0. \qquad \text{(b)} \quad E|\hat\sigma^2_{\hat R,q} - \hat\sigma^2_{R,q}| \to 0. \tag{8.3}$$

Proof of (8.3)(a). The proof here is similar to that of Theorem 9.4.1, p279-280 of GKS. Let

$$\tilde\sigma^2_{R,q} := n^{-1}\sum_{|k|\le q}(1-\frac{|k|}{q})\sum_{1\le t,s\le n, t-s=k}\tilde R_t\tilde R_s, \quad \tilde R_t := R_t - ER_t. \tag{8.4}$$

Since $E\hat\sigma^2_{R,q}\to\sigma^2_R$, (8.3)(a) follows from

$$\text{(a)}\quad \mathrm{Var}(\tilde\sigma^2_{R,q})\to 0, \qquad \text{(b)}\quad E|\hat\sigma^2_{R,q}-\tilde\sigma^2_{R,q}|\to 0. \tag{8.5}$$

Proof of (8.5)(b). Let

$$\hat\gamma_k := n^{-1}\sum_{i=1}^{n-|k|}(R_i-\bar R)(R_{i+|k|}-\bar R), \qquad \tilde\gamma_k := n^{-1}\sum_{i=1}^{n-|k|}\tilde R_i\tilde R_{i+|k|}, \quad |k|<n.$$

Then

$$\hat\sigma^2_{R,q} = \sum_{|k|\le q}(1-\frac{|k|}{q})\hat\gamma_k, \quad \tilde\sigma^2_{R,q} = \sum_{|k|\le q}(1-\frac{|k|}{q})\tilde\gamma_k, \quad |\hat\sigma^2_{R,q}-\tilde\sigma^2_{R,q}|\le\sum_{|k|\le q}\big|\hat\gamma_k-\tilde\gamma_k\big|.$$

Moreover,

$$\hat\gamma_k-\tilde\gamma_k = \frac{n-|k|}{n}(\bar{\tilde R})^2 - \frac{\bar{\tilde R}}{n}\Big(\sum_{i=1}^{n-|k|}\tilde R_i + \sum_{i=1}^{n-|k|}\tilde R_{i+|k|}\Big).$$

Because $\sum_{t\in\mathbb Z}|\gamma_R(t)|<\infty$, $E\big(\sum_{i=1}^n\tilde R_i\big)^2\le Cn$. This together with the stationarity and the Cauchy-Shwarz inequality yield

$$\begin{aligned}
E\big|\hat\gamma_k-\tilde\gamma_k\big| &\le Cn^{-1}+2n^{-1}E^{1/2}(\bar{\tilde R})^2E^{1/2}\Big(\sum_{i=1}^{n-|k|}\tilde R_i\Big)^2\\
&\le C\big[n^{-1}+n^{-1}(1-\frac{|k|}{n})^{1/2}\big]\le Cn^{-1},\\
E|\hat\sigma^2_{R,q}-\tilde\sigma^2_{R,q}| &\le C(q/n)\to 0,
\end{aligned}$$

thereby completing the proof of (8.5)(b).

Proof of (8.5)(a). Let $\tilde X_t = X_t - EX_t$. Since $R_t = \varepsilon_t\tilde X_t - \beta u_t\tilde X_t + \varepsilon_t u_t - \beta u_t^2$, we rewrite

$$\begin{aligned}
\tilde\sigma^2_{R,q} &= \tilde\sigma^2_{\varepsilon\tilde X,q}+\beta^2\tilde\sigma^2_{u\tilde X,q}+\tilde\sigma^2_{\varepsilon u,q}+\beta^2\tilde\sigma^2_{u^2,q}-2\beta\tilde\sigma_{\varepsilon\tilde X,u\tilde X,q}\\
&\quad+2\tilde\sigma_{\varepsilon\tilde X,\varepsilon u,q}-2\beta\tilde\sigma_{\varepsilon\tilde X,u^2,q}+2\beta^2\tilde\sigma_{u\tilde X,u^2,q}-2\beta\tilde\sigma_{\varepsilon u,u^2,q},
\end{aligned}\tag{8.6}$$

where

$$\begin{aligned}
\tilde\sigma^2_{\varepsilon\tilde X,q} &:= n^{-1}\sum_{|k|\le q}(1-\frac{|k|}{q})\sum_{1\le t,s\le n,t-s=k}\varepsilon_t\tilde X_t\,\varepsilon_s\tilde X_s,\\
\tilde\sigma_{\varepsilon\tilde X,u\tilde X,q} &:= n^{-1}\sum_{|k|\le q}(1-\frac{|k|}{q})\sum_{1\le t,s\le n,t-s=k}\varepsilon_t\tilde X_t\,u_s\tilde X_s,
\end{aligned}$$

and the other quantities in (8.6) are defined analogously. Clearly, (8.5)(a) will follow if we show that the variance of each term on the r.h.s. of (8.6) tends to zero. Since the terms on the r.h.s. of (8.6) are particular cases of (8.7) below, these convergence results will follow from Lemma 8.1(i).

Lemma 8.1 *Let* $X_{t,i} = \sum_{k=0}^{\infty} b_{k,i}\xi_{t-k,i}$, $t \in \mathbb{Z}$, $i = 1,2,3,4$ *be four LMMA processes with* $b_{k,i} \sim \kappa_i k^{d_i-1}$, $d_i \in (0,1/2)$, $\kappa_i > 0$, $\xi_{t,i}$ *standardized i.i.d.,* $E\xi_{t,i}^8 < \infty$ *(independence of* $\xi_{t,i}, i = 1,2,3,4$ *is not assumed). Let* $\delta_{\max} := \max\{d_1+d_2, d_3+d_4\}$, $X_{t,12} := X_{t,1}X_{t,2}$, $X_{t,34} := X_{t,3}X_{t,4}$ *and*

$$(8.7) \qquad \tilde{\sigma}_{X_{12},X_{34},q} = n^{-1}\sum_{|k|\le q}(1-\frac{|k|}{q})\sum_{1\le t,s\le n, t-s=k}(X_{t,12}-EX_{t,12})(X_{s,34}-EX_{s,34}).$$

(i) *If* $\delta_{\max} < 1/2$, *then* $\sum_{t\in\mathbb{Z}}|\mathrm{Cov}(X_{0,12}X_{t,34})| < \infty$ *and*

$$(8.8) \quad \text{(a)} \quad E\tilde{\sigma}_{X_{12},X_{34},q} \to \sigma_{X_{12},X_{34}} = \sum_{t\in\mathbb{Z}}\mathrm{Cov}(X_{0,12},X_{t,34}), \qquad \text{(b)} \quad \mathrm{Var}(\tilde{\sigma}_{X_{12},X_{34},q}) \to 0.$$

Particularly, $\tilde{\sigma}_{X_{12},X_{34},q}$ *is a consistent estimator of* $\sigma_{X_{12},X_{34}}$.

(ii) *If* $\delta_{\max} \ge 1/2$, *then*

$$(8.9) \qquad \tilde{\sigma}_{X_{12},X_{34},q} = O_p\big(q^{2\delta_{\max}-1}(1+(\log q)I(\delta_{\max}=1/2))\big).$$

The proof of this lemma will be given later. Here we show how it is used to conclude (8.5)(a). Recall (8.6). In Lemma 8.1, take $X_{t,1} \equiv \varepsilon_t \equiv X_{t,3}$, $X_{t,2} \equiv \tilde{X}_t \equiv X_{t,4}$. Then $\tilde{\sigma}_{X_{12},X_{34},q} = \tilde{\sigma}^2_{\varepsilon\tilde{X},q}$, $d_1 = d_\varepsilon = d_3$, $d_2 = d_X = d_4$, and $\delta_{\max} = \max\{d_1+d_2, d_3+d_4\} = d_\varepsilon + d_X$. The assumption $D_{\max} = \max\{d_X+d_\varepsilon, d_X+d_u, d_u+d_\varepsilon, 2d_u\} < 1/2$ of Proposition 4.1 implies that $\delta_{\max} < 1/2$. Hence by Lemma 8.1(i)(b), $\mathrm{Var}(\tilde{\sigma}^2_{\varepsilon\tilde{X},q}) \to 0$.

Next, take $X_{t,1} \equiv \varepsilon_t$, $X_{t,2} \equiv \tilde{X}_t \equiv X_{t,4}$, $X_{t,3} \equiv u_t$. Then $\tilde{\sigma}_{X_{12},X_{34},q} = \tilde{\sigma}_{\varepsilon\tilde{X},u\tilde{X},q}$, $d_1 = d_\varepsilon$ $d_2 = d_X$, $d_3 = d_4 = d_u$, $\delta_{\max} = \max\{d_1+d_2, d_3+d_4\} = \max\{d_\varepsilon+d_X, 2d_u\}$, and again $D_{\max} < 1/2$ implies $\delta_{\max} < 1/2$ and we obtain $\mathrm{Var}(\tilde{\sigma}_{\varepsilon\tilde{X},u\tilde{X},q}) \to 0$, by Lemma 8.1(i)(b). Similarly, the variances of the other terms on the right hand side of (8.6) are shown to tend to zero, thereby completing the proof of (8.5)(a). This also completes the proof of (8.3)(a).

Proof of (8.3)(b). Let $\bar{e} = n^{-1}\sum_{t=1}^n e_t$, $\bar{Z}^2 := n^{-1}\sum_{t=1}^n Z_t^2$. Use $Y_t = \alpha + \beta Z_t + e_t$ to write

$$\begin{aligned}
\hat{e}_t - e_t &= Y_t - \hat{\alpha} - \hat{\beta}Z_t - e_t = (\alpha-\hat{\alpha}) + (\beta-\hat{\beta})Z_t, \\
\hat{R}_t &= e_t(Z_t-\bar{Z}) + (\alpha-\hat{\alpha})(Z_t-\bar{Z}) + (\beta-\hat{\beta})Z_t(Z_t-\bar{Z}), \\
&= R_t + e_t(EZ-\bar{Z}) + (\alpha-\hat{\alpha})(Z_t-\bar{Z}) + (\beta-\hat{\beta})Z_t(Z_t-\bar{Z}), \\
&= R_t + g_{n,t}, \\
\bar{\hat{R}} &= \bar{R} + \bar{e}(EZ-\bar{Z}) + (\beta-\hat{\beta})(\bar{Z}^2-(\bar{Z})^2),
\end{aligned}$$

where

$$g_{n,t} = e_t(EZ - \bar{Z}) + (\alpha - \hat{\alpha})(Z_t - \bar{Z}) + (\beta - \hat{\beta})Z_t(Z_t - \bar{Z}).$$

Then $\hat{\sigma}^2_{\hat{R},q} - \hat{\sigma}^2_{R,q} = 2\hat{\sigma}_{R,g_n,q} + \hat{\sigma}^2_{g_n,q}$ where

$$\hat{\sigma}_{R,g_n,q} := n^{-1} \sum_{\ell=1-q}^{n-1} \big(\sum_{1\le t\le n: 1+\ell\le t\le q+\ell} (R_t - \bar{R}) \big) \big(\sum_{1\le s\le n: 1+\ell\le s\le q+\ell} (g_{n,s} - \bar{g}_n) \big)$$

satisfies $|\hat{\sigma}_{R,g_n,q}| \le \sqrt{\hat{\sigma}^2_{R,q}}\sqrt{\hat{\sigma}^2_{g_n,q}}$, see (8.2), also Koul et al (2016), p.176. Hence (8.3)(b) follows from $\hat{\sigma}^2_{R,q} = O_p(1)$ (which is implied by (8.3)(a)) and

(8.10) $$\hat{\sigma}^2_{g_n,q} \ \to_p \ 0.$$

By the first inequality in (8.2), the proof of (8.10) reduces to the corresponding statements for processes $e_t(EZ_0 - \bar{Z}), (\alpha - \hat{\alpha})(Z_t - \bar{Z}), (\beta - \hat{\beta})Z_t^2$, and $-(\beta - \hat{\beta})\bar{Z}\, Z_t$ whose sum is $g_{n,t}$. More precisely, (8.10) follows from

(8.11) $$\begin{aligned} &\text{(i)} \quad (EZ_0 - \bar{Z})^2 \hat{\sigma}^2_{e,q} \ \to_p \ 0, \qquad &&\text{(ii)} \quad (\hat{\alpha} - \alpha)^2 \hat{\sigma}^2_{Z,q} \ \to_p \ 0, \\ &\text{(iii)} \quad (\hat{\beta} - \beta)^2 \hat{\sigma}^2_{Z^2,q} \ \to_p \ 0, \qquad &&\text{(iv)} \quad (\beta - \hat{\beta})^2 (\bar{Z})^2 \hat{\sigma}^2_{Z,q} \ \to_p \ 0. \end{aligned}$$

Consider (8.11)(i). We have $E(EZ_0 - \bar{Z})^2 \le 2\mathrm{Var}(\bar{X}) + 2\mathrm{Var}(\bar{u}) \le C(n^{2d_X - 1} + n^{2d_u - 1})$ and $\hat{\sigma}^2_{e,q} \le 2\hat{\sigma}^2_{\varepsilon,q} + 2\beta^2\hat{\sigma}^2_{u,q}$ where $q^{-2d_\varepsilon}\hat{\sigma}^2_{\varepsilon,q} = O_p(1), q^{-2d_u}\hat{\sigma}^2_{u,q} = O_p(1)$, see GKS Thm. 9.4.1 or Lavancier et al (2010) (under $(2+\epsilon)$-condition on the innovations as in Corollary 4.1). Therefore (8.11)(i) holds provided n, q satisfy

$$q^{2d_\varepsilon} = o(n^{1-2d_u}), \quad q^{2d_\varepsilon} = o(n^{1-2d_X}), \quad q^{2d_u} = o(n^{1-2d_u}), \quad q^{2d_u} = o(n^{1-2d_X}),$$

or $(q/n)^{2d_\varepsilon} = o(n^{1-2(d_u+d_\varepsilon)}) = o(n^{1-2\delta_{\max}}), (q/n)^{2d_\varepsilon} = o(n^{1-2(d_u+d_X)}) = o(n^{1-2\delta_{\max}}), (q/n)^{2d_u} = o(n^{1-4d_u}) = o(n^{1-2\delta_{\max}}), (q/n)^{2d_u} = o(n^{1-2(d_u+d_X)}) = o(n^{1-2\delta_{\max}})$, which clearly hold by $q/n \to 0, \delta_{\max} < 1/2$, proving (8.11)(i).

Consider (8.11)(ii). Since $\hat{\alpha} - \alpha = O_p(n^{1-2(d_u \vee d_\varepsilon)})$ by Cor. 4.1(v), (8.11)(ii) follows from

$$q^{2d_X} = o(n^{1-2(d_u\vee d_\varepsilon)}), \quad q^{2d_u} = o(n^{1-2(d_u\vee d_\varepsilon)})$$

which again reduce to $q/n \to 0, \delta_{\max} < 1/2$. The proof of (8.11)(iv) is analogous since $\hat{\beta}$ converges faster than $\hat{\alpha}$, see Corollary 4.1(v).

It remains to prove (8.11)(iii) which follows from

(8.12) $$\hat{\sigma}^2_{Z^2,q} = o_p(n)$$

since $(\hat{\beta} - \beta)^2 = O(n^{-1})$ by Corollary 4.1(v). We have $\hat{\sigma}^2_{Z^2,q} \le 2(\hat{\sigma}^2_{X^2,q} + \hat{\sigma}^2_{u^2,q})$. Arguing as for the proof of (8.3)(a)-(b) one can show that $\hat{\sigma}^2_{u^2,q} \to \sigma^2_{u^2}$. Hence (8.12) follows from

(8.13) $$\hat{\sigma}^2_{X^2,q} = o_p(n).$$

For this, we use Lemma 8.1 with $X_{t,1} = ... = X_{t,4} = X_t$ and $d_1 = ... = d_4 = d_X, \delta_{\max} = 2d_X$. If $d_X < 1/4$ then (i) of the above lemma applies, yielding $\hat{\sigma}^2_{X^2,q} = O_p(1) = o_p(n)$. If $d_X > 1/4$ then by part (ii), $\hat{\sigma}^2_{X^2,q} = O_p(q^{4d_X-1}) = o_p(q) = o_p(n)$ and for $d_X = 1/4$ we get $\hat{\sigma}^2_{X^2,q} = O_p(\log(q)) = o_p(n)$, proving (8.13). This proves (8.11)(iii) and completes the proof of (8.3)(a), hence that of the Proposition 4.1. □

The proof of Lemma 8.1 is facilitated by the following lemma, which provides inequalities that are often used in the sequel.

Lemma 8.2 *(i) For $a > 0, 0 < b < 1, a + b > 1$*

$$\sum_{s\in\mathbb{Z}} |s|_+^{-a}|t-s|_+^{-b} \le C \begin{cases} |t|_+^{1-a-b}, & 0 < a < 1, \\ |t|_+^{-b}, & a > 1, \end{cases} \qquad t \in \mathbb{Z}. \tag{8.14}$$

(ii) For $a > 1, 0 < b < 1, q \ge 1, t \in \mathbb{Z}$

$$\sum_{|s|\ge q} |s|^{-a}|t-s|_+^{-b} \le Cq^{a-1}(q \vee |t|_+)^{-b}, \tag{8.15}$$

$$\sum_{|s|\le q} |t-s|_+^{-b} \le Cq(q \vee |t|_+)^{-b}. \tag{8.16}$$

(iii) For $a > 0, b > 0, c > 0, q \ge 1$

$$\sum_{|t|,|s|\le q} |t|_+^{-a}|s|_+^{-b}|t-s|_+^{-c} \le C\big(q^{(1-a)_+ + (1-b-c)_+} + q^{(1-b)_+ + (1-a-c)_+}\big)\big(1 + \log_{a,b,c}(q)\big), \tag{8.17}$$

where $\log_{a,b,c}(q) := \log(q)\big(\mathbf{1}(a=1) + \mathbf{1}(b+c=1) + \mathbf{1}(b=1) + \mathbf{1}(a+c=1)\big)$, $(x)_+ := x \vee 0, x \in \mathbb{R}$.

Proof. (i) follows from (10.2.53) of GKS.
(ii) To check (8.15), let $t \ge 1$ w.l.g. First, let $q \ge t$ then $\sum_{|s|\ge q} |s|^{-a}|t-s|_+^{-b} \le 2\sum_{s\ge q} s^{-a}(s-t)^{-b} = \sum_{s=0}^{\infty}(s+q)^{-a}(s+q-t)_+^{-b} \le \sum_{s=0}^{\infty}(s+q)^{-a}s_+^{-b} \le \int_0^{\infty}(x+q)^{-a}x^{-b}dx = Cq^{a+b-1}$. Next, let $1 \le q \le t$ then $\sum_{s\ge q} s^{-a}(s-t)^{-b} = \sum_{q\ge s<t/2} s^{-a}(s-t)^{-b} + \sum_{t/2\le s\le 3t/2} s^{-a}(s-t)^{-b} + \sum_{s>3t/2} s^{-a}(s-t)^{-b} \le (t/2)^{-b}\sum_{s\ge q} s^{-a} + (t/2)^{-a}\sum_{|s|\le t/2} |s|^{-b} + C\sum_{s\ge 3t/2} s^{-a-b} \le C(t^{-b}q^{1-a} + t^{1-a-b}) \le Ct^{-b}q^{1-a}$, proving (8.15).
To show (8.16), let $|t| \le 2q$ then $\sum_{|s|\le q} |t-s|_+^{-b} \le \sum_{|s|\le 3q} |s|_+^{-b} \le Cq^{1-b}$. Next, let $|t| > 2q$ then $\sum_{|s|\le q} |t-s|_+^{-b} \le C\sum_{|t|\le s\le |t|+q} |s|^{-b} \le Cq|t|^{-b}$, proving (8.16).
(iii) Let $\log_{a,b,c}(q) = 0$ for simplicity. Split J_q(= the l.h.s. of (8.17)) as $J_q = \sum_{i=1}^{3} J_{q,i}$ according to whether $|s| < |t|/2, |s| > 2|t|$ and $|t|/2 \le |s| \le 2|t|$, respectively. Note $|s| < |t|/2$ implies $|s-t| > |t|/2$ while $|s| > 2|t|$ implies $|s-t| > |s|/2$. Hence $J_{q,1} \le C\sum_{|t|,|s|\le q} |t|_+^{-a-c}|s|_+^{-b} \le Cq^{(1-a-c)_+ + (1-b)_+}$ and similarly, $J_{q,3} \le C\sum_{|t|,|s|\le q} |t|_+^{-a}|s|_+^{-b-c} \le$

$Cq^{(1-a)_+ + (1-b-c)_+}$ which agree with (8.17).
Consider $J_{q,3} = J^+_{q,3} + J^-_{q,3}$ where the $\pm$ subscripts refer to $\mathrm{sign}(t) = \mathrm{sign}(s)$ and $\mathrm{sign}(t) \neq \mathrm{sign}(s)$. Clearly, $J^-_{q,3} \le J^+_{q,3} \le 2\sum_{0\le t,s\le q, t/2\le s\le 2t} t_+^{-a} s_+^{-b} |t-s|_+^{-c}$. Note $t/2 \le s \le 2t$ implies $|t-s| \le |t|$ and hence $J_{q,3} \le C\sum_{0\le t,s\le q} s_+^{-b}|t-s|_+^{-a-c} \le Cq^{(1-a-c)_+ + (1-b)_+}$. For $\log_{a,b,c}(q) \ge \log(q) \neq 0$ (8.17) follows similarly. This proves (8.17) and the lemma, too. □

Proof of Lemma 8.1. *Proof of* (i). We have $X_{t,12} - EX_{t,12} = \sum_{s_1,s_2\le t} b_{t-s_1,1} b_{t-s_2,2} \xi_{s_1,1}\xi_{s_2,2} = Y^0_{t,12} + Y_{t,12} + Y_{t,21}$, where

$$\begin{aligned} (8.18) \qquad Y^0_{t,12} &:= \sum_{s\le t} b_{t-s,1} b_{t-s,2} (\xi_{s,1}\xi_{s,2} - E\xi_{s,1}\xi_{s,2}), \\ Y_{t,12} &:= \sum_{s_2<s_1\le t} b_{t-s_1,1} b_{t-s_2,2} \xi_{s_1,1}\xi_{s_2,2}, \qquad Y_{t,21} := \sum_{s_1<s_2\le t} b_{t-s_1,1} b_{t-s_2,2} \xi_{s_1,1}\xi_{s_2,2} \end{aligned}$$

are stationary processes with zero means, finite variances and respective covariance functions $\mathrm{Cov}(Y^0_{t,12}, Y^0_{u,12}) = \mathrm{Var}(\xi_{0,1}\xi_{0,2}) \sum_{s\le t\wedge u} b_{t-s,1} b_{t-s,2} b_{u-s,1} b_{u-s,2}$,

$$\begin{aligned} \mathrm{Cov}(Y_{t,12}, Y_{u,12}) &= \sum_{s_2<s_1\le t\wedge u} b_{t-s_1,1} b_{t-s_2,2} b_{u-s_1,1} b_{u-s_2,2}, \\ \mathrm{Cov}(Y_{t,21}, Y_{u,21}) &= \sum_{s_1<s_2\le t\wedge u} b_{t-s_1,1} b_{t-s_2,2} b_{u-s_1,1} b_{u-s_2,2}. \end{aligned}$$

Similarly, $X_{t,34} - EX_{t,34} = \sum_{s_3,s_4\le t} b_{t-s_3,3} b_{t-s_4,4} \xi_{s_3,3}\xi_{s_4,4} = Y^0_{t,34} + Y_{t,34} + Y_{t,43}$, where $Y^0_{t,34}, Y_{t,34}, Y_{t,43}$ are defined analogously as (8.18). Since

$$\sum_{k=0}^\infty |b_{k,1} b_{k,2}| \le C \sum_{k=0}^\infty k_+^{d_1+d_2-2} < \infty, \quad \sum_{k=0}^\infty |b_{k,3} b_{k,4}| \le C \sum_{k=0}^\infty k_+^{d_3+d_4-2} < \infty,$$

we conclude that $Y^0_{t,12}, Y^0_{t,34}$ are linear processes with summable covariances and cross covariances, i.e., $\sum_{t\in\mathbb{Z}} (|\mathrm{Cov}(Y^0_{t,12}, Y^0_{0,12})| + |\mathrm{Cov}(Y^0_{t,34}, Y^0_{0,34})| + |\mathrm{Cov}(Y^0_{t,12}, Y^0_{0,34})|) < \infty$.

Next, let

$$\Delta := d_1 + d_2 + d_3 + d_4. \tag{8.19}$$

Because $\delta_{\max} = \max\{d_1+d_2, d_3+d_4\} < 1/2$, $0 < \Delta \le 2\delta_{\max} < 1$ and, by (8.14),

$$|\mathrm{Cov}(Y_{t,12}, Y_{0,34})| + |\mathrm{Cov}(Y_{t,12}, Y_{0,43})| + |\mathrm{Cov}(Y_{t,21}, Y_{0,34})| + |\mathrm{Cov}(Y_{t,21}, Y_{0,43})| \le C|t|_+^{\Delta-2}.$$

Also note that for all $t,u \in \mathbb{Z}$, $\mathrm{Cov}(Y^0_{t,12}, Y_{u,34}) = \mathrm{Cov}(Y^0_{t,12}, Y_{u,43}) = \mathrm{Cov}(Y_{t,12}, Y^0_{u,34}) = \mathrm{Cov}(Y_{t,21}, Y^0_{u,34}) = 0$. These facts imply that

$$\sum_{\tau\in\mathbb{Z}} |\mathrm{Cov}\big(X_{t,12}, X_{0,12}\big)| < \infty, \quad \lim_{n\to\infty} n^{-1} \mathrm{Cov}\Big(\sum_{t=1}^n X_{t,12}, \sum_{s=1}^n X_{s,34}\Big) = \sigma_{X_{12},X_{34}},$$

with $\sigma_{X_{12},X_{34}}$ defined at (8.8). As in GKS (9.4.8), these facts in turn yield (8.8)(a).

It remains to prove (8.8)(b). Similarly to (8.6) we rewrite

$$\begin{aligned}\tilde\sigma_{X_{12},X_{34},q} &= \tilde\sigma_{Y_{12},Y_{34},q}+\tilde\sigma_{Y_{12},Y_{43},q}+\tilde\sigma_{Y_{12},Y^0_{34},q}+\tilde\sigma_{Y_{21},Y_{34},q}+\tilde\sigma_{Y_{21},Y_{43},q}\\ &\quad +\tilde\sigma_{Y_{21},Y^0_{34},q}+\tilde\sigma_{Y^0_{12},Y_{34},q}+\tilde\sigma_{Y^0_{12},Y_{43},q}+\tilde\sigma_{Y^0_{12},Y^0_{34},q}.\end{aligned} \tag{8.20}$$

Clearly (8.8)(b) follows once we prove that the variance of each term on the r.h.s. of (8.20) vanishes in the limit. The subsequent discussion is limited to the proof of

$$\text{(a)}\quad \mathrm{Var}(\tilde\sigma_{Y_{12},Y_{34},q}) \ \to\ 0 \qquad \text{and} \qquad \text{(b)}\quad \mathrm{Var}(\tilde\sigma_{Y_{12},Y^0_{34},q}) \ \to\ 0, \tag{8.21}$$

since the remaining variances can be evaluated in a similar fashion.

Consider the claim (8.21)(a). Write $Y_{t,12} =: Y_t,\ Y_{t,34} =: Y'_t$ for brevity. Then

$$\begin{aligned}\mathrm{Var}(\tilde\sigma_{Y_{12},Y_{34},q}) &= n^{-2}\sum_{|k_1|,|k_2|\le q}(1-\frac{|k_1|}{q})(1-\frac{|k_2|}{q})\sum_{1\le t_1,t_2\le n}\mathrm{Cov}\big(Y_{t_1}Y_{t_1+k_1},Y'_{t_2}Y'_{t_2+k_2}\big)\\ &\le n^{-2}\sum_{|k_1|,|k_2|\le q}\ \sum_{1\le t_1,t_2\le n}\big|\mathrm{Cov}\big(Y_{t_1}Y_{t_1+k_1},Y'_{t_2}Y'_{t_2+k_2}\big)\big|\\ &\le n^{-1}\sum_{|k_1|,|k_2|\le q}\ \sum_{|t|\le n}\big|\mathrm{Cov}\big(Y_0Y_{k_1},Y'_tY'_{t+k_2}\big)\big| \ \to\ 0,\end{aligned} \tag{8.22}$$

provided for some $\nu > 0$

$$\mathcal{M}_{n,q} := \sum_{|k_1|,|k_2|\le q}\ \sum_{|t|\le n}\big|\mathrm{Cov}\big(Y_0Y_{k_1},Y'_tY'_{t+k_2}\big)\big| \ \le\ Cn(q/n)^{\nu}. \tag{8.23}$$

As in GKS, p.281, use the bound $|\mathrm{Cov}(Y_0,Y'_t)| \le C|t|_+^{\Delta-2}$, see above, and the identity

$$\begin{aligned}&\mathrm{Cov}(Y_0Y_{k_1},Y'_tY'_{t+k_2})\\ &\quad= \mathrm{Cum}(Y_0,Y_{k_1},Y'_t,Y'_{t+k_2})+\mathrm{Cov}(Y_0,Y'_t)\mathrm{Cov}(Y_{k_1},Y'_{t+k_2})+\mathrm{Cov}(Y_0,Y'_{t+k_2})\mathrm{Cov}(Y_{k_1},Y'_t),\end{aligned}$$

to obtain $\mathcal{M}_{n,q} \le \sum_{i=1}^3 \mathcal{M}_{n,q,i}$, where

$$\begin{aligned}\mathcal{M}_{n,q,1} &:= \sum_{|k_1|,|k_2|\le q}\ \sum_{|t|\le n}\big|\mathrm{Cum}\big(Y_0,Y_{k_1},Y'_t,Y'_{t+k_2}\big)\big|,\\ \mathcal{M}_{n,q,2} &\le C\sum_{|k_1|,|k_2|\le q,|t|\le n}|t|_+^{\Delta-2}|t+k_2-k_1|_+^{\Delta-2} \le Cq(\sum_{t=1}^\infty t^{\Delta-2})(\sum_{k=1}^\infty k^{\Delta-2}) \le Cq,\\ \mathcal{M}_{n,q,3} &\le C\sum_{|k_1|,|k_2|\le q,|t|\le n}|t+k_2|_+^{\Delta-2}|t-k_1|_+^{\Delta-2} \le Cq.\end{aligned}$$

Whence (8.23) with $\nu = 1$ follows if we prove that

$$\mathcal{M}_{n,q,1} \ \le\ Cq. \tag{8.24}$$

We note that (8.24) is weaker than Assumption M in GKS thm. 9.4.1, viz,

$$\max_{t_3} \sum_{t_1,t_2=-n}^{n} |\mathrm{Cum}(Y_0, Y_{t_1}, Y_{t_2}, Y_{t_3})| \ \le \ C,$$

for the consistency of the HAC estimator which apparently is not satisfied by $X_{t,12}$ and some other processes discussed here. Let

$$B_{0,k_1,t,t+k_2}(u_1, s_1, \cdots, u_4, s_4) := b_{-s_1,1} b_{-u_1,2} b_{k_1-s_2,1} b_{k_1-u_2,2} b_{t-s_3,3} b_{t-u_3,4} b_{t+k_2-s_4,3} b_{t+k_2-u_4,4}.$$

By the multilinearity property of cumulants

$$\begin{aligned} \text{(8.25)}\quad \mathrm{Cum}(Y_0, Y_{k_1}, Y'_t, Y'_{t+k_2}) \ &= \sum_{s_i<u_i, i=1,2,3,4} B_{0,k_1,t,t+k_2}(u_1, s_1, \cdots, u_4, s_4) \\ &\qquad \times \mathrm{Cum}(\xi_{s_1,1}\xi_{u_1,2},\ \xi_{s_2,1}\xi_{u_2,2},\ \xi_{s_3,3}\xi_{u_3,4},\ \xi_{s_4,3}\xi_{u_4,4}). \end{aligned}$$

To proceed further we need to introduce the tables

$$\text{(8.26)}\qquad T_1 := \begin{pmatrix} s_1 & u_1 \\ s_2 & u_2 \\ s_3 & u_3 \\ s_4 & u_4 \end{pmatrix}, \qquad T_2 := \begin{pmatrix} s_1 & u_1 \\ s_2 & s_2 \\ s_3 & u_3 \\ s_4 & s_4 \end{pmatrix}.$$

Using GKS (14.1.15), we rewrite

$$\text{(8.27)}\qquad \mathrm{Cum}(\xi_{s_1,1}\xi_{u_1,2},\ \xi_{s_2,1}\xi_{u_2,2},\ \xi_{s_3,3}\xi_{u_3,4},\ \xi_{s_4,3}\xi_{u_4,4}) = \sum_{\{V\}\subset\Gamma^c_{T_1}} I_V,$$

where $I_V = \prod_{k=1}^r \mathrm{Cum}(\xi_{s_i,1},\ \xi_{s_i,3},\ \xi_{u_j,2},\ \xi_{u_j,4}; s_i \in V_k, u_j \in V_k)$ and the sum is taken over all connected diagrams $\{V\} = (V_1, \cdots, V_r)$ (partitions) of the table T_1. Since random vectors $(\xi_{s,1}, \xi_{s,2},\ \xi_{s,3}, \xi_{s,4}), s \in \mathbb{Z}$ are independent and $s_i < u_i, i = 1, 2, 3, 4$ this implies that $\mathrm{Cum}(\xi_{s_i,1},\ \xi_{s_i,3},\ \ \xi_{u_j,2},\ \xi_{u_j,4}; s_i \in V, u_j \in V) = 0$ for any V which contains both elements from a single row of T_1; in other words, diagrams $\{V\} = (V_1, \cdots, V_r)$ with $I_V \neq 0$ consist of 'vertical' partitions connecting different rows of T_1. There are four types of such partitions designated as D1) $\{V\} = (2, 2, 2, 2)$, D2) $\{V\} = (4, 2, 2)$, D3) $\{V\} = (3, 3, 2)$ and D4) $\{V\} = (4, 4)$. More precisely, D1) corresponds to $\{V\} = (V_1, V_2, V_3, V_4), |V_1| = |V_2| = |V_3| = |V_4| = 2$, D1) to $\{V\} = (V_1, V_2, V_3), |V_1| = 4, |V_2| = |V_3| = 2$, D3) to $\{V\} = (V_1, V_2, V_3), |V_1| = |V_2| = 3, |V_2| = 2$, and D4) to $\{V\} = (V_1, V_2), |V_1| = |V_2| = 4$.

By (8.25)–(8.27), the l.h.s. of (8.25) can be written as $\sum_{\{V\}\subset\Gamma^c_{T_1}} M_V(0, k_1, t, t+k_2)$, where

$$\text{(8.28)}\qquad M_V(0, k_1, t, t + k_2) := \sum_{s_i<u_i, i=1,2,3,4} B_{0,k_1,t,t+k_2}(u_1, s_1, \cdots, u_4, s_4).$$

Below we estimate $M_V(0, k_1, t, t + k_2)$ in each case D1)-D4), which will prove (8.24).

Case D1) It suffices to consider the three diagrams: $u_1 = s_2 =: u, u_2 = s_3 =: v, u_3 = s_4 =: w, u_4 = s_1 =: s$ D1a), $u_1 = u_4 =: u, s_1 = s_3 =: s, u_2 = u_3 =: v, s_2 = s_4 =: w$ D1b), and $u_1 = u_4 =: u$, $s_1 = s_3 =: s$, $s_2 = u_3 =: v$, $s_4 = u_2 =: w$ D1c) in table T_1 in (8.26). (Here and below, we identify a diagram with a set of equalities between the variables in T_1. E.g., D1c) corresponds to partition $\{V\} = (V_1, V_2, V_3, V_4), V_1 = \{u_1, u_4\}, V_2 = \{s_1, s_3\}, V_3 = \{s_2, u_3\}, V_4 = \{s_4, u_2\}$.)

For D1a), by (8.14),

$$\begin{aligned} |M_V(0, k_1, t, t + k_2)| &\le C \sum_{s,u,v,w} |b_{-s,1} b_{-u,2} b_{k_1-u,1} b_{k_1-v,2} b_{t-v,3} b_{t-w,4} b_{t+k_2-w,3} b_{t+k_2-s,4}| \\ &\le C|t + k_2|_+^{-(1-d_1-d_4)} |k_1|_+^{-(1-d_1-d_2)} |k_1 - t|_+^{-(1-d_3-d_2)} |k_2|_+^{-(1-d_3-d_4)}. \end{aligned}$$

Hence, using (8.14) with $a = 1 - d_1 - d_4, b = 1 - d_2 - d_3, a + b = 2 - \Delta > 1$, see (8.19),

$$\begin{aligned} (8.29) \qquad & \sum_{|t|\le n, |k_1|, |k_2| \le q} |M_V(0, k_1, t, t + k_2)| \\ &\le C \sum_{t\in\mathbb{Z}, |k_1|, |k_2|\le q} \frac{1}{|t + k_2|_+^{1-d_1-d_4} |k_1|_+^{1-d_1-d_2} |k_1 + t|_+^{1-d_2-d_3} |k_2|_+^{1-d_3-d_4}} \\ &\le C \sum_{|k_1|, |k_2|\le q} \frac{1}{|k_1|_+^{1-d_1-d_2} |k_2|_+^{1-d_3-d_4}} \le Cq^{\Delta} \le Cq. \end{aligned}$$

For D1b), use (8.14) w.r.t. $s, u, v, w \in \mathbb{Z}$ to obtain

$$\begin{aligned} (8.30) \quad & |M_V(0, k_1, t, t + k_2)| \\ &\le C \sum_{s,u,v,w} |b_{-s,1} b_{t-s,3} b_{-u,2} b_{t+k_2-u,4} b_{k_1-v,2} b_{t-v,4} b_{k_1-w,1} b_{t+k_2-w,3}| \\ &\le C|t|_+^{-(1-d_1-d_3)} |t + k_2|_+^{-(1-d_2-d_4)} |t - k_1|_+^{-(1-d_2-d_4)} |t + k_2 - k_1|_+^{-(1-d_1-d_3)}. \end{aligned}$$

Next, use (8.14) w.r.t. $t \in \mathbb{Z}$ and $k_1 \in \mathbb{Z}$ to obtain

$$(8.31) \sum_{|t|\le n, |k_1|, |k_2|\le q} |M_V(0, k_1, t, t + k_2)| \le C \sum_{|k_2|\le q} |k_2|_+^{2(\Delta-2)} \le Cq^{(2\Delta-1)_+} \le Cq.$$

For D1c), exactly similarly as above,

$$\begin{aligned} & |M_V(0, k_1, t, t + k_2)| \\ &\le C \sum_{s,u,v,w} |b_{-s,1} b_{t-s,3} b_{-u,2} b_{t+k_2-u,4} b_{k_1-v,1} b_{t-v,4} b_{k_1-w,2} c_{t+k_2-w,3}| \\ &\le C|t|_+^{-(1-d_1-d_3)} |t + k_2|_+^{-(1-d_2-d_4)} |t - k_1|_+^{-(1-d_1-d_4)} |t + k_2 - k_1|_+^{-(1-d_2-d_3)} \end{aligned}$$

resulting in the same bound as (8.30) above.

Case D2). It suffices to consider two diagrams $s_1 = s_2 = s_3 = s_4 =: s$, $u_1 = u_2 =: u$, $u_3 = u_4 =: v$ D2a), and $s_1 = s_2 = s_3 = s_4 =: s$, $u_1 = u_3 =: u$, $u_2 = u_4 =: v$ D2b). For D2a),

$$\begin{aligned}|M_V(0,k_1,t,t+k_2)| &\le C\sum_{s,u,v}|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{t+k_2-s,3}b_{-u,2}b_{k_1-u,2}b_{t-v,4}b_{t+k_2-v,4}| \\ &\le \frac{C}{|k_1|_+^{1-2d_2}|k_2|_+^{1-2d_4}}\sum_s|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{t+k_2-s,3}|.\end{aligned}$$

Because $\delta_{\max} < 1/2$, the above bound and (8.14) imply

$$\begin{aligned}&\sum_{|t|\le n,|k_1|,|k_2|\le q}|M_V(0,k_1,t,t+k_2)| \\ &\le C\sum_{|k_1|,|k_2|\le q}\frac{1}{|k_1|_+^{1-2d_2}|k_2|_+^{1-2d_4}}\sum_{t,s\in\mathbb{Z}}|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{t+k_2-s,3}| \\ &\le C\sum_{k_1,k_2\in\mathbb{Z}}\frac{1}{|k_1|_+^{2(1-d_1-d_2)}|k_2|_+^{2(1-d_3-d_4)}} \le C. \end{aligned} \tag{8.32}$$

Similarly, for D2b),

$$\begin{aligned}|M_V(0,k_1,t,t+k_2)| &\le C\sum_{s,u,v}|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{t+k_2-s,3}b_{-u,2}b_{t-u,4}b_{k_1-v,2}b_{t+k_2-v,4}| \\ &\le C|t|_+^{-(1-d_2-d_4))}|t+k_2-k_1|_+^{-(1-d_2-d_4)}\sum_s|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{t+k_2-s,3}|,\end{aligned}$$

and hence, by (8.14) and the fact $2-\Delta > 1$,

$$\begin{aligned}&\sum_{|t|\le n,|k_1|,|k_2|\le q}|M_V(0,k_1,t,t+k_2)| \\ &\le C\sum_{t,s,k_1,k_2}|t|_+^{d_2+d_4-1}|t+k_2-k_1|_+^{d_2+d_4-1}|s|_+^{d_1-1}|k_1-s|_+^{d_1-1}|t-s|_+^{d_3-1}|t+k_2-s|_+^{d_3-1} \\ &\le C\sum_{t,s,k_1}\frac{1}{|t|_+^{1-d_2-d_4}|s|_+^{1-d_1}|k_1-s|_+^{2-\Delta}|t-s|_+^{1-d_3}} \\ &\le C\sum_{t,s}\frac{1}{|t|_+^{1-d_2-d_4}|s|_+^{1-d_1}|t-s|_+^{1-d_3}} \le C\sum_t\frac{1}{|t|_+^{2-\Delta}} \le C.\end{aligned} \tag{8.33}$$

Case D3) Consider the two diagrams $s_1 = s_2 = s_3 =: s, u_1 = s_4 =: u, u_2 = u_3 = u_4 =: v$ D3a) and $s_1 = s_2 = s_3 =: s, u_1 = u_2 = u_4 =: u, u_3 = s_4 =: v$ D3b).

For D3a),

$$\begin{aligned}|M_V(0,k_1,t,t+k_2)| &\le C\sum_{s,u,v}|b_{-s,1}b_{k_1-s,1}b_{t-s,3}b_{-u,2}b_{t+k_2-u,3}b_{k_1-v,2}b_{t-v,4}b_{t+k_2-v,4}| \\ &\le C|t+k_2|_+^{-(1-d_2-d_3)}\sum_s|b_{-s,1}b_{k_1-s,1}b_{t-s,3}|\sum_v|b_{k_1-v,2}b_{t-v,4}b_{t+k_2-v,4}|.\end{aligned}$$

Therefore, $\sum_{|t|\le n,|k_1|,|k_2|\le q}|M_V(0,k_1,t,t+k_2)|\le CJ$ where

$$(8.34)\qquad J:=\sum_{|t|\le n,|k_1|,|k_2|\le q,s,v}|t+k_2|_+^{d_2+d_3-1}|s|_+^{d_1-1}|k_1-s|_+^{d_1-1}|t-s|_+^{d_3-1}\times|k_1-v|_+^{d_2-1}|t-v|_+^{d_4-1}|t+k_2-v|_+^{d_4-1}=:J_1+J_2,$$

and $J_1:=\sum_{|t|\le n,|k_1|,|k_2|\le q,v,|s|\ge 2|k_1|}\cdots$, $J_2:=\sum_{|t|\le n,|k_1|,|k_2|\le q,v,|s|<2|k_1|}\cdots$, split according to $|s|\ge 2|k_1|$ and $|s|<2|k_1|$. In J_1, we have that $|s|_+^{d_1-1}|k_1-s|_+^{d_1-1}\le C|s|_+^{2(d_1-1)}$ where $2(1-d_1)>1$. Therefore applying inequality (8.15) with $a=2(1-d_1), b=1-d_3$ results in

$$J_1\le C\sum_{|t|\le n,|k_1|,|k_2|\le q,v}|t+k_2|_+^{d_2+d_3-1}|k_1|_+^{2d_1-1}(|t|_+\vee|k_1|_+)^{d_3-1}\times|k_1-v|_+^{d_2-1}|t-v|_+^{d_4-1}|t+k_2-v|_+^{d_4-1}.$$

We further split the last sum as $J_1=J_{11}+J_{12}$ according to whether $|t|_+>6q$ or $|t|_+\le 6q$. By change $k_2+t\to\tilde k_2$ it follows that $|\tilde k_2|\le 8q$ in J_{12}. Therefore J_{12} can be bounded as

$$(8.35)\quad J_{12}\ \le\ C\sum_{|t|\le q,|k_1|,|k_2|\le 8q,v}|k_2|_+^{d_2+d_3-1}|k_1|_+^{2d_1+d_3-2}|k_1-v|_+^{d_2-1}|t-v|_+^{d_4-1}|k_2-v|_+^{d_4-1}.$$

Finally, split the last sum as $J_{12}\le C(J_{12}^++J_{12}^-)$ according to whether $|v|\le 3q$ or $|v|>3q$. Use the elementary fact that $\sum_{|t|\le q}|t-v|_+^{d_4-1}\le Cq^{d_4}$ uniformly in $|v|\le 3q$, see also (8.16), and (8.14) w.r.t. $v\in\mathbb{Z}$ to obtain

$$(8.36)\quad \begin{aligned} J_{12}^+ &\le Cq^{d_4}\sum_{|k_1|,|k_2|\le 2q,|v|\le 3q}|k_2|_+^{d_2+d_3-1}|k_1|_+^{2d_1+d_3-2}|k_1-v|_+^{d_2-1}|k_2-v|_+^{d_4-1}\\ &\le Cq^{d_4}\sum_{|k_1|,|k_2|\le 2q}|k_2|_+^{d_2+d_3-1}|k_1|_+^{2d_1+d_3-2}|k_1-k_2|_+^{d_2+d_4-1}\ \le\ Cq^{\lambda_{12}^+},\end{aligned}$$

where $\lambda_{12}^+:=\max\{d_2+d_3+d_4+(\Delta+d_1-2)_+, d_4+(2d_1+d_3-1)_++(2d_2+d_3+d_4-1)_+\}$ and where we used (8.17) with $a=1-d_2-d_3, b=2-2d_1-d_3, c=1-d_2-d_4$. By definition,

$$(8.37)\qquad \lambda_{12}^+\le\max\{2\Delta-2, d_2+d_3+d_4, 2d_1+d_3+d_4-1, 2d_2+d_3+2d_4-1\}.$$

Observe that $\delta_{\max}<1/2$ implies that each term on the r.h.s. is less than 1 and hence $\lambda_{12}^+<1$, implying $J_{12}^+\le Cq$.

Consider J_{12}^-. In this case $|k_1-v|,|t-v|,|k_2-v|>|v|$ in (8.35) and we obtain

$$(8.38)\qquad J_{12}^-\ \le\ C\sum_{|t|,|k_1|,|k_2|\le 3q}|k_2|_+^{d_2+d_3-1}|k_1|_+^{2d_1+d_3-2}\sum_{|v|>3q}|v|^{d_2+2d_4-3}\ \le\ Cq^{\lambda_{12}^-}$$

where

$$(8.39)\ \lambda_{12}^-:=2d_2+d_3+2d_4-1+(2d_1+d_3-1)_+\le\max\{2\Delta-2, 2d_2+d_3+d_4-1\}.$$

Observe again that $\delta_{\max} < 1/2$ implies $\lambda_{12}^- < 1$ and hence $J_{12}^- \le Cq$ and $J_{12} \le Cq$.

Next, consider

$$J_{11} \le C \textstyle\sum_{8q<|t|\le n, |k_1|, |k_2| \le q, v} |t|_+^{d_2+2d_3-2} |k_1|_+^{2d_1-1} |k_1 - v|_+^{d_2-1} |t-v|_+^{d_4-1} |t+k_2-v|_+^{d_4-1},$$

where we used the fact that $|t+k_2| \ge |t|_+/2$ for $|t| > 8q, |k_2| \le q$. Split $J_{11} \le J_{11}^+ + J_{11}^-$ according to whether $|v| \le 2q$ or $|v| > 2q$. Note $|t-v|_+ \ge |t|_+/2, |t+k_2-v|_+ \ge |t|_+/2$ in J_{12}^+. Therefore,

$$\begin{aligned} J_{11}^+ &\le C \sum_{8q<|t|\le n, |k_1|, |k_2|, |v| \le 2q} |t|_+^{d_2+2d_3+2d_4-4} |k_1|_+^{2d_1-1} |k_1-v|_+^{d_2-1} \\ &\le Cq^{1+2d_1+d_2} \sum_{|t|>8q} |t|^{d_2+2d_3+2d_4-4} \le Cq^{\lambda_{11}^+}, \qquad \lambda_{11}^+ := 2\Delta - 2. \end{aligned} \tag{8.40}$$

On the other hand, for J_{12}^- using $\sum_{|k_2|\le q} |t+k_2-v|_+^{d_4-1} \le Cq|t-v|_+^{d_4-1}$, see (8.16), $|k_1 - v|_+ > |v|_+/2$, the facts $2(1-d_4) > 1, 2d_2 + 2d_3 < 2$ and inequality (8.14) we obtain

$$\begin{aligned} J_{11}^- &\le Cq^{1+2d_1} \sum_{8q<|t|\le n, v} |t|_+^{d_2+2d_3-2} |v|_+^{d_2-1} |t-v|_+^{2(d_4-1)} \\ &\le Cq^{1+2d_1} \sum_{|t|>8q} |t|^{2d_2+2d_3-3} \le Cq^{\lambda_{11}^-}, \qquad \lambda_{11}^- := 2(d_1+d_2+d_3) - 1. \end{aligned} \tag{8.41}$$

It remains to evaluate J_2. Note $|s| \le 2|k_1|$ implies $|s| \le 2q$. By taking the sum over k_1 and using (8.14) we obtain

$$J_2 \le C \sum_{|t|\le n, |k_2|\le q, |s| \le 2q, v} |t+k_2|_+^{d_2+d_3-1} |s|_+^{d_1-1} |t-s|_+^{d_3-1} |s-v|_+^{d_1+d_2-1} |t-v|_+^{d_4-1} |t+k_2-v|_+^{d_4-1}.$$

Split the last sum $J_2 \le C(J_{21} + J_{22})$ according to $|t| > 3q$ and $|t| \le 3q$. Note the former assumption implies $|t+k_2| \le |t|/2, |t-s| \ge |t|/3$. Hence

$$\begin{aligned} J_{21} &\le C \sum_{3q<|t|\le n, |k_2|\le q, |s|\le 2q, v} |t|_+^{d_2+2d_3-2} |s|_+^{d_1-1} |s-v|_+^{d_1+d_2-1} |t-v|_+^{d_4-1} |t+k_2-v|_+^{d_4-1} \\ &\le Cq \sum_{3q<|t|\le n, |s|\le 2q, v} |t|_+^{d_2+2d_3-2} |s|_+^{d_1-1} |s-v|_+^{d_1+d_2-1} |t-v|_+^{2d_4-2} \\ &\le Cq \sum_{3q<|t|\le n, |s|\le 2q} |t|_+^{d_2+2d_3-2} |s|_+^{d_1-1} |t-s|_+^{d_1+d_2-1} \\ &\le Cq \sum_{3q<|t|\le n, |s|\le 2q} |t|_+^{d_1+2d_2+2d_3-3} |s|_+^{d_1-1} \qquad (8.42) \\ &\le Cq^{\lambda_{21}}, \qquad \lambda_{21} := 2d_1 + 2d_2 + 2d_3 - 1 \qquad \text{since } 3 - d_1 - 2d_2 - 2d_3 > 1. \qquad (8.43) \end{aligned}$$

Finally, consider J_{22}. By change $t+k_2 \to \tilde{k}_2, |\tilde{k}_2| \le 3q$ rewrite

$$
\begin{aligned}
(8.44)\qquad J_{22} &\le C \sum_{|t|\le 3q, |k_2|\le 3q, |s|\le 2q, v} |k_2|_+^{d_2+d_3-1} |s|_+^{d_1-1} |t-k_2-s|^{d_3-1} |s-v|_+^{d_1+d_2-1} \\
&\qquad\qquad \times |t-k_2-v|_+^{d_4-1} |k_2-v|_+^{d_4-1} \\
&\le C \sum_{|k_2|\le 3q, |s|\le 2q, v} |k_2|_+^{d_2+d_3-1} |s|_+^{d_1-1} |s-v|_+^{\Delta-2} |k_2-v|_+^{d_4-1},
\end{aligned}
$$

where the last inequality follows by application of (8.14) w.r.t. $t \in \mathbb{Z}$. Since $\Delta < 1$ use (8.14) w.r.t. $v \in \mathbb{Z}$ to obtain

$$
J_{22} \le C \sum_{|k_2|\le 3q, |s|\le 2q} |k_2|_+^{d_2+d_3-1} |s|_+^{d_1-1} |k_2-s|_+^{d_4-1}.
$$

Now apply (8.17) with $a = 1-d_1, b = 1-d_4, c = 1-d_2-d_3$ to obtain

$$
(8.45)\qquad J_{22} \le Cq^{\lambda_{22}}, \qquad \lambda_{22} := \max\{d_1 + (d_2+d_3-1)_+, d_4 + (d_1+d_2+d_3-1)_+\}.
$$

Combining the bounds in (8.36)–(8.45) we get $J \le Cq^{\lambda}$ where

$$
\lambda := \max\{\lambda_{12}^+, \lambda_{12}^-, \lambda_{11}^+, \lambda_{11}^-, \lambda_{12}^-, \lambda_{21}, \lambda_{22}\} < 1
$$

and the last inequality follows $\delta_{\max} < 1/2$ by the explicit form of $\lambda_{ij}^{\pm}$'s, see (8.37), (8.39), (8.40), (8.41), (8.43), (8.45). This proves (8.24) for D3a).

For D3b),

$$
\begin{aligned}
|M_V(0,k_1,t,t+k_2)| &\le C \sum_{s,u,v} |b_{-s,1} b_{k_1-s,1} b_{t-s,3} b_{-u,2} b_{k_1-u,2} b_{t+k_2-u,4} b_{t-v,4} b_{t+k_2-v,3}| \\
&\le C |k_2|_+^{-(1-d_3-d_4)} \sum_s |b_{-s,1} b_{k_1-s,1} b_{t-s,3}| \sum_u |b_{-u,2} b_{k_1-u,2} b_{t+k_2-u,4}|.
\end{aligned}
$$

Therefore, $2(1-d_1) > 1, 2(1-d_2) > 1, 2(1-d_3-d_4) > 1$ and (8.14) yield

$$
\begin{aligned}
(8.46)\qquad \sum_{|t|\le n, |k_1|, |k_2|\le q} |M_V(0,k_1,t,t+k_2)| &\le q \sum_{t,k_2} |M_V(0,0,t,t+k_2)| \\
&\le Cq \sum_{t,k_2,s,u} |k_2|_+^{d_3+d_4-1} |s|_+^{2(d_1-1)} |t-s|_+^{d_3-1} |u|_+^{2(d_2-1)} |t+k_2-u|_+^{d_4-1} \\
&\le Cq \sum_{k_2,t} |k_2|_+^{d_3+d_4-1} |t|_+^{d_3-1} |t+k_2|_+^{d_4-1} \le Cq \sum_{k_2} |k_2|_+^{-2(1-d_3-d_4)} \le Cq.
\end{aligned}
$$

Case D4) It suffices to consider the diagram $s_1 = s_2 = s_3 = s_4 =: s, u_1 = u_2 = u_3 = u_4 =: u$. Then $|M_V(0,k_1,t,t+k_2)| \le C \sum_{s,u} |b_{-s,1} b_{k_1-s,1} b_{t-s,3} b_{t+k_2-s,3} b_{-u,2} b_{k_1-u,2} b_{t-u,4} b_{t+k_2-u,4}|$.

Because $\sum_{u=0}^{\infty} b_{u,i}^2 < \infty, i = 1,2,3,4$, by the C-S inequality and (8.14)

$$
\begin{aligned}
(8.47)\quad & \sum_{|t|\le n, |k_1|, |k_2| \le q} |M_V(0,k_1,t,t+k_2)| \\
& \le \sum_{|t|\le n, k_1, k_2, s, u} |b_{-s,1}| b_{t-s,3} b_{t+k_2-s,3} b_{-u,2} b_{k_1-u,2} b_{t-u,4} b_{t+k_2-u,4}| \\
& \le \prod_{i=1}^{4} \|b_i\| \sum_{|t|\le n, s, u} |s|_+^{d_1-1} |t-s|_+^{d_3-1} |u|_+^{d_2-1} |t-u|_+^{d_4-1} \le C \sum_{|t|\le n} |t|_+^{\Delta-2} \ \le \ C, \quad \text{for } \Delta < 1.
\end{aligned}
$$

This proves (8.21)(a).

Next, we shall prove (8.21)(b), which follows similarly from (8.23) or

$$
(8.48)\qquad \mathcal{M}_{n,q,1}^0 := \sum_{|k_1|,|k_2|\le q} \sum_{|t|\le n} \left|\mathrm{Cum}\big(Y_0, Y'_{k_1}, Y_t, Y'_{t+k_2}\big)\right| \le Cn(q/n)^{\nu}
$$

with $Y_t := Y_{t,12} =: Y_t,\ Y'_t := Y^0_{t,34}$, for some $\nu > 0$. As in (8.25)–(8.27) we can write

$$
\begin{aligned}
(8.49)\qquad \mathrm{Cum}(Y_0, Y'_{k_1}, Y_t, Y'_{t+k_2}) &= \textstyle\sum_{s_i<u_i, i=1,3,s_2,s_4} B^0_{0,k_1,t,t+k_2}(s_1,u_1,s_3,u_3,s_2,s_4) \\
&\quad \times \mathrm{Cum}(\xi_{s_1,1}\xi_{u_1,2}, \xi_{s_2,1}\xi_{s_2,2}, \xi_{s_3,3}\xi_{u_3,4}, \xi_{s_4,3}\xi_{s_4,4}), \\
B^0_{0,k_1,t,t+k_2}(s_1,u_1,s_2,s_3,u_3,s_4) &:= B_{0,k_1,t,t+k_2}(s_1,u_1,s_2,s_2,s_3,u_3,s_4,s_4)
\end{aligned}
$$

where $\mathrm{Cum}(\xi_{s_1,1}\xi_{u_1,2}, \xi_{s_2,1}\xi_{s_2,2}, \xi_{s_3,3}\xi_{u_3,4}, \xi_{s_4,3}\xi_{s_4,4})$ is written as a sum in (8.27) over all connected diagrams $\{V\} = (V_1, \cdots, V_r) \subset \Gamma^c_{T_2}$ over the table T_2 given at (8.26). Then similarly as above $\mathrm{Cum}(Y_0, Y_{k_1}, Y'_t, Y'_{t+k_2}) = \sum_{\{V\}\subset\Gamma^c_{T_2}} M_V(0,k_1,t,t+k_2)$, where $M_V(0,k_1,t,t+k_2) := \sum_{s_i<u_i, i=1,2,s_3,s_4} B^0_{0,k_1,t,t+k_2}(s_1,u_1,s_2,s_3,u_3,s_4)$.

Consider the three diagrams: D5: $s_1 = s_2 =: s, u_1 = u_3 =: u, s_3 = s_4 =: v$; D6: $s_1 = s_4 =: s, u_1 = u_3 =: u, s_2 = s_3 =: v$; and D7: $s_1 = s_3 =: s, u_1 = u_3 = s_2 = s_4 =: u$.

Case D5). Here,

$$
\begin{aligned}
|M_V(0,k_1,t,t+k_2)| &\le C \sum_{s,u,v} |b_{-s,1} b_{-u,2} b_{k_1-s,1} b_{k_1-s,2} b_{t-v,3} b_{t-u,4} b_{t+k_2-v,3} b_{t+k_2-v,4}| \\
&\le C|t|_+^{-(1-d_2-d_4)} \sum_{s,v} |b_{-s,1} b_{k_1-s,1} b_{k_1-s,2} b_{t-v,3} b_{t+k_2-v,3} b_{t+k_2-v,4}|,
\end{aligned}
$$

and hence, by using (8.14) and $2-d_1-d_2 > 1, 2-d_3-d_4 > 1$,

$$
\begin{aligned}
& \sum_{|t|\le n, |k_1|, |k_2|\le q} |M_V(0,k_1,t,t+k_2)| \\
& \le \ C \sum_{|t|\le n, |k_1|,|k_2|\le q, s, v} \frac{1}{|t|_+^{1-d_2-d_4} |s|_+^{1-d_1} |k_1-s|_+^{2-d_1-d_2} |t-v|_+^{1-d_3} |t+k_2-v|_+^{2-d_3-d_4}} \\
& \le \ C \sum_{|t|\le n, |k_1|,|k_2|\le q} \frac{1}{|t|_+^{1-d_2-d_4} |k_1|_+^{1-d_1} |k_2|_+^{1-d_3}} \\
& \le \ Cn^{d_2+d_4} q^{d_1+d_3} \ = \ C(q/n)^{d_1+d_3} n^{\Delta} \le C(q/n)^{d_1+d_3} n. \qquad \Delta \text{ as in (8.19)}.
\end{aligned}
$$

Case D6) Here, we have

$$\begin{aligned} |M_V(0,k_1,t,t+k_2)| &\le C\sum_{s,u,v} |b_{-s,1}b_{-u,2}b_{k_1-v,1}b_{k_1-v,2}b_{t-v,3}b_{t-u,4}b_{t+k_2-s,3}b_{t+k_2-s,4}| \\ &\le C|t|_+^{-(1-d_2-d_4)}\sum_{s,v} |b_{-s,1}b_{k_1-v,1}b_{k_1-v,2}b_{t-v,3}b_{t+k_2-s,3}b_{t+k_2-s,4}|, \end{aligned}$$

and hence, similarly as above,

$$\begin{aligned} &\sum_{|t|\le n,|k_1|,|k_2|\le q} |M_V(0,k_1,t,t+k_2)| \\ &\quad\le C\sum_{|t|\le n,|k_1|,|k_2|\le q,s,v} \frac{1}{|t|_+^{1-d_2-d_4}|s|_+^{1-d_1}|k_1-v|_+^{2-d_1-d_2}|t-v|_+^{1-d_3}|t+k_2-s|_+^{2-d_3-d_4}} \\ &\quad\le C\sum_{|t|\le n,|k_1|,|k_2|\le q} \frac{1}{|t|_+^{1-d_2-d_4}|t+k_2|_+^{1-d_1}|t-k_1|_+^{1-d_3}}. \end{aligned}$$

Split the last sum over $|t|\le n$ into two sums I_1+I_2, where $I_1 := \sum_{|t|\le 2q}$, $I_2 := \sum_{2q<|t|\le n}$. Then $I_1 \le C\sum_{|t|\le 2q}|t|_+^{-(1-d_2-d_4)}\sum_{|k_1|,|k_2|\le 2q}|k_2|_+^{-(1-d_1)}|k_1|_+^{-(1-d_3)} \le Cn^{d_2+d_4}q^{d_1+d_3}$ $\le Cn^{\Delta}(q/n)^{d_1+d_3} \le Cn(q/n)^{d_1+d_3}$ and $I_2 \le Cq^2\sum_{t=2q}^{\infty} t^{-(3-\Delta)} \le C(q/n)^2 n^{\Delta} \le Cn(q/n)^2$.

Case D7) We have

$$\begin{aligned} |M_V(0,k_1,t,t+k_2)| &\le C\sum_{s,u} |b_{-s,1}b_{-u,2}b_{k_1-u,1}b_{k_1-u,2}b_{t-s,3}b_{t-u,4}b_{t+k_2-u,3}b_{t+k_2-u,4}| \\ &\le C|t|_+^{-(1-d_1-d_3)}\sum_{u} |b_{-u,2}b_{k_1-u,1}b_{k_1-u,2}b_{t-u,4}b_{t+k_2-u,3}b_{t+k_2-u,4}| \end{aligned}$$

and hence

$$\begin{aligned} &\sum_{|t|\le n,|k_1|,|k_2|\le q} |M_V(0,k_1,t,t+k_2)| \\ &\quad\le C\sum_{k_1,k_2,u,t} \frac{1}{|t|_+^{1-d_1-d_3}|u|_+^{1-d_2}|k_1-u|_+^{2-d_1-d_2}|t-u|_+^{1-d_4}|t+k_2-u|_+^{2-d_3-d_4}} \\ &\quad\le C\sum_{k_1,u,t} \frac{1}{|t|_+^{1-d_1-d_3}|u|_+^{1-d_2}|k_1-u|_+^{2-d_1-d_2}|t-u|_+^{1-d_4}} \\ &\quad\le C\sum_{k_1,u} \frac{1}{|u|_+^{2-\Delta}|k_1-u|_+^{2-d_1-d_2}} \;\le\; C<\infty. \end{aligned}$$

This completes the proof of (8.48) and, hence, also that of (8.21) and Lemma 8.1(i).

Proof of Lemma 8.1(ii). Recall the decomposition in (8.20). Clearly, it suffices to show (8.9) for each term on the r.h.s. of (8.20). Note $\Delta \le 2\delta_{\max}$ where Δ is defined in (8.19). In order to simplify subsequent evaluations we restrict the discussion to the case

$$1/2 < \delta_{\max} < 1, \qquad 1 < \Delta < 2. \tag{8.50}$$

Following (8.21) we confine the subsequent proof to showing

$$(8.51) \qquad \tilde\sigma_{Y_{12},Y_{34},q} \;=\; O_p(q^{2\delta_{\max}-1}) \qquad \text{and} \qquad \tilde\sigma_{Y_{12},Y^0_{34},q} \;=\; O_p(q^{2\delta_{\max}-1}).$$

Clearly, the first relation in (8.51) follows from the following two relations:

$$(8.52) \quad \text{(a)} \quad |E\tilde\sigma_{Y_{12},Y_{34},q}| \;=\; O(q^{2\delta_{\max}-1}), \qquad \text{(b)} \quad \mathrm{Var}(\tilde\sigma_{Y_{12},Y_{34},q}) = O(q^{2(2\delta_{\max}-1)}).$$

Consider (8.52)(a). We have $|\mathrm{Cov}(Y_{t,12}, Y_{s,34})| = \big|\sum_{u_2<u_1\le t\wedge s} b_{t-u_1,1}b_{t-u_s,2}b_{s-u_1,3}b_{s-u_2,4}\big| \le C|t-s|^{\Delta-1}$ and hence

$$|E\tilde\sigma_{Y_{12},Y_{34},q}| \le \sum_{|k|\le q} |\mathrm{Cov}(Y_{k,12}, Y_{0,34})| \le C\sum_{|k|\le q} |k|^{\Delta-2} = O(q^{\Delta-1}), \quad \text{implying (8.52)}(a).$$

Consider (8.52)(b). By evaluating the variance as in (8.22) we see that (8.52)(b) follows from

$$(8.53) \qquad \mathcal{M}_{n,q} := \sum_{|k_1|,|k_2|\le q}\sum_{|t|\le n} \big|\mathrm{Cov}\big(Y_0Y_{k_1}, Y'_tY'_{t+k_2}\big)\big| \;\le\; Cn\,q^{2(2\delta_{\max}-1)}.$$

Next $\mathcal{M}_{n,q} \le \sum_{i=1}^3 \mathcal{M}_{n,q,i}$ as the proof of (8.23), where $\mathcal{M}_{n,q,2} \le Cq\sum_{|k|\le q,|t|\le n} |k|_+^{\Delta-2}|t+k|_+^{\Delta-2} \le Cq^{\Delta}n^{\Delta-1} \le Cnq^{2\Delta-2}$ $(q \le n)$ and

$$\begin{aligned}(8.54) \quad \mathcal{M}_{n,q,3} \;&\le\; Cq\sum_{|k|\le q,|t|\le n} |t|_+^{\Delta-2}|t+k|_+^{\Delta-2} \\ &\le\; Cq^2(I(\Delta<3/2) + \log(n/q)I(\Delta=3/2) + n^{2\Delta-3}I(\Delta>3/2)) \le Cnq^{2\Delta-2}\end{aligned}$$

do not exceed the r.h.s. of (8.53). Hence (8.52)(b) follows from

$$(8.55) \qquad \mathcal{M}_{n,q,1} = \sum_{|k_1|,|k_2|\le q,|t|\le n} \big|\mathrm{Cum}\big(Y_0, Y_{k_1}, Y'_t, Y'_{t+k_2}\big)\big| \;\le\; Cn\,q^{2(2\delta_{\max}-1)}.$$

To prove (8.55), we rewrite the l.h.s. as the sum $\mathcal{M}_{n,q,1} = \sum_{\{V\}\subset\Gamma^c_{T_2}}\sum_{|k_1|,|k_2|\le q,|t|\le n} M_V(0,k_1, t, t+k_2)$ over all connected diagrams over table T_1 of (8.26) as in the proof of (8.24) and evaluate the last sum for each diagram in the latter proof. We designate the following evaluations as Cases D1′) - D4′) to distinguish from the previous Cases D1) - D4). Obviously, the differences between these evaluations are due to $\delta_{\max} < 1/2$ in the latter case and $\delta_{\max} \ge 1/2$ in the former case.

Case D1′) For D1′a, exactly as in (8.29) we get $\sum_{|k_1|,|k_2|\le q,|t|\le n} M_V(0,k_1,t,t+k_2) \le Cq^{\Delta}$, proving the bound in (8.55) for this case.

For D1′b), and by (8.30) we get $\sum_{|k_1|,|k_2|\le q,|t|\le n} M_V(0,k_1,t,t+k_2) \le Cq\sum_{|k|\le q,|t|\le n}|t|_+^{\Delta-2}|t+k|_+^{\Delta-2} \le Cnq^{2\Delta-2}$ as in (8.54) proving the bound in (8.55) for this case.

For D1′c), the same bound applies, see D1).

Case D2′) For D2′a), by following (8.32) we obtain

$$\sum_{|t|\le n, |k_1|, |k_2|\le q} |M_V(0,k_1,t,t+k_2)| \le Cq^{2\Delta-2},$$

thereby proving the bound in (8.55) for this case.

For D2′b) following (8.33) we obtain

$$\begin{aligned}\sum_{|t|\le n, |k_1|, |k_2|\le q} |M_V(0,k_1,t,t+k_2)| \ &\le \ C \sum_{|t|\le n, |k_1|\le q, s} |t|_+^{d_2+d_4-1}|s|_+^{d_1-1}|k_1-s|_+^{\Delta-2}|t-s|_+^{d_3-1} \\ &\le \ Cq \sum_{|t|\le n, s} |t|_+^{d_2+d_4-1}|s|_+^{d_1+\Delta-3}|t-s|_+^{d_3-1} =: J,\end{aligned}$$

since $\sum_{|k_1|\le q} |k_1-s|_+^{\Delta-2} \le Cq|s|_+^{\Delta-2}, s\in\mathbb{Z}$ follows from (8.16). Using (8.14) we obtain

$$J \le \ Cq \begin{cases} \sum_{|t|\le n} |t|_+^{d_2+d_3+d_4-2}, & d_1+\Delta<2, \\ \sum_{|t|\le n} |t|_+^{2\Delta-4}, & d_1+\Delta>2. \end{cases}$$

Whence one can easily check that J satisfies the bound in (8.55). E.g. for $d_1+\Delta>2, \Delta>3/2$ we get $J\le Cqn^{2\Delta-3}\le Cnq^{2\Delta-2}$ where the last inequality is equivalent to $q^{3-2\Delta}\le Cn^{4-2\Delta}$ which trivially holds due to $3-2\Delta<0, 4-2\Delta>0$. For $d_1+\Delta>2, \Delta<3/2$ we get $J\le Cq\le Cnq^{2\Delta-2}$ due to $q\le n, 3-2\Delta<1$; see (8.50).

Case D3′) For D3′a) following (8.34) recall the decompositions $J=J_1+J_2\le\sum_{i,j=1}^2 J_{ij}, J_{11}\le C(J_{11}^+ + J_{11}^-), J_{12}\le C(J_{12}^+ + J_{12}^-)$ in the evaluation of D3a) above. The respective evaluations (8.36), (8.38), (8.40), (8.41), (8.43), (8.45) must be updated in view of $\delta_{\max}>1/2$.

For $J_{12}^\pm$ bounds (8.36), (8.38) remain valid and (8.55) reduces to

$$\lambda_{12}^\pm \le 2\Delta-1. \tag{8.56}$$

Note $\lambda_{12}^+\le\max\{2\Delta-2,\Delta\}, \lambda_{12}^-\le\max\{2\Delta-2,2\Delta-1\}$, implying (8.56) by $\Delta>1$. For $J_{11}^\pm$ bounds (8.40), (8.41) remain valid with $\lambda_{11}^\pm\le\max\{2\Delta-2,\Delta-1\}$ and (8.55) holds. For J_{21} (8.43) need not be true but (8.42) leads to $J_{21}\le Cq^{d_1}n^{(d_1+2d_2+2d_3-2)_+}(1+\log(n)\mathbf{1}(d_1+2d_2+2d_3=2))$. First, let $d_1+2d_2+2d_3>2$. Then $J_{21}\le Cnq^{2\Delta-2}$ and (8.55) follow from $d_1+2-2\Delta<3-d_1-2d_2-2d_3$ which is equivalent to $1>-2d_4$. Next, let $d_1+2d_2+2d_3\le 2$. Then the same conclusion follows from $d_1+2-2\Delta<1$ which is immediate by $\Delta>1$. For J_{22} following (8.44) and using $\Delta>1$, (8.14), (8.17) we get $J_{22}\le Cq^{\lambda_{22}}$ with $\lambda_{22}=\max\{d_1+(d_2+d_3+d_4-2)_+, d_2+d_3+(d_1+\Delta-2)_+\}<2\Delta-1$ as in (8.56). This proves the required bound $J\le Cnq^{2\delta-2}$ and hence (8.55) for Case D3′a).

For D3′b) in (8.46) we get $\sum_{|t|\le n,|k_1|,|k_2|\le q}|M_V(0,k_1,t,t+k_2)|\le Cq\sum_{|k_2|\le q}|k_2|_+^{-2(1-d_3-d_4)}\le Cq^{2(d_3+d_4)}\le Cq^{2\delta_{\max}}\le Cnq^{2(2\delta_{\max}-1)}$ since $2\delta_{\max}>1, q\le n$.
Case D4′). Following (8.47) we see that this sum is bounded by $Cn^{\Delta-1}$ $(\Delta>1)$ and (8.55) holds by $2-2\Delta<2-\Delta$.

The above calculations prove (8.55), (8.52)(b) and the first relation in (8.51). Using (8.2) we see that the second relation in (8.51) can be reduced to

$$\tilde\sigma_{Y_{12},q}\,\tilde\sigma_{Y^0_{34},q} \;=\; O_p(q^{2\delta_{\max}-1}). \tag{8.57}$$

As noted in the beginning of the proof of Lemma 8.1, $\{Y^0_{t,34}\}$ is a linear process with summable covariance function and finite variance. By applying the criterion in (8.23) (with $Y_t=Y'_t=Y^0_{34}$) it easily follows that $\tilde\sigma_{Y^0_{34},q}=O_p(1)$ provided $q=o(n)$ (GKS Thm. 9.4.1 provides such a result under slightly more stringent condition on q). We also have from the first relation in (8.51) that $\tilde\sigma_{Y_{12},q}=O_p(q^{(2\delta_{\max}-1)/2})$. These facts prove (8.57) and complete the proof of (8.51) and Lemma 8.1. □

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9 References

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